Adomian decomposition method to find the approximate solutions for the fractional PDEs

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Abstract: By introducing the fractional derivatives in the sense of caputo, we use the Adomian decomposition method to construct the approximate solutions for some fractional partial differential equations with time and space fractional derivatives via the time and space fractional derivatives wave equation, the time and space fractional derivatives reduced wave equation and the (1+1)-dimensional Burger's equation . The result of this problems reveal that the Adomian decomposition method is very powerful, effective, convenient and quite accurate to systems of nonlinear fractional equation.

Key–Words: Adomian decomposition method, Fractional calculus, The fractional nonlinear partial differential equations.

1 Introduction

In recent years, there has been a great deal of interest in fractional differential equations. First there were almost no practical applications of fractional calculus, and it was considered by many as an abstract area containing only mathematical manipulations of little or no use. Nearly 30 years ago, the paradigm began to shift from pure mathematical formulations to applications in various fields. During the last decade Fractional Calculus has been applied to almost every field of science, engineering, and mathematics. Several fields of application of fractional differentiation and fractional integration are already well established, some others have just started. Many applications of fractional calculus can be found in turbulence and fluid dynamics, stochastic dynamical system, plasma physics and controlled thermonuclear fusion, nonlinear control theory, image processing, nonlinear biological systems, astrophysics [1]–[11]. Historical summaries of the developments of fractional calculus can be found in [1]-[3]. There has been some attempt to solve linear problems with multiple fractional derivatives (the so-called multi-term equations) [2, 12]. Not much work has been done for nonlinear problems and only a few numerical schemes have been proposed to solve nonlinear fractional differential equations. More recently, applications have included classes of nonlinear equation with multi-order fractional derivative and this motivates us to develop a numerical scheme for their solutions [13]. Numerical and analytical methods have included Adomian decomposition method (ADM) [14]-[16], variational iteration method (VIM) [17], and homotopy perturbation method [19]-[20].

The main objective of the present paper is to use the Adomian decomposition method [14]-[16] to calculate the approximate solutions of the following fractional partial differential equations of the form:

(i) The time and space fractional wave equation with the variable coefficient [21]:

$$\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}} = G(x)\frac{\partial^{2\beta}u}{\partial x^{2\beta}},\ \ t>0, 0<\alpha,\beta\leqslant 1, \tag{1}$$

where $G(x)$ is an arbitrary function of x .

(ii) The time and space fractional reduced wave equation with the variable coefficient [22]:

$$\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}} = \frac{1}{2}y^2\frac{\partial^{2\beta}u}{\partial x^{2\beta}}+\frac{1}{2}x^2\frac{\partial^{2\gamma}u}{\partial y^{2\gamma}},\ \ t>0,\ 0<\alpha,\beta,\gamma\leqslant 1. \tag{2}$$

(iii) The time and space fractional nonlinear

Burger's equation [23]:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \mu \frac{\partial^{2\beta} u}{\partial x^{2\beta}} - \lambda u \frac{\partial^\beta u}{\partial x^\beta}, \qquad t > 0, \quad 0 < \alpha, \beta \leqslant 1, \tag{3}$$

where μ and λ are arbitrary constants.

The function $u(x,t)$ is assumed to be a causal function of time and space, i.e. vanishing for $t < 0$ and $x < 0$. The fractional derivatives are considered in the Caputo sense. The general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various responses.

2 Preliminaries and notations

We give some basic definitions and properties of the fractional calculus theory which are used further in [2, 3, 24, 25].

Definition 1 *A real function $f(x), x > 0$, is said to be in the space $C_\mu, \mu \in R$, if there exists a real number $(p > \mu)$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C(0, \infty)$, and it is said to be in the space C_μ^m if $f^m \in C_\mu, m \in N$.*

Definition 2 *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_\mu, \mu \geq -1$, is defined as*

$$\begin{aligned} & J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt, \\ & \qquad \alpha > 0, \ \ x > 0, \\ & J^0 f(x) = f(x). \end{aligned} \tag{4}$$

Properties of the operator J^α can be found in [2, 3, 24, 25], we mention only the following:

For $f \in C_\mu, \ \mu \geq -1, \ \alpha, \beta \geq 0$ and $\gamma > -1$:

(1) $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$,

(2) $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$,

(3) $J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$.

The Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D^α proposed by M. Caputo in his work on the theory of viscoelasticity [2, 3, 24, 25].

Definition 3 *The fractional derivative of $f(x)$ in the Caputo sense is defined as*

$$D_t^\alpha u(x,t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m u(x,\tau)}{\partial \tau^m} d\tau, \\ \quad for\ m-1 < \alpha < m \\ \frac{\partial^m u(x,t)}{\partial t^m}, \quad for\ \alpha = m \in N \end{cases} \tag{5}$$

For more information on the mathematical properties of fractional derivatives and integrals one can consult the mentioned references.

3 Basic idea of the Adomain decomposition method (ADM)

In this section, we illustrate the idea of the Adomain decomposition method [14]-[16]. Let us consider the nonlinear differential equation

$$L(u) + R(u) + N(u) - g(r) = 0, \tag{6}$$

where L is the highest order derivative which assumed to be invertible, R is a linear differentiable operator of order less than L, N is a nonlinear differentiable operator. Applying the inverse operator L^{-1} to both sides of (6) and using the given condition

$$u = f - L^{-1}[R(u) + N(u)] \tag{7}$$

where the function f represents the terms arising from the integrating the source term g and by using the given condition. Adomain's decomposition method [14]-[16] defines the solution $u(x)$ by the series:

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t) \tag{8}$$

where the components $u_k(x,t)$ are usually determined recurrently by using the relation:

$$\begin{aligned} & u_0(x,t) = f \\ & u_{k+1}(x,t) = -L^{-1}[R(u_k) + N(u_k)], k \geq 0. \end{aligned} \tag{9}$$

The nonlinear operator $N(u)$ can be decomposed into an infinite series of a polynomials given by

$$N(u) = \sum_{k=0}^{\infty} A_k \tag{10}$$

where A_k are so called the Adomains polynomials which given by

$$A_k = \frac{1}{k!} \left[\frac{d^k}{d\lambda^k} \left\{ N \left(\sum_{k=o}^{\infty} \lambda^i u_i(x,t) \right) \right\} \right]_{\lambda=0} \tag{11}$$

4 ADM for time and space fractional wave equation with the constant coefficient

In this section, we use the Adomain decomposition method to calculate the approximate solution of the time and space fractional wave equation in the following form:

$$\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}} = C^2\frac{\partial^{2\beta}u}{\partial x^{2\beta}},\ \ t>0,\ 0<\alpha,\beta\le 1. \quad (12)$$

with the initial conditions

$$u(x,0)=e^{-x},\ \ \frac{\partial^{\alpha}u}{\partial t^{\alpha}}(x,0)=0, \quad (13)$$

where C is an arbitrary constant. By using the inverse operator of D_t^{α} to both sides of Eq.(12), we have

$$\begin{aligned} u(x,t)= \ & u(x,0)+\tfrac{t^{\alpha}}{\Gamma(\alpha+1)}\tfrac{\partial^{\alpha}u}{\partial t^{\alpha}}(x,0) \\ & +C^2J^{2\alpha}[D_x^{\beta}(D_x^{\beta}u(x,t))], \end{aligned} \quad (14)$$

where $D_x^{\beta}=\frac{\partial^{\beta}}{\partial x^{\beta}}$, $D_t^{\alpha}=\frac{\partial^{\alpha}}{\partial t^{\alpha}}$ and J^{α} is the inverse operator of D_t^{α}.

The Adomain decomposition method leads to

$$u_0(x,t)=u(x,0)+\frac{t^{\alpha}}{\Gamma(\alpha+1)}\frac{\partial^{\alpha}u}{\partial t^{\alpha}}(x,0) \quad (15)$$

$$u_{k+1}(x,t)=C^2J^{2\alpha}[D_x^{\beta}(D_x^{\beta}u_k(x,t))],\ k\ge 0. \quad (16)$$

Applying the recursive relation (16) and the initial conditions (13), we get the following results:

$$u_0(x,t)=e^{-x},$$

$$u_1(x,t)=\frac{C^2t^{2\alpha}}{\Gamma(2\alpha+1)}\sum_{n=1}^{\infty}\frac{(-1)^n x^{n-2\beta}}{\Gamma(n+1-2\beta)},$$

$$u_2(x,t)=\frac{C^4t^{4\alpha}}{\Gamma(4\alpha+1)}\sum_{n=1}^{\infty}\frac{(-1)^n x^{n-4\beta}}{\Gamma(n+1-4\beta)},$$

$$u_3(x,t)=\frac{C^6t^{6\alpha}}{\Gamma(6\alpha+1)}\sum_{n=1}^{\infty}\frac{(-1)^n x^{n-6\beta}}{\Gamma(n+1-6\beta)},$$

....

$$u_k(x,t)=\frac{C^{2k}t^{2k\alpha}}{\Gamma(2k\alpha+1)}\sum_{n=1}^{\infty}\frac{(-1)^n x^{n-2k\beta}}{\Gamma(n+1-2k\beta)}, \quad (17)$$

Thus the approximate solution of Eq.(12) in a series form is given by

$$\begin{aligned} & u(x,t)=u_0+u_1+u_2+....+u_k+... \\ & u(x,t)=e^{-x}+\sum_{n=1}^{\infty}(-1)^n\left\{\tfrac{C^2t^{2\alpha}x^{n-2\beta}}{\Gamma(2\alpha+1)\Gamma(n+1-2\beta)}\right. \\ & +\tfrac{C^4t^{4\alpha}x^{n-4\beta}}{\Gamma(4\alpha+1)\Gamma(n+1-4\beta)} \\ & +\tfrac{C^6t^{6\alpha}x^{n-6\beta}}{\Gamma(6\alpha+1)\Gamma(n+1-6\beta)}+.... \\ & +\tfrac{C^{2k}t^{2k\alpha}x^{n-2k\beta}}{\Gamma(2k\alpha+1)\Gamma(n+1-2k\beta)}+.... \end{aligned} \quad (18)$$

Figure 1 and Table 1 illustrate the behavior of the analytic approximate solutions (18).

x/t	0.1	0.2	0.3
0.1	0.904457	0.903508	0.902071
0.2	0.818103	0.81654	0.814173
0.3	0.739995	0.737946	0.734844
0.4	0.669336	0.666889	0.663186
0.5	0.605414	0.602636	0.598432
0.6	0.547583	0.544528	0.539906
0.7	0.495263	0.544528	0.487001
0.8	0.447928	0.491975	0.395936
0.9	0.405102	0.444444	0.439174
1	0.366356	0.401454	0.395936

x/t	0.4	0.5	0.6
0.1	0.900177	0.897839	0.895064
0.2	0.811054	0.807211	0.802658
0.3	0.730759	0.725729	0.719772
0.4	0.658311	0.652309	0.645207
0.5	0.5929	0.586093	0.578039
0.6	0.533824	0.526342	0.517494
0.7	0.480458	0.47241	0.462894
0.8	0.432243	0.423719	0.413643
0.9	0.388678	0.379754	0.369208
1	0.349313	0.340055	0.329115

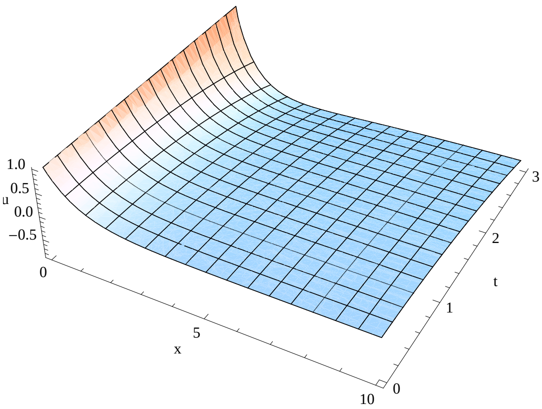
x/t	0.7	0.8	0.9
0.1	0.891855	0.888206	0.884112
0.2	0.797397	0.791429	0.784745
0.3	0.712898	0.705106	0.696389
0.4	0.637015	0.627735	0.617361
0.5	0.568754	0.558242	0.546497
0.6	0.507296	0.495754	0.482865
0.7	0.45193	0.439526	0.425679
0.8	0.402036	0.388909	0.374259
0.9	0.357063	0.34333	0.328008
1	0.316519	0.30228	0.286397

Table 1.show the approximate solution for the different values of $0<x\le 1$ and $0<t<1$ when $\beta=0.1,\ \alpha=0.9,\ k=0.5,\ N=100, 0<x<3.$

5 ADM for time and space fractional wave equation with the variable coefficient

In this section, we use the Adomain decomposition method to calculate the approximate solution of the time and space fractional wave equation with the variable coefficient in the following form:

$$\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}} = x^2\frac{\partial^{2\beta}u}{\partial x^{2\beta}}, \qquad t>0,\ 0<\alpha,\beta\le 1, \quad (19)$$



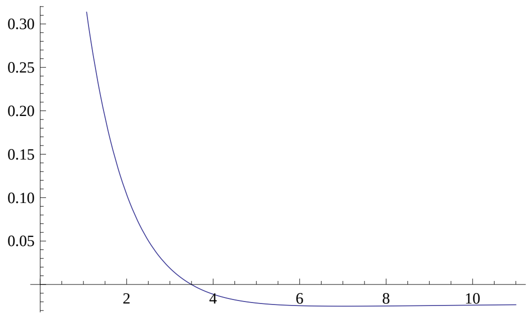


Fig 1 The surface shows the approximate solution $u(x,t)$ for Eq (18) :

(a) $\beta = 0.1,\ \alpha = 0.9,\ k = 0.5,\ N = 100, 0 < t < 3, 0 < x < 3,$

(b) The projection of the surface when $\beta = 0.1,\ \alpha = 0.9,\ k = 0.5,\ N = 100, 0 < x < 3, t = 0.5,$

with the initial conditions

$$u(x,0) = \cos(x), \quad \frac{\partial^\alpha u}{\partial t^\alpha}(x,0) = 0, \tag{20}$$

By using the inverse operator of D_t^α to bath sides of Eq.(19), we have

$$\begin{aligned} u(x,t) = \quad & u(x,0) + \tfrac{t^\alpha}{\Gamma(\alpha+1)} \tfrac{\partial^\alpha u}{\partial t^\alpha}(x,0) \\ & + J^{2\alpha}[x^2 D_x^\beta (D_x^\beta(x,t))], \quad k \geq 0. \end{aligned} \tag{21}$$

The Adomain decomposition method leads to obtain

$$u_0(x,t) = u(x,0) + \frac{t^\alpha}{\Gamma(\alpha+1)} \frac{\partial^\alpha u}{\partial t^\alpha}(x,0), \tag{22}$$

$$u_{k+1}(x,t) = J^{2\alpha}[x^2 D_x^\beta (D_x^\beta u_k(x,t))], \quad k \geq 0. \tag{23}$$

Applying the recursive relation (23) and the initial conditions (20), we get the following results:

$$\begin{aligned} u_0(x,t) &= \cos(x), \\ u_1(x,t) &= \tfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \sum_{n=1}^{\infty} \tfrac{(-1)^n C_{0n} x^{2n+2-2\beta}}{\Gamma(2n+1-2\beta)}, \\ u_2(x,t) &= \tfrac{t^{4\alpha}}{\Gamma(4\alpha+1)} \sum_{n=1}^{\infty} \tfrac{(-1)^n C_{1n} x^{2n+4-4\beta}}{\Gamma(2n+3-4\beta)}, \\ u_3(x,t) &= \tfrac{t^{6\alpha}}{\Gamma(6\alpha+1)} \sum_{n=1}^{\infty} \tfrac{(-1)^n C_{2n} x^{2n+6-6\beta}}{\Gamma(2n+5-6\beta)}, \\ & \dots \\ u_k(x,t) &= \tfrac{t^{2k\alpha}}{\Gamma(2k\alpha+1)} \sum_{n=1}^{\infty} \tfrac{(-1)^n C_{kn} x^{2n+2k-2k\beta}}{\Gamma(2n+2k-1-2k\beta)}, \end{aligned} \tag{24}$$

where

$$\begin{aligned} C_{0n} &= 1, \\ C_{1n} &= (2n-2\beta+1)(2n-2\beta+2)C_{0n}, \\ C_{2n} &= (2n-4\beta+3)(2n-4\beta+4)C_{1n}, \\ C_{3n} &= (2n-6\beta+5)(2n-6\beta+6)C_{2n}, \\ & \dots\dots \\ C_{kn} &= (2n-2k\beta+2k-1)(2n-2k\beta+2k)C_{k-1n} \end{aligned} \tag{25}$$

Thus the approximate solution of (19) in a series form is given by:

$$\begin{aligned} u(x,t) = \cos(x) + \sum_{n=1}^{\infty} (-1)^n \Big\{ & \tfrac{C_{0n} t^{2\alpha} x^{2n+2-2\beta}}{\Gamma(2\alpha+1)\Gamma(2n+1-2\beta)} \\ & + \tfrac{C_{1n} t^{4\alpha} x^{2n+4-4\beta}}{\Gamma(4\alpha+1)\Gamma(2n+3-4\beta)}, \\ & + \tfrac{C_{2n} t^{6\alpha} x^{2n+6-6\beta}}{\Gamma(2n+5-6\beta)\Gamma(6\alpha+1)} + \dots \\ & + \tfrac{C_{kn} t^{2k\alpha} x^{2n+2k-2k\beta}}{\Gamma(2k\alpha+1)\Gamma(2n+2k-1-2k\beta)} + \cdots \end{aligned} \tag{26}$$

Figure 2 and Table 2 illustrate the behavior of the analytic approximate solutions (26).

Table 2

x/t	0.1	0.2	0.3
0.1	0.904457	0.903508	0.902071
0.2	0.818103	0.81654	0.814173
0.3	0.739995	0.737946	0.734844
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Table 2.show the approximate solution for the different values of $0 < x \le 1$ and $0 < t < 1$ when $\beta = 0.1,\ \alpha = 0.9,\ k = 0.5,\ N = 100, 0 < x < 3.$

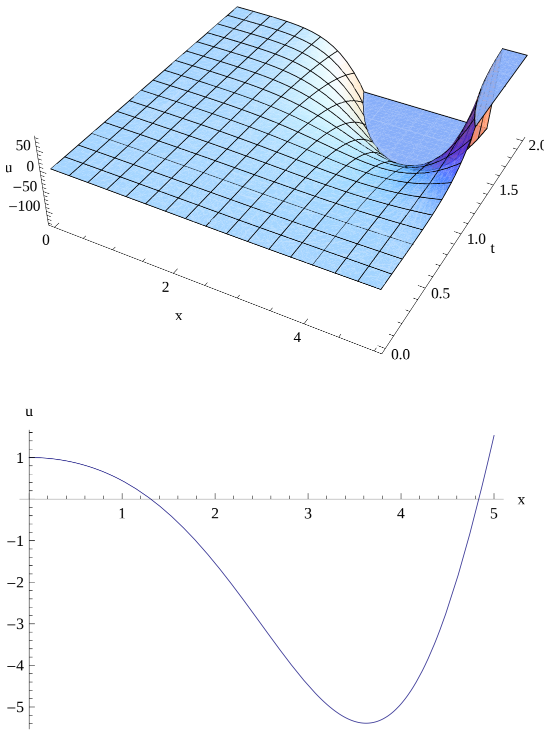


Fig 2 The surface shows the approximate solution $u(x,t)$ for Eq (26):

(a) $\beta = 0.1,\ \alpha = 0.9,\ ,\ N = 10, 0 < t < 2, 0 < x < 5,$

(b) The projection of the surface when $\beta = 0.1,\ \alpha = 0.9,\ \ N = 10, 0 < x < 5, t = 0.5.$

6 ADM for time and space fractional reduced wave equation with the variable coefficients

In this section, we use the Adomain decomposition method to calculate the approximate solution of the time and space fractional reduced wave equation with the variable coefficient in the following form:

$$\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} = \frac{1}{2} y^2 \frac{\partial^{2\beta} u}{\partial x^{2\beta}} + \frac{1}{2} x^2 \frac{\partial^{2\gamma} u}{\partial y^{2\gamma}}, \quad t > 0,\ 0 < \alpha, \beta, \gamma \le 1, \tag{27}$$

with the initial conditions

$$u(x,y,0) = x^2 + y^2, \quad \frac{\partial^\alpha u}{\partial t^\alpha}(x,y,0) = 0. \tag{28}$$

By using the inverse operator of D_t^α to bath sides of Eq.(27), we have

$$\begin{aligned} u(x,y,t) &= u(x,y,0) + \tfrac{t^\alpha}{\Gamma(\alpha+1)} \tfrac{\partial^\alpha u}{\partial t^\alpha}(x,y,0) \\ &+ J^{2\alpha}[\tfrac{1}{2} y^2 D_x^\beta (D_x^\beta u(x,y,t)) + \tfrac{1}{2} x^2 D_y^\gamma (D_y^\gamma u(x,y,t))], \end{aligned} \tag{29}$$

where $D_x^\beta = \frac{\partial^\beta}{\partial x^\beta}$, $D_y^\gamma = \frac{\partial^\gamma}{\partial y^\gamma}$, $D_t^\alpha = \frac{\partial^\alpha}{\partial t^\alpha}$ and J^α is the inverse operator of D_t^α. The Adomain decomposition method leads to get

$$u0(x,y,t) = u(x,y,0) + \tfrac{t^\alpha}{\Gamma(\alpha+1)} \tfrac{\partial^\alpha u}{\partial t^\alpha}(x,y,0) \tag{30}$$

$$\begin{aligned} u_{k+1}(x,y,t) &= J^{2\alpha}[\tfrac{1}{2} y^2 D_x^\beta (D_x^\beta u_k(x,y,t)) \\ &+ \tfrac{1}{2} x^2 D_y^\gamma (D_y^\gamma u_k(x,y,t))], \quad k \ge 0, \end{aligned} \tag{31}$$

Applying the recursive relation (31) and the initial conditions (28), we get the following results:

$$\begin{aligned} & u_0(x,y,t) = x^2 + y^2, \\ & u_1(x,y,t) = \tfrac{t^{2\alpha}}{2\Gamma(2\alpha+1)} \left[\tfrac{\Gamma(3) y^2 x^{2-2\beta}}{\Gamma(3-2\beta)} + \tfrac{\Gamma(3) x^2 y^{2-2\gamma}}{\Gamma(3-2\gamma)} \right], \\ & u_2(x,y,t) = \tfrac{t^{4\alpha}}{2^2\Gamma(4\alpha+1)} \left[\tfrac{\Gamma(3) y^2 x^{2-4\beta}}{\Gamma(3-4\beta)} \right. \\ & \left. + \tfrac{2\Gamma^2(3) y^{2-2\gamma} x^{2-2\beta}}{\Gamma(3-2\beta)\Gamma(3-2\gamma)} + \tfrac{\Gamma(3) x^2 y^{2-4\gamma}}{\Gamma(3-4\gamma)} \right], \end{aligned}$$

$$u_3(x,y,t)=\frac{t^{6\alpha}}{2^3\Gamma(6\alpha+1)}\left[\frac{\Gamma(3)y^2x^{2-6\beta}}{\Gamma(3-6\beta)}\right.+\frac{3\Gamma^2(3)y^{2-2\gamma}x^{2-4\beta}}{\Gamma(3-4\beta)\Gamma(3-2\gamma)}+\frac{3\Gamma^2(3)y^{2-4\gamma}x^{2-2\beta}}{\Gamma(3-2\beta)\Gamma(3-4\gamma)}\left.+\frac{\Gamma(3)x^2y^{2-6\gamma}}{\Gamma(3-6\gamma)}\right], \quad \ldots \tag{32}$$

and

$$u_k(x,y,t)=\frac{t^{2k\alpha}}{2^k\Gamma(2k\alpha+1)}\left[\frac{\Gamma(3)y^2x^{2-2k\beta}}{\Gamma(3-2k\beta)}\right.+\frac{k\Gamma^2(3)y^{2-2\gamma}x^{2-2(k-1)\beta}}{\Gamma(3-2(k-1)\beta)\Gamma(3-2\gamma)}+\frac{k(k-1)\Gamma^2(3)y^{2-4\gamma}x^{2-2(k-2)\beta}}{2\Gamma(3-4\gamma)\Gamma(3-2(k-2)\beta)}+\ldots.\left.+\frac{k\Gamma^2(3)y^{2-2(k-1)\gamma}x^{2-2\beta}}{\Gamma(3-2(k-1)\gamma)\Gamma(3-2\beta)}+\frac{\Gamma(3)x^2y^{2-2k\gamma}}{\Gamma(3-2k\gamma)}\right]. \tag{33}$$

Thus the approximate solution of Eq.(27) in a series form is given by

$$\begin{aligned}u(x,y,t)&=x^2+y^2+\frac{t^{2\alpha}}{2\Gamma(2\alpha+1)}\left[\frac{\Gamma(3)y^2x^{2-2\beta}}{\Gamma(3-2\beta)}+\frac{\Gamma(3)x^2y^{2-2\gamma}}{\Gamma(3-2\gamma)}\right]\\&+\frac{t^{4\alpha}}{2^2\Gamma(4\alpha+1)}\left[\frac{\Gamma(3)y^2x^{2-4\beta}}{\Gamma(3-4\beta)}+\frac{2\Gamma^2(3)y^{2-2\gamma}x^{2-2\beta}}{\Gamma(3-2\beta)\Gamma(3-2\gamma)}+\frac{\Gamma(3)x^2y^{2-4\gamma}}{\Gamma(3-4\gamma)}\right]\\&+\frac{t^{6\alpha}}{2^3\Gamma(6\alpha+1)}\left[\frac{\Gamma(3)y^2x^{2-6\beta}}{\Gamma(3-6\beta)}+\frac{3\Gamma^2(3)y^{2-2\gamma}x^{2-4\beta}}{\Gamma(3-4\beta)\Gamma(3-2\gamma)}+\frac{3\Gamma^2(3)y^{2-4\gamma}x^{2-2\beta}}{\Gamma(3-2\beta)\Gamma(3-4\gamma)}+\frac{\Gamma(3)x^2y^{2-6\gamma}}{\Gamma(3-6\gamma)}\right]+\ldots\\&+\frac{t^{2k\alpha}}{2^k\Gamma(2k\alpha+1)}\left[\frac{\Gamma(3)y^2x^{2-2k\beta}}{\Gamma(3-2k\beta)}+\frac{k\Gamma^2(3)y^{2-2\gamma}x^{2-2(k-1)\beta}}{\Gamma(3-2(k-1)\beta)\Gamma(3-2\gamma)}+\frac{k(k-1)\Gamma^2(3)y^{2-4\gamma}x^{2-2(k-2)\beta}}{2\Gamma(3-4\gamma)\Gamma(3-2(k-2)\beta)}+\ldots.\right.\\&\left.+\frac{k\Gamma^2(3)y^{2-2(k-1)\gamma}x^{2-2\beta}}{\Gamma(3-2(k-1)\gamma)\Gamma(3-2\beta)}+\frac{\Gamma(3)x^2y^{2-2k\gamma}}{\Gamma(3-2k\gamma)}\right]+\cdots\end{aligned} \tag{34}$$

Figure 3 illustrate the behavior of the analytic approximate solutions (34).

7 ADM for time and space fractional nonlinear Burger's equation

In this section, we use the Adomain decomposition method to calculate the approximate solution of the time and space fractional nonlinear Burger's equation in the following form:

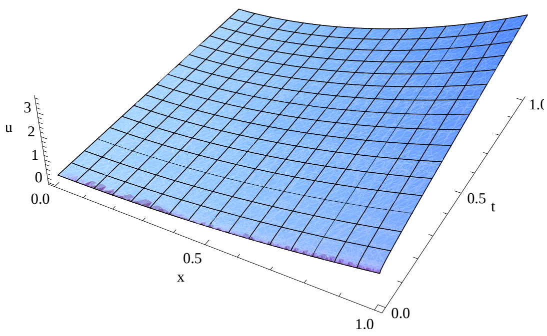
$$D_t^{\alpha}u=\upsilon D_x^{\beta}(D_x^{\beta}u)-\lambda u\,(D_x^{\beta}u),\quad t>0,\quad 0<\alpha,\beta\leqslant 1, \tag{35}$$

with the initial condition

$$u(x,0)=x^2, \tag{36}$$

Applying the inverse operator to both sides of the system (35) , we get

$$u(x,t)=x^2+J^{\alpha}[\upsilon D_x^{\beta}(D_x^{\beta}u)-\lambda G(u)],\quad t>0,\quad 0<\alpha,\beta\leqslant 1, \tag{37}$$



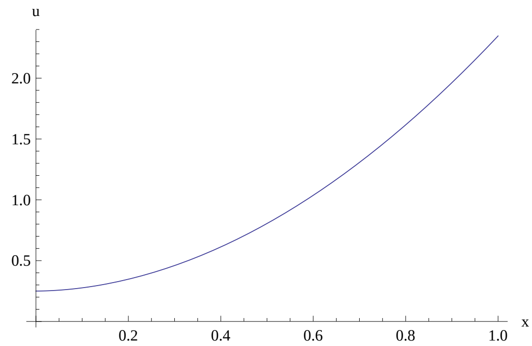


Fig 3 The surface shows the approximate solution $u(x,t)$ for Eq (34) when $\beta=0.1,\ \alpha=0.2, \gamma=0.3, y=0.5, 0<t<1, 0<x<1$, and the projection of the surface when $t=0.5$.

where J^{α} is the inverse operator of D_t^{α} and $G(u)=u\,(D_x^{\beta}u)$ is the nonlinear term in (37). According to the Adomain decomposition method, we assume that a series solution of the function $u(x,t)$ is given by

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t). \tag{38}$$

The nonlinear term $G(u)$ can be decomposed into an infinite series of polynomials given by

$$G(x,t)=\sum_{n=0}^{\infty}A_n, \tag{39}$$

where the component $u_n(x,t)$ will be determined recursively while A_n's are the so called Adomian polynomials of u_n's respectively.

Specific algorithms have been set in [14]-[16] for calculating Adomian's polynomials for nonlinear term $G(u)$

$$A_n=\frac{1}{n!}\left\{\frac{d^n}{d\lambda^n}\left[\left(\sum_{k=0}^{n}\lambda^k u_k\right)D_x^{\beta}\left(\sum_{k=0}^{n}\lambda^k u_k\right)\right]\right\}_{\lambda=0}, \tag{40}$$

Thus, we obtain

$$\begin{aligned} A_0 &= u_0 D_x^\beta u_0, \\ A_1 &= u_1 D_x^\beta u_0 + u_0 D_x^\beta u_1, \\ A_2 &= u_2 D_x^\beta u_0 + u_1 D_x^\beta u_1 + u_0 D_x^\beta u_2 \\ &\dots \end{aligned} \quad (41)$$

and so on.

The components u_n for $n \geq 0$ are given by the following recursive relationships:

$$\begin{aligned} u_0 &= u(x,0) = x^2, \\ u_1 &= J^\alpha[\upsilon D_x^\beta(D_x^\beta u_0) - \lambda A_0], \\ u_2 &= J^\alpha[\upsilon D_x^\beta(D_x^\beta u_1) - \lambda A_1], \\ u_3 &= J^\alpha[\upsilon D_x^\beta(D_x^\beta u_2) - \lambda A_2], \\ &\vdots \\ u_{n+1} &= J^\alpha[\upsilon D_x^\beta(D_x^\beta u_n) - \lambda A_n], \quad n \geq 0 \end{aligned} \quad (42)$$

Using the above recursive relationships, we obtain the following results:

$$\begin{aligned} u_0 =& \; x^2, \\ u_1 =& \; \frac{t^\alpha}{\Gamma(\alpha+1)}\left\{\frac{\upsilon\Gamma(3)}{\Gamma(3-2\beta)}x^{2-2\beta} - \frac{\lambda\Gamma(3)}{\Gamma(3-\beta)}x^{4-\beta}\right\}, \\ u_2 =& \; \frac{t^{2\alpha+1}}{\Gamma(2\alpha+1)}\left\{f_{11}x^{2-4\beta} + f_{12}x^{4-3\beta} + f_{13}x^{4-4\beta}\right. \\ & \left. + f_{14}x^{6-3\beta} + f_{15}x^{6-2\beta}\right\}, \\ u_3 =& \; \frac{t^{3\alpha+1}}{\Gamma(3\alpha+1)}\left\{f_{21}x^{2-6\beta} + f_{22}x^{4-5\beta} + f_{23}x^{4-6\beta}\right. \\ & + f_{24}x^{6-5\beta} + f_{25}x^{6-4\beta} + f_{26}x^{8-4\beta} \\ & \left. + f_{27}x^{8-3\beta}\right\}, \\ \dots \end{aligned} \quad (43)$$

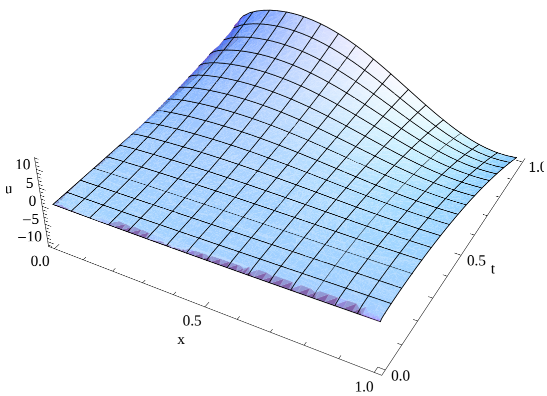
where

$$\begin{aligned} & f_{11} = \frac{\upsilon^2\Gamma(3)}{\Gamma(3-4\beta)}, \;\; f_{13} = -\frac{\upsilon\lambda\Gamma(3)}{\Gamma(3-4\beta)}, \\ & f_{14} = \frac{\lambda^2\Gamma(3)\Gamma(5-\beta)}{\Gamma(3-\beta)\Gamma(5-3\beta)}, \;\; f_{15} = \frac{\lambda^2\Gamma^2(3)}{\Gamma^2(3-\beta)}, \\ & f_{12} = -\frac{\upsilon\lambda\Gamma(3)\Gamma(5-\beta)}{\Gamma(3-\beta)\Gamma(5-3\beta)} - \frac{\upsilon\lambda\Gamma^2(3)}{\Gamma(3-2\beta)\Gamma(3-\beta)}, \\ & f_{21} = \frac{\upsilon f_{11}\Gamma(3-4\beta)}{\Gamma(3-6\beta)}, \;\; f_{23} = \frac{\upsilon f_{13}\Gamma(5-4\beta)}{\Gamma(5-6\beta)}, \\ & f_{22} = \frac{\upsilon f_{12}\Gamma(5-3\beta)}{\Gamma(5-5\beta)} - \frac{\lambda\Gamma(3)f_{11}}{\Gamma(3-\beta)} - \frac{\lambda\Gamma(3-4\beta)f_{11}}{\Gamma(3-5\beta)} \\ & -\frac{\upsilon^2\lambda\Gamma(2\alpha+1)\Gamma^2(3)}{\Gamma^2(\alpha+1)\Gamma(3-2\beta)\Gamma(3-3\beta)}, \\ & f_{24} = \frac{\upsilon f_{14}\Gamma(7-3\beta)}{\Gamma(7-5\beta)} - \frac{\lambda f_{13}\Gamma(3)}{\Gamma(3-\beta)} - \frac{\lambda f_{13}\Gamma(5-4\beta)}{\Gamma(5-5\beta)}, \\ & f_{25} = \frac{\upsilon f_{15}\Gamma(7-2\beta)}{\Gamma(7-4\beta)} - \frac{\lambda f_{12}\Gamma(3)}{\Gamma(3-\beta)} \\ & + \frac{\lambda^2\upsilon\Gamma(2\alpha+1)\Gamma^2(3)\Gamma(5-\beta)}{\Gamma^2(\alpha+1)\Gamma(3-2\beta)\Gamma(3-\beta)\Gamma(5-2\beta)} \\ & - \frac{\lambda f_{12}\Gamma(5-3\beta)}{\Gamma(5-4\beta)} + \frac{\lambda^2\upsilon\Gamma(2\alpha+1)\Gamma^2(3)}{\Gamma^2(\alpha+1)\Gamma(3-3\beta)\Gamma(3-\beta)}, \\ & f_{26} = -\frac{\lambda f_{14}\Gamma(3)}{\Gamma(3-\beta)} - \frac{\lambda f_{14}\Gamma(7-3\beta)}{\Gamma(7-4\beta)}, \\ & f_{27} = -\frac{\lambda f_{15}\Gamma(3)}{\Gamma(3-\beta)} - \frac{\lambda f_{15}\Gamma(7-2\beta)}{\Gamma(7-3\beta)} \\ & - \frac{\lambda^3\Gamma(2\alpha+1)\Gamma^2(3)\Gamma(5-\beta)}{\Gamma^2(\alpha+1)\Gamma^2(3-\beta)\Gamma(5-2\beta)}. \end{aligned} \quad (44)$$

Thus the approximate solution of (35) in a series form is given by:

$$\begin{aligned} u(x,t) =& \; x^2 + \frac{t^\alpha}{\Gamma(\alpha+1)}\left\{\frac{\upsilon\Gamma(3)}{\Gamma(3-2\beta)}x^{2-2\beta}\right. \\ & \left. - \frac{\lambda\Gamma(3)}{\Gamma(3-\beta)}x^{4-\beta}\right\} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+1)}\left\{f_{11}x^{2-4\beta}\right. \\ & \left. + f_{12}x^{4-3\beta} + f_{13}x^{4-4\beta} + f_{14}x^{6-3\beta} + f_{15}x^{6-2\beta}\right\} \\ & + \frac{t^{3\alpha+1}}{\Gamma(3\alpha+1)}\left\{f_{21}x^{2-6\beta} + f_{22}x^{4-5\beta} + f_{23}x^{4-6\beta}\right. \\ & \left. + f_{24}x^{6-5\beta} + f_{25}x^{6-4\beta} + f_{26}x^{8-4\beta} + f_{27}x^{8-3\beta}\right\} + \\ & \dots \end{aligned} \quad (45)$$

Figure 4 illustrate the behavior of the analytic approximate solutions (45).



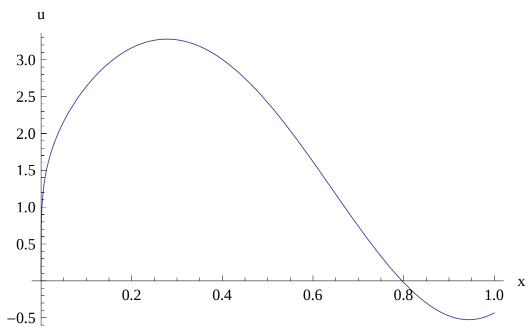


Fig 4 The surface shows the approximate solution $u(x,t)$ for Eq (45) when $\beta = 0.3$, $\alpha = 0.4, \lambda = 1, \nu = 2, 0 < t < 1, 0 < x < 1$, and the projection of the surface when $t = 0.5$.

8 Conclusions

In this paper, the application of Adomain decomposition method was extended to explicit the numerical solutions of the time- and space-fractional partial differential equations in mathematical physics with initial conditions . The Adomain decomposition method was clearly very efficient and powerful technique in finding the approximate solutions of the proposed equations. The obtained results demonstrate the reliability of the algorithm and its wider applicability to fractional nonlinear evolution equations.

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