Some Results on Tenacity of Graphs*

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Abstract: The tenacity of an incomplete connected graph G is defined as $T(G) = \min\{\frac{|S|+m(G-S)}{\omega(G-S)} : S \subset V(G), \omega(G-S) > 1\}$, where $\omega(G-S)$ and $m(G-S)$, respectively, denote the number of components and the order of a largest component in $G-S$. This is a reasonable parameter to measure the vulnerability of networks, as it takes into account both the amount of work done to damage the network and how badly the network is damaged. In this paper, we firstly give some results on the tenacity of gear graphs. After that, the tenacity of the lexicographic product of some special graphs are calculated. We also give the exact values for the tenacity of powers of paths. Finally, the relationships between the tenacity and some vulnerability parameters, namely the integrity, toughness and scattering number are established.

Key–Words: Tenacity, Vulnerability, Cartesian product, Gear graph, Powers of graphs, T-set.

1 Introduction

The vulnerability of a communication network, composed of processing nodes and communication links, is of prime importance to network designers. As the network begins losing links or nodes, eventually there is a loss in its effectiveness. Thus, communication networks must be constructed to be as stable as possible, not only with respect to the initial disruption, but also with respect to the possible reconstruction of the network.

The communication network often has as considerable an impact on a network's performance as the processors themselves. Performance measures for communication networks are essential to guide the designers in choosing an appropriate topology.

In order to measure the performance, we are interested in the following performance metrics (there may be others):

(1) the number of elements that are not functioning,

(2) the number of remaining connected sub-networks,

(3) the size of a largest remaining group within which mutual communication can still occur.

The communication network can be represented as an undirected and unweighted graph, where a processor (station) is represented as a node and a communication link between processors (stations) as an edge between corresponding nodes. If we use a graph to model a network, there are many graph theoretical parameters used to describe the vulnerability of communication networks.

Most notably, the vertex-connectivity and edge-connectivity have been frequently used. The difficulty with these parameters is that they do not take into account what remains after the graph is disconnected. Consequently, a number of other parameters have been introduced that attempt to cope with this difficulty, including toughness and edge-toughness in [10,20,25], integrity and edge-integrity in [4,5,6], tenacity and edge-tenacity in [1,8,9,11,12,16,18,21], rupture degree in [14,15,17] and scattering number in [22,23,25]. Unlike the connectivity measures, each of these parameters shows not only the difficulty to break down the network but also the damage that has been caused.

For convenience, we recall some parameters of [2]. Let G be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. For $S \subseteq V(G)$, let $\omega(G-S)$ and $m(G-S)$, respectively, denote the number of components and the order of a largest component in $G-S$. A set $S \subseteq V(G)$ is a *cut set* of G, if either $G-S$ is disconnected or $G-S$ has only one vertex. We shall use $\lceil x \rceil$ for the smallest integer not smaller than x, and $\lfloor x \rfloor$ for the largest integer not larger than x. An edge is said to be *subdivided* when it is replaced by a path of length two connecting its ends, and the internal vertex in this path is a new vertex.

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A subset S of V is called an *independent set* of G if no two vertices of S are adjacent in G. An independent set S is a maximum if G has no independent set S' with $|S'| > |S|$. The *independence number* of G, $\beta(G)$, is the number of vertices in a maximum independent set of G. A subset S of V is called a *covering* of G if every edge of G has at least one end in S. A covering S is a minimum covering if G has no covering S' with $|S'| > |S|$. The *covering number*, $\alpha(G)$, is the number of vertices in a minimum covering of G.

We use Bondy and Murty [2] for terminology and notations not defined here. For comparing, the following graph parameters are listed.

The connectivity is a parameter defined based on Quantity (1). The *connectivity* of an incomplete graph G is defined by

$$\kappa(G) = \min\{|S| : S \subset V(G), \omega(G-S) > 1\},$$

and that of the complete graph K_n is defined as $n-1$.

Both toughness and scattering number take into account Quantities (1) and (2). The *toughness* and *scattering number* of an incomplete connected graph G are defined by

$$t(G) = \min\{\frac{|S|}{\omega(G-S)} : S \subset V(G), \omega(G-S) > 1\}$$

and

$$s(G) = \max\{\omega(G-S) - |S| : S \subset V(G), \omega(G-S) > 1\},$$

respectively.

The integrity is defined based on Quantities (1) and (3). The *integrity* of a graph G is defined by

$$I(G) = \min\{|S| + m(G-S) : S \subset V(G)\}.$$

Both the tenacity and rupture degree take into account all the three quantities. The *tenacity* and *rupture degree* of an incomplete connected graph G are defined by

$$T(G) = \min\{\frac{|S| + m(G-S)}{\omega(G-S)} : S \subset V(G), \omega(G-S) > 1\}$$

and

$$r(G) = \max\{\omega(G-X) - |X| - m(G-X) : X \subset V(G), \omega(G-X) > 1\},$$

respectively. And the tenacity and rupture degree of the complete graph K_n is defined as n and $n-1$, respectively.

The corresponding edge analogues of these concepts are defined similarly, see [6, 20, 21].

From the above definitions, we can see that the connectivity of a graph reflects the difficulty in breaking down a network into several pieces. This invariant is often too weak, since it does not take into account what remains after the corresponding graph is disconnected. Unlike the connectivity, each of the other vulnerability measures, i.e., toughness, scattering number, integrity, tenacity and rupture degree, reflects not only the difficulty in breaking down the network but also the damage that has been caused. Further, we can easily see that the tenacity and rupture degree are the two most advanced ones among these parameters when measuring the vulnerability of networks. Among the above parameters, the tenacity is a reasonable parameter can be used for measuring the vulnerability of networks [1,7,8,10,11,17,18].

A vertex cut set S of a graph G is called a *T-set* of G if it satisfies that $T(G) = \frac{|S|+m(G-S)}{\omega(G-S)}$.

In [11], Moazzami *et al.* compared the integrity, connectivity, binding number, toughness, and tenacity for several classes of graphs. In [9], Choudum *et al.* studied the tenacity of complete graph products and grids. In [16], Li, Ye and Li discussed the tenacity and rupture degree for permutation graphs of complete bipartite graphs. Li, Ye and Sheng gave exact values of rupture degree for some useful graphs in [15]. Cheng *et al.* [8] determined the maximum tenacity of trees and unicyclic graphs with given order and show the corresponding extremal graphs. These results are helpful in constructing stable networks with lower costs. Cozzens *et al.*[11] studied the tenacity of Harary Graphs. In [18], Mann proved that computing the tenacity of a graph is NP-hard in general. So, it is an interesting problem to determine tenacity for some special graphs.

In this paper, we consider the problem of computing the tenacity of graphs. In Section 2, we give some results on the tenacity of gear graphs. After that, the tenacity of the lexicographic product of some special graphs, such as $P_n[P_2]$, $C_n[P_2]$ and $K_{1,n-1}[P_2]$ are calculated in sections 3. We also give the exact values for the tenacity of powers of paths in section 4. Finally, the relationships between the tenacity and some vulnerability parameters, namely the integrity, toughness and scattering number are established in sections 5.

2 Computing the Tenacity of Gear Graphs

Geared systems are used in dynamic modelling. These are graph theoretic models that are obtained by using gear graphs. Similarly the cartesian product of gear graphs, the complement of a gear graph, and the line graph of a gear graph can be used to design a gear network. We know that the tenacity is a reasonable parameter to measure the vulnerability among these parameters. Consequently these considerations motivated us to investigate the tenacity of gear graphs. Now we give the following definitions.

Definition 1 *[2] The* **wheel graph** *with n spokes, W_n, is the graph that consists of an n-cycle and one additional vertex, say u, that is adjacent to all the vertices of the cycle.*

In Figure 1, we display W_6.

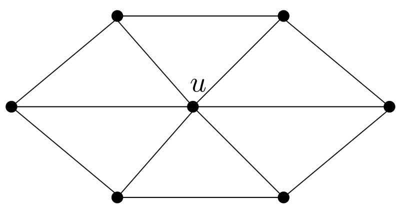

Figure 1: Wheel graph W_6

Definition 2 *[7] The* **gear graph** *G_n is a graph obtained from the wheel graph W_n by subdividing each edge of the outer n-cycle of the W_n just once.*

It is easily seen that the gear graph G_n has $2n+1$ vertices and $3n$ edges. In Figure 2 we display G_6 and we call the vertex u *center vertex* of G_n. Now we give the tenacity of a gear graph.

Theorem 3 *Let G_n be a gear graph. Then*

$$T(G_n) = 1.$$

Proof. On one hand, G_n is an incomplete connected graph with $|V(G_n)| = 2n+1$ number of vertices, let S be a cut set of G_n with $|S| = x$, then the remaining graph $G_n - S$ has at most $x+1$ components, and so,

$$m(G_n - S) \geq \frac{2n+1-x}{x+1}.$$

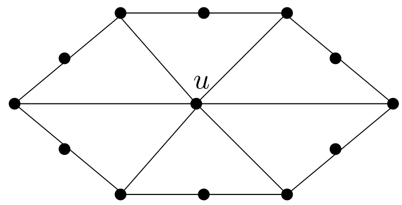

Figure 2: Gear graph G_6

Since

$$\frac{2n+1-x}{x+1} \geq 1,$$

So, x must be at most n.

So we get that

$$T(G_n) \geq min\{\frac{\frac{2n+1-x}{x+1}+x}{x+1}\},$$

where $x \leq n$.

Now we consider the function

$$f(x) = \frac{\frac{2n+1-x}{x+1}+x}{x+1} = \frac{x^2+(2n+1)}{(x+1)^2}.$$

It is easy to see that

$$f'(x) = \frac{2(x-(2n+1))}{(x+1)^3}.$$

Since $x \leq n < 2n+1$, we have $f'(x) < 0$, and so $f(x)$ is a decreasing function.

Thus the function

$$f(x) = \frac{x^2+(2n+1)}{(x+1)^2}.$$

takes its minimum value at $x = n$, and $f_{\min}(x) = 1$. Since this value can be achieved for each n, we have

$$T(G_n) \geq 1. \quad (1)$$

On the other hand, we let S' denotes the covering set of G_n. Then $|S'| = \alpha(G_n) = n$ and $\omega(G_n - S') = \beta(G_n) = n+1$. So $m(G_n - S') = 1$. From the definition of tenacity, we have

$$T(G_n) \leq \frac{|S'| + m(G_n - S')}{\omega(G_n - S')} = \frac{n+1}{n+1} = 1. \quad (2)$$

Consequently, by using (1) and (2), we have $T(G_n) = 1$. $\square$

In the next, we will study the tenacity of complement of gear graph G_n. Firstly, we introduce the concept of the complement of a graph.

Definition 4 *[2] The* **complement** *of a graph G is a graph $\overline{G}$ on the same vertices such that two vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G.*

Theorem 5 *Let G_n be a gear graph. Then*

$$T(\overline{G_n}) = n.$$

Proof. We know that a gear graph G_n can be constructed from a wheel graph W_n by subdividing each edge of the outer cycle of the W_n just once. Let S' be a set of vertices of the outer n-cycle in W_n, and let S'' be a set of vertices which are added to the outer n-cycle in G_n. Let u be a center vertex. Since S' is an independent set of G_n, these vertices form a complete graph with order n in $\overline{G_n}$. Similarly, since $S'' \cup \{u\}$ is an independent set of G_n, these vertices form a complete graph with order $n+1$ in $\overline{G_n}$. Moreover the graph $\overline{G_n}$ contains some edges joining K_{n+1} to K_n. It is obvious that the vertex u in $\overline{G_n}$ is not adjacent to any vertex in K_n. So we have two cases:

Case 1. If we remove the vertices of S' in $\overline{G_n}$, then we have only one component which is graph K_{n+1}. Then

$$m(\overline{G_n} - S') = |V(K_{n+1})| = n + 1$$

and so

$$\frac{|S'| + m(\overline{G_n} - S')}{\omega(\overline{G_n} - S')} = 2n + 1. \tag{3}$$

Case 2. If we remove the vertices of S'' in $\overline{G_n}$, then we have two components which are graphs K_n and K_1. Then

$$m(\overline{G_n} - S'') = |V(K_n)| = n$$

and so

$$\frac{|S''| + m(\overline{G_n} - S'')}{\omega(\overline{G_n} - S'')} = n. \tag{4}$$

By using (3) and (4) we have $T(\overline{G_n}) = \min\{n, 2n + 1\} = n$. □

Now we consider the tenacity of the cartesian product of two graphs.

Definition 6 (2) *The* **Cartesian product** *of two graphs G_1 and G_2, denoted by $G_1 \times G_2$, is defined as follows:*

$$V(G_1 \times G_2) = V(G_1) \times V(G_2),$$

two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2 or u_1 is adjacent to v_1 in G_1 and $u_2 = v_2$.

Observe that if G_1 and G_2 are connected, then $G_1 \times G_2$ is connected.

Lemma 7 *[25] Let G_n be a gear graph. Then*

$$t(K_2 \times G_n) = 1.$$

Lemma 8 *[25] Let $m \geq 3$ and $n \geq 3$ be positive integers. Then*

$$t(G_m \times G_n) = \frac{2mn + m + n}{2mn + m + n + 1}.$$

Lemma 9 *[12] If G is an incomplete connected graph, $\beta(G)$ is the independence number of G and $t(G)$ is the toughness of G, then we have*

$$T(G) \geq t(G) + \frac{1}{\beta(G)}.$$

Theorem 10 *Let G_n be a gear graph. Then*

$$T(K_2 \times G_n) = \frac{2n + 2}{2n + 1}.$$

Proof. It is easy to see that $\beta(K_2 \times G_n) = 2n + 1$. On one hand, by lemmas 7 and 9, we know that

$$\begin{aligned} T(K_2 \times G_n) &\geq t(K_2 \times G_n) + \frac{1}{\beta(K_2 \times G_n)} \\ &= \frac{2n + 2}{2n + 1}. \end{aligned} \tag{5}$$

On the other hand, we can take the covering set S of $K_2 \times G_n$ instead of a vertex cut of $K_2 \times G_n$. Then $|S| = \alpha(K_2 \times G_n) = 2n + 1$ and $\omega(K_2 \times G_n - S) = \beta(K_2 \times G_n) = 2n + 1$. So $m(K_2 \times G_n - S) = 1$. From the definition of tenacity, we have

$$\begin{aligned} T(K_2 \times G_n) &\leq \frac{|S| + m(K_2 \times G_n - S)}{\omega(K_2 \times G_n - S)} \\ &= \frac{2n + 2}{2n + 1}. \end{aligned} \tag{6}$$

Consequently, by using (5) and (6), we have $T(K_2 \times G_n) = \frac{2n+2}{2n+1}$. □

Theorem 11 *Let $m \geq 3$ and $n \geq 3$ be positive integers. Then*

$$T(G_m \times G_n) = 1.$$

Proof. It is obvious that $\beta(G_m \times G_n) = 2mn + m + n + 1$, $\alpha(G_m \times G_n) = 2mn + m + n$. On one hand, by lemmas 8 and 9, we know that

$$T(G_m \times G_n) \geq t(G_m \times G_n) + \frac{1}{\beta(G_m \times G_n)}$$

$$= \frac{2mn+m+n+1}{2mn+m+n+1} = 1. \quad (7)$$

On the other hand, we let S denotes the covering set of $G_m \times G_n$. Then $|S| = \alpha(G_m \times G_n) = 2mn + m + n$ and $\omega(G_m \times G_n - S) = \beta(G_m \times G_n) = 2mn + m + n + 1$. So $m(G_m \times G_n - S) = 1$. From the definition of tenacity, we have

$$\begin{aligned} T(G_m \times G_n) &\leq \frac{|S| + m(G_m \times G_n - S)}{\omega(G_m \times G_n - S)} \\ &= \frac{2mn+m+n+1}{2mn+m+n+1} = 1. \quad (8) \end{aligned}$$

Consequently, by using (7) and (8), we have $T(G_m \times G_n) = 1$. □

Definition 12 *[2] The* **line graph** $L(G)$ *of a graph* G *is a graph such that each vertex of* $L(G)$ *represents an edge of* G*, and any two vertices of* $L(G)$ *are adjacent if and only if their edges are incident, meaning they share a common end vertex in* G*.*

Theorem 13 *Let* G_n *be a gear graph. Then*

$$T(L(G_n)) = \frac{2n+1}{n}.$$

Proof. It is obvious that $\beta(L(G_n)) = n$ and $\alpha(L(G_n)) = 2n$. On one hand, let S be a cut set of $L(G_n)$ with $|S| = x$, then the remaining graph $L(G_n) - S$ has at most $\frac{x}{2}$ components, i.e.

$$\omega(L(G_n) - S) \leq \frac{x}{2},$$

and so

$$m(L(G_n) - S) \geq \frac{3n - x}{\frac{x}{2}}.$$

Since $\frac{3n-x}{\frac{x}{2}} \geq 1$, x must be at most $2n$. Thus we get that

$$T(L(G_n)) \geq \min\{\frac{x + \frac{3n-x}{\frac{x}{2}}}{\frac{x}{2}}\},$$

where $x \leq 2n$.

Now we consider the function

$$f(x) = \frac{x + \frac{3n-x}{\frac{x}{2}}}{\frac{x}{2}} = \frac{2(x^2 - 2x + 6n)}{x^2}.$$

It is easy to see that

$$f^{'}(x) = \frac{4(x - 6n)}{x^3}.$$

Since $x \leq 2n < 6n$, we have $f^{'}(x) < 0$, and so $f(x)$ is a decreasing function. So, the function

$$f(x) = \frac{2(x^2 - 2x + 6n)}{x^2}$$

takes its minimum value at $x = 2n$, and $f_{\min}(x) = \frac{2n+1}{n}$. Since this value can be achieved for each n, we have

$$T(L(G_n)) \geq \frac{2n+1}{n}. \quad (9)$$

On the other hand, we let S' denotes the covering set of $L(G_n)$. Then $|S'| = \alpha(L(G_n)) = 2n$ and $\omega(L(G_n) - S') = \beta(L(G_n)) = n$. So $m(L(G_n) - S') = 1$. From the definition of tenacity, we have

$$\begin{aligned} T(L(G_n)) &\leq \frac{|S'| + m(L(G_n) - S')}{\omega(L(G_n) - S')} \\ &= \frac{2n+1}{n}. \quad (10) \end{aligned}$$

Consequently, by using (9) and (10), we have $T(L(G_n)) = \frac{2n+1}{n}$. □

3 Computing the Tenacity of Lexicographic Product of Graphs

In this section, the tenacity of the lexicographic product of some special graphs, $P_n[P_2]$, $C_n[P_2]$ and $K_{1,n-1}[P_2]$ are calculated.

Definition 14 *[5] The* **lexicographic product** $G_1[G_2]$ *of two graphs* G_1 *and* G_2 *is a graph such that:*

$$V(G_1[G_2]) = V(G_1) \times V(G_2),$$

two distinct vertices (u_1, v_1) *and* (u_2, v_2) *of* $G_1[G_2]$ *are adjacent if and only if either* u_1 *is adjacent to* u_2 *in* G_1 *or* $u_1 = u_2$ *and* v_1 *is adjacent to* v_2 *in* G_2*.*

The lexicographic product is also known as the composition. The lexicographic product is not commutative and is connected whenever G_1 is connected.

Theorem 15 *Let* P_n *be the path with order* n $(n \geq 4)$*, then the tenacity of* $P_n[P_2]$ *is*

$$T(P_n[P_2]) = \begin{cases} \frac{2(n+2)}{n} & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. $P_n[P_2]$ is a connected graph with $|V(P_n[P_2])| = 2n$ number of vertices. We distinguish two cases:

Case 1. When n is even. Let S be a cut set of $P_n[P_2]$ with $|S| = x$, it is obvious that $x \geq 2$.

Subcase 1.1 When $x \geq 3$, we note that $P_n[P_2] - S$ has at most $\lceil \frac{x}{2} \rceil$ components. This implies

$$m(P_n[P_2] - S) \geq \frac{2n - x}{\lceil \frac{x}{2} \rceil}.$$

Since $\frac{2n-x}{\frac{x}{2}} \geq 2$, x must be at most n. Hence, we have that

$$T(P_n[P_2] \geq \min\{\frac{\frac{2n-x}{\frac{x}{2}}+x}{\frac{x}{2}}\},$$

where $x \leq n$.

Now we consider the function

$$f(x) = \frac{\frac{2n-x}{\frac{x}{2}}+x}{\frac{x}{2}} = \frac{2(4n+x^2-2x)}{x^2}.$$

It is easy to see that

$$f^{'}(x) = \frac{4(x-4n)}{x^3}.$$

Since $x \leq n < 4n$, we have $f^{'}(x) < 0$, and so $f(x)$ is a decreasing function. Thus the function

$$f(x) = \frac{2(4n+x^2-2x)}{x^2}$$

takes its minimum value at $x = n$, and $f_{\min}(x) = \frac{2(n+2)}{n}$. Since this value can be achieved for each even n, we have

$$T(P_n[P_2]) \geq \frac{2(n+2)}{n}.$$

Subcase 1.2 When $x = 2$, we note that $P_n[P_2]-S$ has exact 2 components. This implies $m(P_n[P_2]-S) \geq n$. Hence, we have that

$$\frac{|S|+m(P_n[P_2]-S)}{\omega(P_n[P_2]-S)} \geq \frac{2+n}{2}.$$

It is easily seen that when $n \geq 4$,

$$\frac{2+n}{2} \geq \frac{2(n+2)}{n}.$$

So, in this case we have

$$T(P_n[P_2]) \geq \frac{2(n+2)}{n}. \tag{11}$$

On the other hand, as shown in Figure 3, we let

$$S' = \{((u_i,v_1),(u_i,v_2))|i = 2,4,\cdots,n-2\},$$

then $|S'| = n-2$, $\omega(P_n[P_2]-S') = \frac{n}{2}$ and $m(P_n[P_2]-S') = 4$. From the definition of tenacity, we have

$$\begin{aligned} T(P_n[P_2]) &\leq \frac{|S'|+m(P_n[P_2]-S')}{\omega(P_n[P_2]-S')} \\ &= \frac{2(n+2)}{n}. \end{aligned} \tag{12}$$

Consequently, by using (11) and (12), we know that in this case

$$T(P_n[P_2]) = \frac{2(n+2)}{n}. \tag{13}$$

Case 2. When n is odd. Let S be a cut set of $P_n[P_2]$ with $|S| = x$, it is obvious that $x \geq 2$. Note that $P_n[P_2]-S$ has at most $\lfloor\frac{x+2}{2}\rfloor$ components. This implies

$$m(P_n[P_2]-S) \geq \frac{2n-x}{\lfloor\frac{x+2}{2}\rfloor}.$$

Since $\frac{2n-x}{\frac{x+2}{2}} \geq 2$, x must be at most $n-1$. We have that

$$T(P_n[P_2] \geq \min\{\frac{\frac{2n-x}{\frac{x+2}{2}}+x}{\frac{x+2}{2}}\},$$

where $x \leq n-1$.

Now we consider the function

$$f(x) = \frac{\frac{2n-x}{\frac{x+2}{2}}+x}{\frac{x+2}{2}} = \frac{2(x^2+4n)}{(x+2)^2}.$$

It is easy to see that

$$f^{'}(x) = \frac{8(x-2n)}{(x+2)^3}.$$

Since $x \leq n-1 < 2n$, we have $f^{'}(x) < 0$, and so $f(x)$ is a decreasing function. Thus the function

$$f(x) = \frac{2(x^2+4n)}{(x+2)^2}$$

takes its minimum value at $x = n-1$, and $f_{min}(x) = 2$. Since this value can be achieved for each odd n, we have

$$T(P_n[P_2]) \geq 2. \tag{14}$$

On the other hand, as shown in Figure 3, we let

$$S' = \{((u_i,v_1),(u_i,v_2))|i = 2,4,\cdots,n-1\},$$

then $|S'| = n-1$, $\omega(P_n[P_2]-S') = \frac{n+1}{2}$ and $m(P_n[P_2]-S') = 2$. From the definition of tenacity, we have

$$T(P_n[P_2]) \leq \frac{|S'|+m(P_n[P_2]-S')}{\omega(P_n[P_2]-S')} = 2. \tag{15}$$

Hence, by combing (14) and (15), we know that in this case

$$T(P_n[P_2]) = 2. \tag{16}$$

Consequently, by using (13) and (16), we have

$$T(P_n[P_2]) = \begin{cases} \frac{2(n+2)}{n} & \text{if n is even} \\ 2 & \text{if n is odd.} \end{cases}$$

□

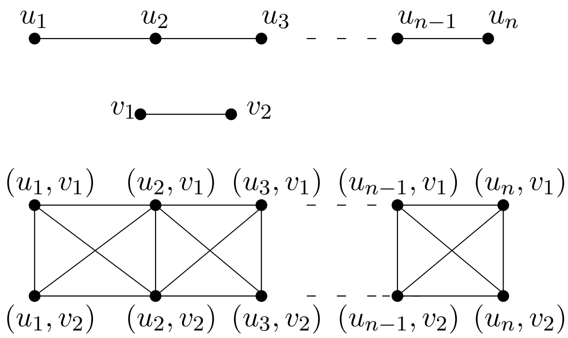

Figure 3: Graphs P_n, P_2 and $P_n[P_2]$

Theorem 16 *Let C_n be the cycle with order n ($n \geq 4$), then the tenacity of $C_n[P_2]$ is*

$$T(C_n[P_2]) = \begin{cases} \frac{2(n+2)}{n} & \text{if } n \text{ is even} \\ \frac{2(n+3)}{n-1} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. $C_n[P_2]$ is a connected graph with $|V(C_n[P_2])| = 2n$ number of vertices. We distinguish two cases to complete the proof.

Case 1. When n is even. Let S be a cut set of $C_n[P_2]$ with $|S| = x$, it is obvious that $x \geq 4$. Note that $C_n[P_2] - S$ has at most $\lfloor \frac{x}{2} \rfloor$ components. This implies

$$m(C_n[P_2] - S) \geq \frac{2n - x}{\lfloor \frac{x}{2} \rfloor}.$$

Since $\frac{2n-x}{\frac{x}{2}} \geq 2$, x must be at most n. Hence, we have that

$$T(C_n[P_2] \geq \min\{\frac{\frac{2n-x}{\frac{x}{2}} + x}{\frac{x}{2}}\},$$

where $x \leq n$.

Now we consider the function

$$f(x) = \frac{\frac{2n-x}{\frac{x}{2}} + x}{\frac{x}{2}} = \frac{2(4n + x^2 - 2x)}{x^2}.$$

It is easy to see that

$$f^{'}(x) = \frac{4(x - 4n)}{x^3}.$$

Since $x \leq n < 4n$, we have $f^{'}(x) < 0$, and so $f(x)$ is a decreasing function. Thus the function

$$f(x) = \frac{2(4n + x^2 - 2x)}{x^2}$$

takes its minimum value at $x = n$, and $f_{min}(x) = \frac{2(n+2)}{n}$. Since this value can be achieved for each even n, we have

$$T(C_n[P_2]) \geq \frac{2(n+2)}{n}. \tag{17}$$

On the other hand, as shown in Figure 4, we let

$$S' = \{((u_i, v_1), (u_i, v_2)) | i = 1, 3, \cdots, n-1\},$$

then $|S'| = n$, $\omega(P_n[P_2] - S') = \frac{n}{2}$ and $m(P_n[P_2] - S') = 2$. From the definition of tenacity, we have

$$\begin{aligned} T(P_n[P_2]) &\leq \frac{|S'| + m(P_n[P_2] - S')}{\omega(P_n[P_2] - S')} \\ &= \frac{2(n+2)}{n}. \end{aligned} \tag{18}$$

Consequently, by using (17) and (18), we know that in case 1

$$T(P_n[P_2]) = \frac{2(n+2)}{n}. \tag{19}$$

Case 2. When n is odd. Let S be a cut set of $C_n[P_2]$ with $|S| = x$, it is obvious that $x \geq 4$.

Subcase 1.1 When $x \geq 5$, we note that $C_n[P_2] - S$ has at most $\lceil \frac{x-2}{2} \rceil$ components. This implies

$$m(C_n[P_2] - S) \geq \frac{2n - x}{\lceil \frac{x-2}{2} \rceil}.$$

Since $\frac{2n-x}{\frac{x-2}{2}} \geq 2$, x must be at most $n + 1$. We have that

$$T(C_n[P_2] \geq \min\{\frac{\frac{2n-x}{\frac{x-2}{2}} + x}{\frac{x-2}{2}}\},$$

where $x \leq n + 1$.

Now we consider the function

$$f(x) = \frac{\frac{2n-x}{\frac{x-2}{2}} + x}{\frac{x-2}{2}} = \frac{2(x^2 - 4x + 4n)}{(x-2)^2}.$$

It is easy to see that

$$f^{'}(x) = \frac{16(1-n)}{(x-2)^3} < 0,$$

so, $f(x)$ is a decreasing function. The function

$$f(x) = \frac{2(x^2 - 4x + 4n)}{(x-2)^2}$$

takes its minimum value at $x = n+1$, and $f_{min}(x) = \frac{2(n+3)}{n-1}$. Since this value can be achieved for each odd n, we have

$$T(C_n[P_2]) \geq \frac{2(n+3)}{n-1}.$$

Subcase 1.2 When $x = 4$, we note that $C_n[P_2] - S$ has exact 2 components. This implies

$$m(C_n[P_2] - S) \geq n - 1.$$

Hence, we have that

$$\frac{|S| + m(C_n[P_2] - S)}{\omega(C_n[P_2] - S)} \geq \frac{3+n}{2}.$$

It is easily seen that when $n \geq 5$,

$$\frac{3+n}{2} \geq \frac{2(n+3)}{n-1}.$$

So, in case 2, we have

$$T(C_n[P_2]) \geq \frac{2(n+3)}{n-1}. \quad (20)$$

On the other hand, as shown in figure 4, we let

$$S' = \{((u_i, v_1), (u_i, v_2))|i = 1, 3, \cdots, n-2\},$$

then $|S'| = n - 1$, $\omega(P_n[P_2] - S') = \frac{n-1}{2}$ and $m(P_n[P_2] - S') = 4$. From the definition of tenacity, we have

$$\begin{aligned} T(P_n[P_2]) &\leq \frac{|S'| + m(P_n[P_2] - S')}{\omega(P_n[P_2] - S')} \\ &= \frac{2(n+3)}{n-1}. \end{aligned} \quad (21)$$

Consequently, by using (20) and (21), we have the following result in case 2

$$T(P_n[P_2]) = \frac{2(n+3)}{n-1}. \quad (22)$$

Hence, by combing (19) and (22), we have

$$T(C_n[P_2]) = \begin{cases} \frac{2(n+2)}{n} & \text{if n is even} \\ \frac{2(n+3)}{n-1} & \text{if n is odd.} \end{cases}$$

□

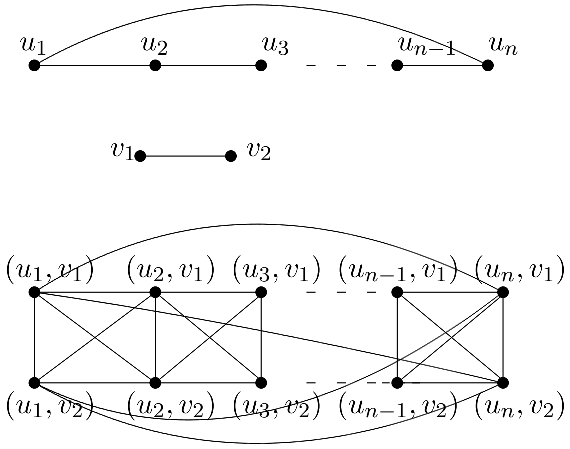

Figure 4: Graphs C_n, P_2 and $C_n[P_2]$

Theorem 17 *Let $K_{1,n-1}$ be the star with order $n(n \geq 4)$, then the tenacity of $K_{1,n-1}[P_2]$ is*

$$T(K_{1,n-1}[P_2]) = \frac{4}{n-1}.$$

Proof. $K_{1,n-1}[P_2]$ is a connected graph with $|V(K_{1,n-1}[P_2])| = 2n$ number of vertices. On one hand, it is easy to see that the connectivity of $K_{1,n}[P_2]$ is $\kappa(K_{1,n-1}[P_2]) = 2$. As shown in Figure 5, the vertex set $S = \{(u, v_1), (u, v_2)\}$ is the unique cut set which satisfies the connectivity of $K_{1,n-1}[P_2]$. If we remove the S from $K_{1,n-1}[P_2]$, then the remaining graph has

$$\omega(K_{1,n-1}[P_2] - S) = n - 1$$

components and the order of the largest component of $K_{1,n-1}[P_2]$ is $m(K_{1,n}[P_2] - S) = 2$. Hence, by the definition of tenacity, we know that

$$\begin{aligned} T(K_{1,n-1}[P_2]) &\leq \frac{|S| + m(K_{1,n-1}[P_2] - S)}{\omega(K_{1,n-1}[P_2] - S)} \\ &= \frac{4}{n-1}. \end{aligned} \quad (23)$$

On the other hand, let S' be a T-set of $K_{1,n}[P_2]$, if S' has more than two vertices, it is easy to see that

$$\omega(K_{1,n-1}[P_2] - S') \leq n - 1$$

and

$$m(K_{1,n}[P_2] - S') \geq 1.$$

Hence, by the definition of tenacity, we know that

$$\begin{aligned} T(K_{1,n-1}[P_2]) &= \frac{|S'| + m(K_{1,n-1}[P_2] - S')}{\omega(K_{1,n-1}[P_2] - S')} \\ &\geq \frac{4}{n-1}. \end{aligned} \quad (24)$$

Consequently, by using(23) and (24), we have $T(K_{1,n-1}[P_2]) = \frac{4}{n-1}$. □

4 Tenacity of Powers of Paths

For an integer $k \geq 1$, the *k-th power* of a graph G, denoted by G^k, is a supergraph with $V(G^k) = V(G)$ and $E(G^k) = \{(u, v) : u, v \in V(G), u \neq v \text{ and } d_G(u, v) \leq k\}$. The second power of a graph is also called its square.

We notice that G^1 is just G itself. So, we let $k \geq 2$ in the following.

As a useful network, power of cycles and paths have arouse interests for many network designers. C.A. Barefoot, et al. gave the exact values of integrity

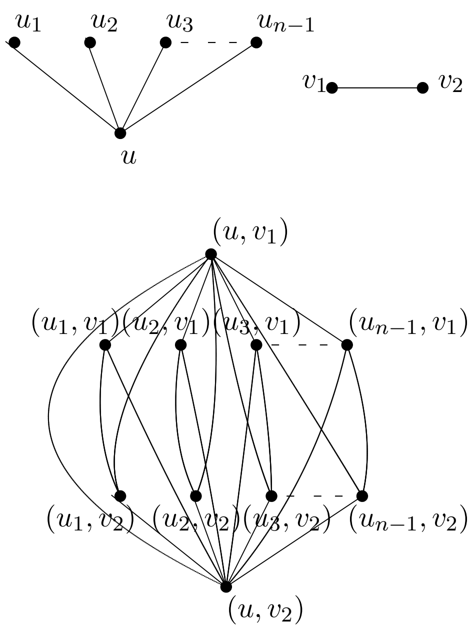

Figure 5: Graphs $K_{1,n-1}$, P_2 and $K_{1,n-1}[P_2]$

of powers of cycles in [3], and determined the connectivity, binding number and toughness of powers of cycles[4]. Vertex-neighbor-integrity of powers of cycles were studied in [13] by Cozzens and Wu. In [19] Moazzami gave the exact values for the tenacity of powers of cycles. Zhang and Yang[24] studied the binding number of the Powers of Paths and cycles.

In this section, we consider the problem of computing the tenacity of powers of paths. It is easy to see that $P_n^k \cong K_n$ if $n \leq k+1$. So, in the following lemmas, we suppose that $2 \leq k \leq n-2$.

Lemma 18 *If S is a minimal T-set for the graph P_n^k, $2 \leq k \leq n-2$, then S consists of the union of sets of k consecutive vertices such that there exists at least one vertex not in S between any two sets of consecutive vertices in S.*

Proof. We assume that the vertices of P_n^k are labeled by $0, 1, 2, \cdots, n-1$. Let S be a minimal T-set of P_n^k and j be the smallest integer such that $T = \{j, j+1, \cdots, j+t-1\}$ is a maximum set of consecutive vertices such that $T \subseteq S$. Relabel the vertices of P_n^k as $v_1 = j, v_2 = j+1, \cdots, v_t = j+t-1$, $\cdots, v_n = j-1$. Since $S \neq V(P_n^k)$ and $T \neq V(P_n^k)$, v_n does not belong to S. Since S must leave at least two components of $G - S$, we have $t \neq n-1$, and so $v_{t+1} \neq v_n$. Therefore, $\{v_{t+1}, v_n\} \cap S = \emptyset$. Now suppose $t < k$. Choose v_i such that $1 \leq i \leq t$, and delete v_i from S yielding a new set $S' = S - \{v_i\}$ with $|S'| = |S| - 1$. By the definition of P_n^k $(1 \leq k \leq \frac{n}{2})$ we know that the edges $v_i v_n$ and $v_i v_{t+1}$ are in $P_n^k - S'$. Consider a vertex v_p adjacent to v_i in $P_n^k - S'$. If $p \geq t+1$, then $p < t+k$. So, v_p is also adjacent to v_{t+1} in $P_n^k - S'$. If $p < n$, then $p \geq n-k+1$ and v_p is also adjacent to v_n in $P_n^k - S'$. Since $t < k$, then v_n and v_{t+1} are adjacent in $P_n^k - S'$. Therefore, we can conclude that deleting the vertex v_i from S does not change the number of components, and so

$$\omega(P_n^k - S') = \omega(P_n^k - S)$$

and

$$m(P_n^k - S') \leq m(P_n^k - S) + 1.$$

Thus, we have

$$\begin{aligned}\frac{|S'| + m(P_n^k - S')}{\omega(P_n^k - S')} &\leq \frac{|S| - 1 + m(P_n^k - S) + 1}{\omega(P_n^k - S)} \\ &= \frac{|S| + m(P_n^k - S)}{\omega(P_n^k - S)} = T(P_n^k).\end{aligned}$$

This is contrary to our choice of S. Thus we must have $t \geq k$. Now suppose $t > k$. Delete v_t from the set S yielding a new set $S_1 = S - \{v_t\}$. Since $t > k$, the edge $v_t v_n$ is not in $P_n^k - S_1$. Consider a vertex v_p adjacent to v_t in $P_n^k - S_1$. Then, $p \geq t+1$ and $p \leq t+k$, and so v_p is also adjacent to v_{t+1} in $P_n^k - S_1$. Therefore, deleting v_t from S yields

$$\omega(P_n^k - S_1) = \omega(P_n^k - S)$$

and

$$m(P_n^k - S_1) \leq m(P_n^k - S) + 1.$$

So,

$$\begin{aligned}\frac{|S_1| + m(P_n^k - S_1)}{\omega(P_n^k - S_1)} &\leq \frac{|S| - 1 + m(P_n^k - S) + 1}{\omega(P_n^k - S)} \\ &= \frac{|S| + m(P_n^k - S)}{\omega(P_n^k - S)} = T(P_n^k),\end{aligned}$$

which is again contrary to our choice of S. Thus, $t = k$, and so S consists of the union of sets of exactly k consecutive vertices. □

Lemma 19 *There is an T-set S for the graph P_n^k, such that all components of $P_n^k - S$ have order $m(P_n^k - S)$ or $m(P_n^k - S) - 1$.*

Proof. Among all T-sets of minimum order, consider those sets with maximum number of minimum order components, and we let s denote the order of a minimum component. Among these sets, let S be one with the fewest components of order s in P_n^k. Suppose $s \leq m(P_n^k - S) - 2$. Note that all of the components must be sets of consecutive vertices. Assume that C_p is a smallest component.

Then $|V(C_p)| = s$, and without loss of generality, let $C_p = \{v_1, v_2, \cdots, v_s\}$. Suppose C_e is a largest component, and so $|V(C_e)| = m(P_n^k - S) = m$ and let $C_e = \{v_j, v_{j+1}, \cdots, v_{j+m-1}\}$. Let $C_1, C_2, \cdots, C_a$ be the components with vertices between v_s of C_k and v_j of C_e, such that $|C_i| = p_i$ for $1 \leq i \leq a$, and let $C_i = \{v_{i_1}, v_{i_2}, \cdots, v_{i_{p_i}}\}$. Now we construct the vertex set S' as

$$S' = S - \{v_{s+1}, v_{1_{p_1+1}}, v_{2_{p_2+1}}, \cdots, v_{a_{p_a+1}}\}$$
$$\cup \{v_{1_1}, v_{2_2}, \cdots, v_{a_1}, v_j\}.$$

Therefore, $|S'| = |S|$,

$$m(P_n^k - S') \leq m(P_n^k - S)$$

and

$$\omega(P_n^k - S') = \omega(P_n^k - S).$$

So we have

$$\frac{|S'| + m(P_n^k - S')}{\omega(P_n^k - S')} \leq \frac{|S| + m(P_n^k - S)}{\omega(P_n^k - S)}.$$

Therefore,

$$T(P_n^k) = \frac{|S'| + m(P_n^k - S')}{\omega(P_n^k - S')}.$$

But, $P_n^k - S'$ has one less components of order s than $P_n^k - S$, a contradiction. Thus, all components of $P_n^k - S$ have order $m(P_n^k - S)$ or $m(P_n^k - S) - 1$. So, $m(P_n^k - S) = \lceil \frac{n-k(\omega-1)}{\omega} \rceil$. □

By the above two lemmas we give the exact values of tenacity of the powers of paths.

Theorem 20 *Let P_n^k be a powers of a path P_n and $n = r(k+1) + s$ for $0 \leq s < k+1$. Then*

$$T(P_n^k) = \begin{cases} n, & \text{if } n \leq k+1 \\ \frac{k(r-1)+\lceil \frac{n-k(r-1)}{r} \rceil}{r}, & \text{if } n > k+1 \text{ and } s=0 \\ \frac{kr+\lceil \frac{n-kr}{r+1} \rceil}{r+1}, & \text{if } n > k+1 \text{ and } s \neq 0. \end{cases}$$

Proof. If $n \leq k+1$, then $P_n^k = K_n$, so, $T(P_n^k) = n$. If $n > k+1$, let S be a minimum T-set of P_n^k. By Lemmas 18 and 19 we know that

$$|S| = k(\omega - 1)$$

and

$$m(P_n^k - S) = \lceil \frac{n - k(\omega - 1)}{\omega} \rceil.$$

We distinguish two cases:

Case 1. If $s = 0$, then $n = r(k+1)$, by

$$m(P_n^k - S) = \lceil \frac{n - k(\omega - 1)}{\omega} \rceil \geq 1.$$

We know that $2 \leq \omega \leq r$. Thus, by the definition of tenacity we have

$$T(P_n^k) = \min\{\frac{k(\omega-1) + \lceil \frac{n-k(\omega-1)}{\omega} \rceil}{\omega} | 2 \leq \omega \leq r\}.$$

Now we consider the function

$$f(\omega) = \frac{k(\omega-1) + \lceil \frac{n-k(\omega-1)}{\omega} \rceil}{\omega}.$$

It is easy to see that

$$f'(\omega) = \frac{k}{\omega^2} + \lceil \frac{k\omega - 2n - 2k}{\omega^3} \rceil = \lceil \frac{2k\omega - 2n - 2k}{\omega^3} \rceil.$$

Since $\omega^3 > 0$, we have $f'(\omega) \leq 0$ if and only if $g(\omega) = 2k\omega - 2n - 2k \leq 0$. Since the root of the equation $g(\omega) = 2k\omega - 2n - 2k = 0$ is $\omega = \frac{n+k}{k}$. When $\omega \leq \frac{n+k}{k}$, it is easily seen that $\frac{n+k}{k} > r$, so, if $2 \leq \omega \leq r$, we have $g(\omega) \leq 0$, so, $f'(\omega) \leq 0$, and so $f(\omega)$ is a decreasing function and the minimum value occurs at the boundary. Thus $\omega = r$. Then,

$$T(P_n^k) = \frac{k(r-1) + \lceil \frac{n-k(r-1)}{r} \rceil}{r}.$$

Case 2. If $s \neq 0$, then $n = r(k+1) + s$, by

$$m(P_n^k - S) = \lceil \frac{n - k(\omega - 1)}{\omega} \rceil \geq 1.$$

We know that $2 \leq \omega \leq r+1$. Thus, by the definition of tenacity we have

$$T(P_n^k) = \min\{\frac{k(\omega-1) + \lceil \frac{n-k(\omega-1)}{\omega} \rceil}{\omega} | 2 \leq \omega \leq r+1\}.$$

Now we consider the function

$$f(\omega) = \frac{k(\omega-1) + \lceil \frac{n-k(\omega-1)}{\omega} \rceil}{\omega}.$$

It is easy to see that

$$f'(\omega) = \frac{k}{\omega^2} + \lceil \frac{k\omega - 2n - 2k}{\omega^3} \rceil = \lceil \frac{2k\omega - 2n - 2k}{\omega^3} \rceil.$$

Since $\omega^3 > 0$, we have $f'(\omega) \leq 0$ if and only if $g(\omega) = 2k\omega - 2n - 2k \leq 0$. Since the root of the equation $g(\omega) = 2k\omega - 2n - 2k = 0$ is $\omega = \frac{n+k}{k}$. When $\omega \leq \frac{n+k}{k}$, it is easily seen that $\frac{n+k}{k} > r+1$, so, if $2 \leq \omega \leq r+1$, we have $g(\omega) \leq 0$, so, $f'(\omega) \leq 0$, and so $f(\omega)$ is a decreasing function and the minimum value occurs at the boundary. Thus $\omega = r+1$. Then,

$$T(P_n^k) = \frac{kr + \lceil \frac{n-kr}{r+1} \rceil}{r+1}.$$

□

5 Relationships Between Tenacity and Some Other Vulnerability Parameters

In this section, the relationships between the tenacity and some vulnerability parameters, namely the integrity, toughness and scattering number are established.

Theorem 21 *[2] For any graph G of order n,*

$$\beta(G) + \alpha(G) = n.$$

Theorem 22 *If G is an incomplete connected graph, $I(G)$ is the integrity of G and $\beta(G)$ is the independence number of G, then we have*

$$T(G) \geq \frac{I(G)}{\beta(G)}.$$

Proof. Suppose that S is a T-set of G. Then, by the definition, we have

$$T(G) = \frac{|S| + m(G-S)}{\omega(G-S)}.$$

It is obvious that $\omega(G-S) \leq \beta(G)$, $|S|+m(G-S) \geq I(G)$. So we have

$$T(G) = \frac{|S| + m(G-S)}{\omega(G-S)} \geq \frac{I(G)}{\beta(G)}.$$

□

The result in Theorem 22 is best possible, this can be shown by the gear graph $G = G_n$.

Lemma 23 *[5] If G is an incomplete connected graph, $I(G) = \kappa(G)+1$ if and only if $\kappa(G) = \alpha(G)$.*

Theorem 24 *Let G be an incomplete connected graph, if $\kappa(G) = \alpha(G)$, then we have*

$$T(G) = \frac{\kappa(G)+1}{\beta(G)}.$$

Proof. Let we select the maximum covering set S be a cut-set of G. Then, $|S| = \alpha(G)$, and by Theorem 14, we have

$$\omega(G-S) = n - \alpha(G) = \beta(G),$$

$m(G-S) = 1$, by the definition of tenacity, we have

$$T(G) \leq \frac{|S| + m(G-S)}{\omega(G-S)} = \frac{\alpha(G)+1}{\beta(G)}$$

$$= \frac{\kappa(G)+1}{\beta(G)}.$$

On the other hand, by Theorem 22 and Lemma 23, we have

$$T(G) \geq \frac{I(G)}{\beta(G)} = \frac{\kappa(G)+1}{\beta(G)}.$$

Thus, when $\kappa(G) = \alpha(G)$, we have $T(G) = \frac{\kappa(G)+1}{\beta(G)}$.
□

Theorem 25 *If G is an incomplete connected graph, $t(G)$ is the toughness of G and $\alpha(G)$ is the covering number of G, then we have*

$$T(G) \geq t(G)(1 + \frac{1}{\alpha(G)}).$$

Proof. Suppose that S is a T-set of G. Then, by the definition, we have

$$T(G) = \frac{|S| + m(G-S)}{\omega(G-S)}.$$

It is obvious that $\omega(G-S) \leq \beta(G)$, $m(G-S) \geq 1$, and $|S| \leq \alpha(G)$. So we have

$$\begin{aligned} T(G) &= \tfrac{|S|+m(G-S)}{\omega(G-S)} \\ &= \tfrac{|S|}{\omega(G-S)}(\tfrac{|S|+m(G-S)}{|S|}) \\ &\geq t(G)(1 + \tfrac{m(G-S)}{|S|}) \\ &\geq t(G)(1 + \tfrac{1}{\alpha(G)}). \end{aligned}$$

□

The result in Theorem 25 is best possible, this can be shown by the graph $G = K_{1,n-1}$.

Theorem 26 *If G is an incomplete connected graph, $s(G)$ is the scattering number of G and $\beta(G)$ is the independence number of G, then we have*

$$s(G) \leq \frac{n+1}{T(G)+1} - \kappa(G).$$

Proof. Let S be a cut-set of G. Then, by the definition of scattering number, we have

$$s(G) \geq \omega(G-S) - |S|.$$

It is easy to see that

$$|S| + m(G-S) \leq n + 1 - \omega(G-S).$$

Then, by the definition of tenacity, we have

$$T(G) \leq \frac{|S| + m(G-S)}{\omega(G-S)} \leq \frac{n+1-\omega(G-S)}{\omega(G-S)}.$$

So, we have

$$\omega(G-S) \leq \frac{n+1}{T(G)+1}.$$

On the other hand, we know that $|S| \geq \kappa(G)$. Thus

$$\omega(G-S) - |S| \leq \frac{n+1}{T(G)+1} - \kappa(G).$$

By the definition of scattering number and the choice of S, we know that

$$s(G) = max\{\omega(G-S) - |S|\} \leq \frac{n+1}{T(G)+1} - \kappa(G).$$

$\square$

The result in Theorem 26 is best possible, this can be shown by the graph $G = K_{1,n-1}$.

6 Conclusion

If a system such as a communication network is modeled by a graph G, there are many graph theoretical parameters used to describe the vulnerability of communication networks including connectivity, integrity, toughness, binding number, tenacity and rupture degree. Two ways of measuring the vulnerability of a network is through the ease with which one can disrupt the network, and the cost of a disruption. Connectivity has the least cost as far as disrupting the network, but it does not take into account what remains after disruption. One can associate the cost with the number of the vertices destroyed to get small components and the reward with the number of the components remaining after destruction. The tenacity measure is compromise between the cost and the reward by minimizing the cost: reward ratio. Thus, a network with a large tenacity performs better under external attack. In this paper, we have obtained the exact values or bounds for the tenacity of some special graphs.

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References:

[1] Vecdi Aytaç, Computing the tenacity of some graphs, *Selçuk J. Appl. Math.*, Vol.10, 2009,(1):pp. 107-120.

[2] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, Macmillan, London and Elsevier, New york, 1976.

[3] C. A. Barefoot, R. Entringer and Henda C. Swart, Integrity of trees and powers of cycles, *Congress. Numer.*, Vol.58, 1987, pp.103-114.

[4] C. A. Barefoot, R. Entringer and H. Swart, Vulnerability in graphs - A comparative survey, *J. Combin. Math. Combin. Comput.*, Vol.1, 1987, pp. 12-22.

[5] K. S. Bagga, L. W. Beineke, W. D. Goddard, M. J. Lipman and R. E. Pippert, A survey of integrity. *Discrete Appl. Math.*, Vol.37/38, 1992, pp. 13-28.

[6] K. S. Bagga, L. W. Beineke, M. J. Lipman and R. E. Pippert, Edge-integrity- a survey, *Discrete math.*, Vol.124, 1994, pp. 3-12.

[7] A. Brandstädt, V. B. Le and J. P. Spinrad, *Graph classes: a survey*, SIAM, Philadelphia, PA, 1999.

[8] T. C. E. Cheng, Yinkui Li, Chuandong Xu and Shenggui Zhang, Extreme tenacity of graphs with given order and size, *arXiv reprint arXiv:1109.4673v1 [math.CO](2011).*

[9] S. A. Choudum and N. Priya, Tenacity of Complete Graph Products and Grids, *Networks*, Vol.34, 1999, pp. 192-196.

[10] V. Chvátal, Tough graph and hamiltonian circuits, *Discrete Math.*, Vol.5, 1973, pp. 215-228.

[11] M. Cozzens D. Moazzami and S. Stueckle, the Tenacity of Harary Graphs, *J.Comb. Math. Comb. Comput.*, Vol.16, 1994, pp. 33-56.

[12] M. Cozzens, D. Moazzami and S. Stueckle, The tenacity of a graph, *Proc. 7th International Conference on the Theory and Applications of Graphs*, Wiley, New York, 1995, pp. 1111-1122.

[13] M. B. Cozzens and Shu-Shih Y. Wu, Vertex-Neighbor-Integrity of Powers of Cycles, *Ars Combinatoria*, Vol.48, 1998, pp. 257-270.

[14] A. Kirlangic, The Rupture Degree and Gear Graphs, *Bull. Malays. Math. Sci. Soc.*, Vol.32, 2009, (1):pp. 31-36.

[15] F. W. Li, Q. F. Ye and B. H. Sheng, Computing rupture degrees of some graphs, *WSEAS Transactions on Mathematics*, Vol.11, 2012, (1): pp. 23-33.

[16] F. W. Li, Q. F. Ye and X. L. Li, Tenacity and rupture degree of permutation graphs of complete bipartite graphs, *Bull. Malays. Math. Sci. Soc.*, Vol. 34, 2011, (3): pp. 423-434.

[17] Y. K. Li, S. G. Zhang and X. L. Li, Rupture degree of graphs, *Int. J. Comput. Math.*, Vol.82, 2005, (7):pp.793-803.

[18] D. E. Mann, *The tenacity of trees*, Ph.D.Thesis, Northeastern University, 1993.

[19] D. Moazzami, Vulnerability in graphs - A comparative survey, *J. Combin. Math. Combin. Comput.*, Vol. 30, 1999, pp. 23-31.

[20] Y. H. Peng, C. C. Chen and K. M. Koh, On the edge-toughness of a graph (I), *Southeast Asian Math. Bull.*, Vol.12, 1988, pp. 109-122.

[21] B. L. Piazza, F. S. Robertst and S. K. Stueckle, Edge-tenacious networks, *Networks*, Vol.25, 1995, pp. 7-17.

[22] S. G. Zhang, X. L. Li and X. L. Han, Computing the scattering number of graphs, *Int. J. Comput. Math*, Vol. 79, 2002,(2):pp.179-187.

[23] S. G. Zhang, Z. G. wang, Scattering number in graphs, *Networks*, Vol.37, 2001, (2):pp. 102-106.

[24] X. K. Zhang and C. M. Yang, The binding number of the Powers of Path and circuit, *Journal of guangdong institute for nationalities*, Vol.4, 1993, pp. 66-70.

[25] E. Aslan and A. Kirlangic, Computing The Scattering Number and The Toughness for Gear Graphs, *Bull.Soc.Math. Banja Luka*, Vol. 18, 2011, pp. 5-15.