A Structured SVD-Like Decomposition

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Abstract: We present in this paper a method that compute a symplectic SVD-like decomposition for a $2n$-by-m rectangular real matrix. This decomposition focus mainly on numerical solution of some linear-quadratic optimal control theory and signal processing problems. In particular the resolution of gyroscopic and linear Hamiltonian systems. Our approach here is based on symplectic reflectors defined on $\mathbb{R}^{2n\times 2}$. We also give an ortho-symplectic SVD-like decomposition of a $2n$-by-$2n$ symplectic real matrix.

Key–Words: SVD, Schur Form, Hamiltonian, Skew-Symmetric, Skew-Hamiltonian and Symplectic Matrices, Symplectic Reflector.

1 Introduction

The singular value decomposition SVD is a generalization of the eigen-decomposition used to analyze rectangular matrices (the eigen-decomposition is defined only for squared matrix). SVD technique is used in scientists working on applied linear algebra, signal and image processing [8, 9]. A symplectic SVD-like decomposition of rectangular matrix of $A \in \mathbb{R}^{2n\times m}$, such that $SAQ = \Sigma$ where S is symplectic and Q is orthogonal, is used as the basic tool to compute the eigenvalues of some structured matrices (Hamiltonian and skew-Hamiltonian). An example [13] is about the eigenvalue problem of the matrix,

$$F = \begin{bmatrix} -C & -G \\ I & 0 \end{bmatrix} = \begin{bmatrix} -C & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & G \end{bmatrix} \tag{1}$$

which is related to the gyroscopic system [5, 7, 11, 13]

$$q" + Cq' + Gq = 0\ ;\ q(0) = q_0\ ;\ q'(0) = q_1 \tag{2}$$

A matrix $G \in \mathbb{R}^{m\times m}$ is symmetric and positive semi-definite it has a full rank factorization $G = LL^T$. And $C \in \mathbb{R}^{m\times m}$ is skew-symmetric. By using the equality

$$\begin{bmatrix} -C & -I \\ I & 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}C & I \\ I & 0 \end{bmatrix} J \begin{bmatrix} \frac{1}{2}C & I \\ I & 0 \end{bmatrix} \tag{3}$$

where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, I_n denotes the $n \times n$ identity matrix, F is similar to the Hamiltonian matrix

$$\begin{aligned} J \begin{bmatrix} \frac{1}{2}C & I \\ I & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & LL^T \end{bmatrix} \begin{bmatrix} -\frac{1}{2}C & I \\ I & 0 \end{bmatrix} \\ = J \begin{bmatrix} -\frac{1}{2}C & I \\ L^T & 0 \end{bmatrix}^T \begin{bmatrix} -\frac{1}{2}C & I \\ L^T & 0 \end{bmatrix} \end{aligned} \tag{4}$$

Therefore the eigenvalue problem of F can be solved by computing a symplectic SVD-like decomposition of $\begin{pmatrix} -\frac{1}{2}C & I \\ L^T & 0 \end{pmatrix}$.

The main purpose of this work is to study a symplectic SVD-like decomposition of a $2n$-by-m real matrix. A method for computing an SVD-like decomposition was given by Hongguo Xu [12, 13] of a n-by-$2m$ real matrix B. He proved that there exists an orthogonal matrix Q and a symplectic matrix S, such that $B = QDS^{-1}$ where D is in the following form,

$$D = \begin{pmatrix} \Sigma & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Sigma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Σ is positive diagonal.

We treat also in this work a new algorithms that compute the symplectic SVD-like decomposition of symplectic matrices based on symplectic and ortho-symplectic reflectors for more details on symplectic and ortho-symplectic reflectors, see [2, 1]. Symplectic matrices appear in at least two active research fields: optimal control theory and the parametric resonance of mechanical systems [6, 10]. We construct an orthogonal matrix $Q \in \mathbb{R}^{m\times m}$ and a symplectic matrix

$S \in \mathbb{R}^{2n\times 2n}$ such that,

$$SAQ = \begin{pmatrix} \Sigma & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \Sigma & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where Σ is positive diagonal. Moreover, we proved that for symplectic matrix $S \in \mathbb{R}^{2n\times 2n}$, there exist an orthogonal symplectic matrices $U, V \in \mathbb{R}^{2n\times 2n}$ such that

$$S = U^T \begin{pmatrix} \omega_1 & & & & & \\ & \ddots & & & & \\ & & \omega_n & & & \\ & & & \omega_1^{-1} & & \\ & & & & \ddots & \\ & & & & & \omega_n^{-1} \end{pmatrix} V$$

An algorithm is given to compute this decomposition.

The paper is organized as follows. In section 2 we introduce some notation, terminology and some basic facts. We present, in section 3, a symplectic *SVD*-like decomposition of a $2n$-by-m real matrix. In section 4 we give an ortho-symplectic SVD-like decomposition of a symplectic matrix. Finally, some numerical examples are provided to illustrate the effectiveness of the proposed algorithms.

2 Terminology, notation and some basic facts

An ubiquitous matrix in this work is skew-symmetric matrix $J_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, where I_n denotes the $n \times n$ identity matrix. In the following, we will drop the subscripts n and $2n$ whenever the dimension of corresponding matrix is clear from the context. By straightforward algebraic manipulation, we can show that a Hamiltonian matrix M is equivalently defined by the property $MJ = (MJ)^T$. Likewise, a matrix W is skew-Hamiltonian if and only if $-WJ = (WJ)^T$. Any matrix $S \in \mathbb{R}^{2n\times 2n}$ satisfying $S^TJS = SJS^T = J$ is called symplectic matrix. The symplectic similarity transformations preserve Hamiltonian, skew-Hamiltonian and symplectic structures.

We introduce some results useful thereafter, we set $E_i = [e_i \; e_{n+i}] \in \mathbb{R}^{2n\times 2}$ for $i = 1, \cdots, n$. We have

$$E_i^J = E_i^T \text{ and } E_i^J E_j = \delta_{ij} I_2 \text{ where } \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Proposition 1 *Let $U = [u_1 \; u_2]$ be a $2n$-by-2 real matrix, where $u_1 = \sum_{i=1}^{2n} u_i^{(1)} e_i$ and $u_2 = \sum_{j=1}^{2n} u_j^{(2)} e_j$. Then U is written in a unique way as linear combination of $(E_i)_{1\le i\le n}$ on the ring $\mathbb{R}^{2\times 2}$.*

$$U = \sum_{i=1}^{n} E_i M_i \;\; where \;\; M_i = \begin{pmatrix} u_i^{(1)} & u_i^{(2)} \\ u_{n+i}^{(1)} & u_{n+i}^{(2)} \end{pmatrix}$$

2.1 Symplectic reflectors

We recall a symplectic reflector [1, 2] on $\mathbb{R}^{2n\times 2}$ which is defined in parallel with elementary reflectors.

Proposition 2 *Let U, V be two $2n$-by-2 real matrices satisfying $U^JU = V^JV = I_2$. If the 2-by-2 matrix $C = I_2 + V^JU$ is nonsingular, then the transformation*
$S = (U + V)C^{-1}(U + V)^J - I_{2n}$ is symplectic and takes U to V. It's called a symplectic reflector. Additionally, if $U^J = U^T$ and $V^J = V^T$, then S is orthogonal and symplectic.

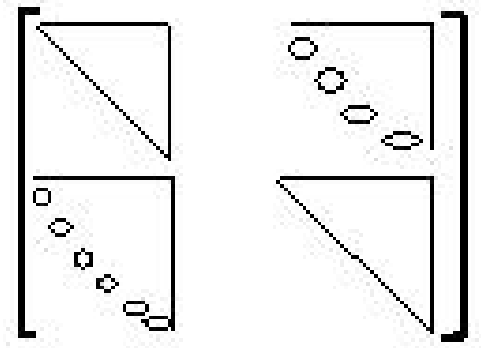
Lemma 3 *Let $U = [u_1 \;\; u_2] \in \mathbb{R}^{2n\times 2}$ be a non-isotropic matrix and $V = Uq(U)^{-1}$ its normalized matrix. Then there exists a symplectic reflector S takes V to E_1 and therefore U to $E_1q(U)$ where , which is in the following form*

$$SU = \begin{pmatrix} * & \mathbf{0} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ \mathbf{0} & * \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \swarrow n+1$$

where

$$q(U) = \begin{cases} \sqrt{\alpha} I_2 & \text{if } \alpha > 0 \\ \sqrt{-\alpha}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \text{if } \alpha < 0 \end{cases}$$
$$\alpha = u_1^H J u_2.$$

Remark 4 *Using symplectic reflector to a matrix $A \in \mathbb{R}^{2n\times 2n}$, we obtain the factorization $A = SR$ where $S \in \mathbb{R}^{2n\times 2n}$ is symplectic and $R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \in \mathbb{R}^{2n\times 2n}$ is J-triangular and in addition R_{12} is an n-by-n strictly upper triangular matrix. Which means R is in the following form*



2.2 Skew-Symmetric Schur-Like decomposition

We present here the Schur like form of real skew-symmetric matrices.

Theorem 5 *[13] Given a $2n$-by-m real matrix A, there exists a real orthogonal matrix Q such that*

$$A^TJA = Q\left(\begin{array}{c|c|c} 0_p & \Sigma_p^2 & 0 \\ -\Sigma_p^2 & 0_p & 0 \\ 0 & 0 & 0_{m-2p} \end{array}\right)Q^T$$

with $\Sigma_p = diag(\sigma_1, \sigma_2, \ldots, \sigma_p)$, $\sigma_i > 0$, $\forall i$ and $p = rank(A^TJA)$

3 Symplectic SVD like Decomposition

It is shown by Xu [12, 13] that for any n-by-$2m$ real matrix B there exists an orthogonal matrix Q and a symplectic matrix S, such that $B = QDS^{-1}$ where D is in the following form

$$D = \begin{pmatrix} \Sigma & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Sigma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and Σ is positive diagonal. The symplectic *SVD*-like decomposition is an effective for computing the Schur-like form of the skew-symmetric matrix BJB^T and the structured canonical form of the Hamiltonian matrix JB^TB. Xu proposed an algorithm to compute eigenvalues of JB^TB and BJB^T using block B_{11} and B_{23} in step 1 of the algorithm (see, section 2 in [13]). As we can see in example 1, although he obtained the eigenvalues (see, [13]), his algorithm doesn't compute the full decomposition.

In this section we will give a new approach to compute the symplectic SVD-like decomposition using a symplectic reflector given in [1, 2]. Firstly, we use the following basic results.

Lemma 6 *Let V be a $2n$-by-m rectangular real matrix such that*

$$V^TJ_{2n}V = \begin{pmatrix} 0_p & I_p & 0 \\ -I_p & 0_p & 0 \\ 0 & 0 & 0_{m-2p} \end{pmatrix}$$

Then there exists a $2n$-by-$2m$ rectangular symplectic real matrix S such that $SV\widetilde{Q}_X =$

$$\begin{pmatrix} I_p & 0_p & 0_{p\times q} & 0_{p\times(m-2p-q} \\ 0_{q\times p} & 0_{q\times p} & I_q & 0_{q\times(m-2p-q)} \\ 0_{r\times p} & 0_{r\times p} & 0_{(r)\times q} & 0_{r\times(m-2p-q)} \\ 0_p & I_p & 0_{p\times q} & 0_{p\times(m-2p-q)} \\ 0_{(n-p)\times p} & 0_{(n-p)\times p} & 0_{(n-p)\times q} & 0_{(n-p)\times(m-2p-q)} \end{pmatrix}$$

$(r = n - p - q)$ where $\widetilde{Q}_X$ is an orthogonal matrix and $rank(V) = 2p + q$.

Proof: Partition V as $V = [V_1\ V_2\ V_3]$ such that $V_1 = [v_{1,1}, \ldots, v_{1,p}] \in \mathbb{R}^{2n\times p}$, $V_2 = [v_{2,1}, \ldots, v_{2,p}] \in \mathbb{R}^{2n\times p}$ and $V_3 = [v_{3,1}, \ldots, v_{3,m-2p}] \in \mathbb{R}^{2n\times(m-2p)}$.

Step 1:
Set $U_1 = [v_{1,1}, v_{2,1}] \in \mathbb{R}^{2n\times 2}$. Since $V^TJ_{2n}V$ is in the form given in lemma, then $U_1^JU_1 = I_2$ and then the symplectic reflector $S_1 = (U_1 + E_1)(I_2 + E_1^JU_1)^{-1}(U_1 + E_1)^J - I_{2n}$ verify $S_1U_1 = E_1$. The $(n + 1)^{th}$-component of $(S_1v_{1,k})$ is equal to zero, for $k = 2, 3, \ldots p$. Indeed, on the one hand $(S_1v_{1,1})^TJS_1v_{1,k} = v_{1,1}^TJv_{1,k} = 0$ and on the other hand $(S_1v_{1,1})^TJS_1v_{1,k} = e_1^TJ(S_1v_{1,k})$ is nothing but the $(n + 1)^{th}$-component of $(S_1v_{1,k})$. The $(n + 1)^{th}$-component of $(S_1v_{2,k})$ and $(S_1v_{3,k})$ vanishing, respectively for $k = 2, 3, \ldots p$ and $k = 2, 3, \ldots m - 2p$. Furthermore, $(S_1v_{2,1})^T J (S_1v_{1,k}) = 0$ and $S_1v_{2,1} = e_{n+1}$, then the first component of $(S_1v_{1,k})$ vanishes for $k = 2, 3, \ldots p$. Likewise the first component of $(S_1v_{2,k})$ and $(S_1v_{3,k})$ vanishes for $k = 2, 3, \ldots p$. Finally we obtain,

$$S_1V = \left(\begin{array}{cc|c|c} & \overbrace{\qquad}^{p} & \overbrace{\qquad}^{p} & \overbrace{\qquad}^{m-2p} \\ \hline & \begin{matrix} 1 & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{matrix} & \begin{matrix} 0 & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{matrix} & \begin{matrix} 0 & \cdots & 0 \\ * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{matrix} \\ n & & & \\ \hline & \begin{matrix} 0 & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{matrix} & \begin{matrix} 1 & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{matrix} & \begin{matrix} 0 & \cdots & 0 \\ * & \cdots & * \\ \vdots & \ddots & * \\ * & \cdots & * \end{matrix} \\ n & & & \end{array}\right)$$

Update the value of V: $V \longleftarrow S_1V$.

Step 2:
Let set $U_2 = [v_{1,2}, v_{2,2}] \in \mathbb{R}^{2n\times 2}$. Since U_2satisfy $U_2^J U_2 = I_2$, then the symplectic reflector $S_2 = (U_2 + E_2)(I_2 + E_2^J U_2)^{-1}(U_2 + E_2)^J - I_{2n}$ has the following form,

$$S_2 = \left(\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & * & * & * & \vdots & * & * & * \\ \vdots & * & * & * & \vdots & * & * & * \\ 0 & * & * & * & 0 & * & * & * \\ \hline 0 & \cdots & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & * & * & * & 0 & * & * & * \\ \vdots & * & * & * & \vdots & * & * & * \\ 0 & * & * & * & 0 & * & * & * \end{array}\right)$$

and verify $S_2 U_2 = E_2$. Likewise step 1, we obtain $S_2 V =$

$$\left(\begin{array}{ccccc|ccccc|ccc} 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & * & \dots & * & 0 & 0 & * & \dots & * & * & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & * & \cdots & * & 0 & 0 & * & \dots & * & * & * & * \\ \hline 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & * & \dots & * & 0 & 0 & * & \dots & * & * & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & * & \dots & * & 0 & 0 & * & \dots & * & * & * & * \end{array}\right)$$

Let's use the QR factorization to W^T,

$$W^T = Q_W \begin{pmatrix} R_{11} \in \mathbb{R}^{q\times q} \\ 0_{(m-2p-q)\times q} \end{pmatrix}$$

where $W = X(1 : q, 1 : m - 2p) \in \mathbb{R}^{q\times(m-2p)}$. $Q_W \in \mathbb{R}^{(m-2p)\times(m-2p)}$ is orthogonal and R_{11} is a nonsingular upper triangular matrix (rank$(W) = q$). Now, $X \longleftarrow XQ_W$ is as follow,

$$X = \begin{pmatrix} R_{11}^T & 0_{q\times(m-2p-q)} \\ 0 & 0 \end{pmatrix}$$

We set $Z = Z_q \cdots Z_2 Z_1$ which is an orthogonal and symplectic matrix. Partition Z conformably,

$$Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}$$

and construct

$$S_{p+1} = \left(\begin{array}{c|c|c|c} I_p & 0 & 0 & 0 \\ 0 & Z_{11} & 0 & Z_{12} \\ \hline 0 & 0 & I_p & 0 \\ 0 & Z_{21} & 0 & Z_{22} \end{array}\right)$$

which is also orthogonal and symplectic. We set $\widetilde{Q}_X = \operatorname{diag}(I_{2p}, Q_X) \in \mathbb{R}^{m\times m}$ which is an orthogonal matrix that commute with

$$\Gamma = \left(\begin{array}{c|c|c} \Sigma_p & 0 & 0 \\ \hline 0 & \Sigma_p & 0 \\ \hline 0 & 0 & I_{m-2p} \end{array}\right)$$

Finally, for V verifying hypothesis of the lemma, we obtain then $S_{p+1} S_p \cdots S_2 S_1 V \widetilde{Q}_X =$

$$\left(\begin{array}{c|c|cc} I_{n\times p} & 0_{n\times p} & \multicolumn{2}{l}{0_{p\times(m-2p)}} \\ & & R_{11}^T & 0_{q\times(m-2p-q)} \\ \hline 0_{n\times p} & I_{n\times p} & 0 & 0 \end{array}\right).$$

By setting $S_{p+2} =$diag$(I_p, R_{11}^{-1}, I_{n-p-q}, I_p, R_{11}, I_{n-p-q})$ which is a symplectic matrix, we have
$\underbrace{S_{p+2} S_{p+1} S_p \cdots S_2 S_1}_{S} V \widetilde{Q}_X = \Delta =$

$$\begin{pmatrix} I_p & 0_p & 0_{p\times q} & 0_{p\times(m-2p-q} \\ 0_{q\times p} & 0_{q\times p} & I_q & 0_{q\times(m-2p-q)} \\ 0_{r\times p} & 0_{r\times p} & 0_{r\times q} & 0_{r\times(m-2p-q)} \\ 0_p & I_p & 0_{p\times q} & 0_{p\times(m-2p-q)} \\ 0_{(n-p)\times p} & 0_{(n-p)\times p} & 0_{(n-p)\times q} & 0_{(n-p)\times(m-2p-q)} \end{pmatrix}$$

$(r = n - p - q)$which is the desired form. □

Theorem 7 ***(Symplectic SVD-like decomposition)*** *Let A be a $2n$-by-m rectangular real matrix. There exists a symplectic matrix $S \in \mathbb{R}^{2n\times 2n}$ and an orthogonal matrix $Q \in \mathbb{R}^{m\times m}$ such that*

$$SAQ = \left(\begin{array}{c|c|c|c} \Sigma_p & 0 & 0 & 0 \\ 0 & 0 & I_q & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & \Sigma_p & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Proof. By using the real Schur decomposition to the skew-symmetric matrix $A^T JA$

$$A^T JA = P \left(\begin{array}{c|c|c} 0_p & \Sigma_p^2 & 0 \\ \hline -\Sigma_p^2 & 0_p & 0 \\ \hline 0 & 0 & 0_{m-2p} \end{array}\right) P^T,$$

we construct $V = AP\Gamma^{-1}$ where

$$\Gamma = \left(\begin{array}{c|c|c} \Sigma_p & 0 & 0 \\ \hline 0 & \Sigma_p & 0 \\ \hline 0 & 0 & I_{m-2p} \end{array}\right)$$

Since

$$\begin{aligned} V^T JV &= \Gamma^{-1} \left(P^T A^T JAP\right) \Gamma^{-1} \\ &= \begin{pmatrix} 0_p & I_p & 0 \\ -I_p & 0_p & 0 \\ 0 & 0 & 0_{m-2p} \end{pmatrix} \end{aligned}$$

then using the previous lemma we have

$$S V \widetilde{Q}_X = \Delta =$$

$$\begin{pmatrix} I_p & 0_p & 0_{p\times q} & 0_{p\times(m-2p-q} \\ 0_{q\times p} & 0_{q\times p} & I_q & 0_{q\times(m-2p-q)} \\ 0_{r\times p} & 0_{r\times p} & 0_{r\times q} & 0_{r\times(m-2p-q)} \\ 0_p & I_q & 0_{p\times q} & 0_{p\times(m-2p-q)} \\ 0_{(n-p)\times p} & 0_{(n-p)\times p} & 0_{(n-p)\times q} & 0_{(n-p)\times(m-2p-q)} \end{pmatrix}$$

with $r = n - p - q$. Using the fact that Γ^{-1} and $\widetilde{Q}_X$ commute, then we obtain

$$SAP\widetilde{Q}_X = \Delta\Gamma = \left(\begin{array}{c|c|c|c} \Sigma_p & 0 & 0 & 0 \\ 0 & 0 & I_q & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & \Sigma_p & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

which is corresponding to the asserted form. □

Algorithm 3.1: Symplectic *SVD*-like algorithm

Input : A matrix $A \in \mathbb{R}^{2n\times m}$ $(n \geq m)$
Output : A symplectic matrix $S \in \mathbb{R}^{2n\times 2n}$, an orthogonal matrix $Q \in \mathbb{R}^{m\times m}$ and the desired SVD-like decomposition

1. We compute the skew-symmetric matrix $M = A^T J A \in \mathbb{R}^{m\times m}$

2. From theorem 5, there exists a real orthogonal matrix P such that

$$P^T M P = \left(\begin{array}{c|c|c} 0_p & \Sigma_p^2 & 0 \\ \hline -\Sigma_p^2 & 0_p & 0 \\ \hline 0 & 0 & 0_{m-2p} \end{array}\right)$$

3. We compute the blok-diagonal matrix

$$\Gamma = \left(\begin{array}{c|c|c} \Sigma_p & 0 & 0 \\ \hline 0 & \Sigma_p & 0 \\ \hline 0 & 0 & I_{m-2p} \\ & & \end{array}\right)$$

4. We compute a $2n$-by-$2m$ rectangular real matrix $V = AP\Gamma^{-1}$

5. From Lemma 6, there exists a real symplectic matrix S such that

$$S V \widetilde{Q}_X = \Delta =$$

$$\begin{pmatrix} I_p & 0_p & 0_{p\times q} & 0_{p\times(m-2p-q} \\ 0_{q\times p} & 0_{q\times p} & I_q & 0_{q\times(m-2p-q)} \\ 0_{r\times p} & 0_{r\times p} & 0_{r\times q} & 0_{r\times(m-2p-q)} \\ 0_p & I_p & 0_{p\times q} & 0_{p\times(m-2p-q)} \\ 0_{(n-p)\times p} & 0_{(n-p)\times p} & 0_{(n-p)\times q} & 0_{(n-p)\times(m-2p-q)} \end{pmatrix}$$

with $r = n - p - q$

6. The matrix Σ

$$\Sigma = \Delta\Gamma = \left(\begin{array}{c|c|c|c} \Sigma_p & 0 & 0 & 0 \\ 0 & 0 & I_q & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & \Sigma_p & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

is verifying $SAQ = \Sigma$ where $Q = P\widetilde{Q}_X$.

4 SVD-like decomposition of symplectic matrices

We treat here an ortho-symplectic *SVD*-like decomposition of a symplectic real matrix. It is shown by Xu [12] that every symplectic matrix $S \in \mathbb{R}^{2m\times 2m}$ has the following real *SVD*-decomposition

$$S = U \begin{pmatrix} \Omega & 0 \\ 0 & \Omega^{-1} \end{pmatrix} V^T$$

where U, V are real orthogonal and symplectic matrices. Our purpose is to give an ortho-symplectic SVD-like decomposition of real symplectic matrices based on orthogonal symplectic reflectors defined in proposition 2. This approach seems to be effective as we can see in example 2. The orthogonal symplectic reflectors preserve conditioning and guaranteed the stability of the algorithm.

The classical SVD decomposition of real symplectic matrices is given by the following theorem.

Theorem 8 *Let $S \in \mathbb{R}^{2n\times 2n}$ be a symplectic matrix. Then there exist two orthogonal matrices P, Q such that*

$$S = P \begin{pmatrix} \omega_1 & & & & & \\ & \ddots & & & & \\ & & \omega_n & & & \\ & & & \omega_1^{-1} & & \\ & & & & \ddots & \\ & & & & & \omega_n^{-1} \end{pmatrix} Q^T$$

with $w_i \geq 1$.

Remark 9 *If σ is a singular value of a symplectic matrix S, then σ^{-1} is a also a singular value of S.*

Indeed, since S is symplectic, then $S = S^{-J} = J^T S^{-T} J$. This prove that S and S^{-T} have the same singular values.

We present below a technical lemma useful thereafter.

Lemma 10 *Using orthogonal matrices P and Q then from theorem 8 given above, we have*

$$S\underbrace{[q_1\, q_2\, \ldots q_n\ \ -Jq_1\ -Jq_2\ \ldots -Jq_n]}_{\widetilde{\mathbf{Q}}} = \underbrace{[p_1\, p_2\, \ldots p_n\ \ -Jp_1\ -Jp_2\ \ldots -Jp_n]}_{\widetilde{\mathbf{P}}}\begin{pmatrix} D & \\ & D^{-1}\end{pmatrix}$$

where $D = \begin{pmatrix}\omega_1 & & \\ & \ddots & \\ & & \omega_n\end{pmatrix}$, $w_i \geq 1$, $q_i = Qe_i$ *and* $p_i = Pe_i$ *for* $i = 1, \cdots, n$. *Here* e_i *denotes the* i^{th} *vector in the canonical basis of* $\mathbb{R}^{2n}$.

Proof: From theorem 8 we have $Sq_i = w_i p_i$ for $i = 1, \cdots, n$ and

$$S^{-T} = P\begin{pmatrix}\omega_1^{-1} & & & & & \\ & \ddots & & & & \\ & & \omega_n^{-1} & & & \\ & & & \omega_1 & & \\ & & & & \ddots & \\ & & & & & \omega_n\end{pmatrix}Q^T$$

Since S is symplectic, then

$$\begin{aligned} S(Jq_i) &= JJ^T S(Jq_i) \\ &= JS^{-T}q_i \\ &= w_i^{-1}Jp_i. \end{aligned}$$

That gives the desired form. □

Theorem 11 ***(Ortho-symplectic SVD-Like decomposition)*** *Let $S \in \mathbb{R}^{2n\times 2n}$ be a symplectic real matrix. There exist an orthogonal symplectic matrices $U, V \in \mathbb{R}^{2n\times 2n}$ such that*

$$S = U\begin{pmatrix}\omega_1 & & & & & \\ & \ddots & & & & \\ & & \omega_n & & & \\ & & & \omega_1^{-1} & & \\ & & & & \ddots & \\ & & & & & \omega_n^{-1}\end{pmatrix}V^T$$

Proof: We proceed by induction on n.

For $n = 1$ we have $SQ = P\begin{pmatrix}\omega_1 & \\ & \omega_1^{-1}\end{pmatrix}$. Let set $U = [p_1\ \ -Jp_1]$ and $V = [q_1\ \ -Jq_1]$. From the above lemma, $S(-Jq_1) = w_1^{-1}Jp_1$ then $SV = U\begin{pmatrix}\omega_1 & \\ & \omega_1^{-1}\end{pmatrix}$ with U and V are orthogonal and symplectic. That gives the desired form for $n = 1$.

Let us assume now that $n > 1$. Consider the n first columns of Q and P ($q_i = Qe_i$, $p_i = Pe_i$, $i = 1, \ldots n$). Let
$U_1 = (P_1+E_1)(I_2+E_1^J P_1)^{-1}(P_1+E_1)^J - I_{2n}$ is the ortho-symplectic reflector that transform $P_1 = [p_1\ \ -Jp_1]$ to $E_1 = [e_1\ \ e_{n+1}]$ and $V_1 = (Q_1 + E_1)(I_2 + E_1^J Q_1)^{-1}(Q_1 + E_1)^J - I_{2n}$ is the ortho-symplectic reflector that transform $Q_1 = [q_1\ \ -Jq_1]$ to E_1. Since $S[q_1\ \ -Jq_1] = [p_1\ \ -Jp_1]\begin{pmatrix}\omega_1 & \\ & \omega_1^{-1}\end{pmatrix}$, then

$$\left(U_1SV_1^T\right)E_1 = E_1\begin{pmatrix}\omega_1 & \\ & \omega_1^{-1}\end{pmatrix}.$$

That prove that $S_1 = U_1SV_1^T$ is in the following form

$$S_1 = \left(\begin{array}{cccc|cccc} \omega_1 & \mathbf{x} & \cdots & \mathbf{x} & \mathbf{0} & \mathbf{x} & \cdots & \mathbf{x} \\ 0 & * & * & * & \vdots & * & * & * \\ \vdots & * & * & * & \vdots & * & * & * \\ 0 & * & * & * & 0 & * & * & * \\ \hline 0 & \mathbf{x} & \cdots & \mathbf{x} & \omega_1^{-1} & \mathbf{x} & \cdots & \mathbf{x} \\ \vdots & * & * & * & 0 & * & * & * \\ \vdots & * & * & * & \vdots & * & * & * \\ 0 & * & * & * & 0 & * & * & * \end{array}\right)$$

Since S_1 remains symplectic, then the components $\mathbf{x}$ in the first and the $(n+1)^{th}$ rows are zero. We obtain then S_1 as follow

$$S_1 = \left(\begin{array}{cccc|cccc} \omega_1 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ 0 & * & * & * & \vdots & * & * & * \\ \vdots & * & * & * & \vdots & * & * & * \\ 0 & * & * & * & 0 & * & * & * \\ \hline 0 & \mathbf{0} & \cdots & \mathbf{0} & \omega_1^{-1} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & * & * & * & 0 & * & * & * \\ \vdots & * & * & * & \vdots & * & * & * \\ 0 & * & * & * & 0 & * & * & * \end{array}\right)$$

Then S_1 can be written as

$$S_1 = E_1\begin{pmatrix}\omega_1 & \\ & \omega_1^{-1}\end{pmatrix}E_1^T + \widetilde{S}_1$$

where $\widetilde{S}_1 = \sum_{i=2}\sum_{j=2} E_i M_{ij} E_j^T$ is symplectic as a restricted matrix on $\mathbb{R}^{2(n-1)}$. Indeed, $S_1 = E_1\begin{pmatrix}\omega_1 & \\ & \omega_1^{-1}\end{pmatrix}E_1^T + \widetilde{S}_1$ therefore $S_1^J = E_1\begin{pmatrix}\omega_1^{-1} & \\ & \omega_1\end{pmatrix}E_1^T + \widetilde{S}_1^J$. Since $I_{2n} =$

$\sum_{i=1} E_i E_i^T = S_1^J S_1 = E_1 E_1^T + \widetilde{S}_1^J \widetilde{S}_1$, then $\widetilde{S}_1^J \widetilde{S}_1 = \sum_{i=2} E_i E_i^T = I_{2(n-1)}$. Using the induction, there exists $\widetilde{U}_1, \widetilde{V}_1 \in \mathbb{R}^{2(n-1)\times 2(n-1)}$ two orthogonal and symplectic matrices and a positive real numbers $\omega_2, \ldots, \omega_n$ such that

$$\widetilde{S}_1 = \widetilde{U}_1 \begin{pmatrix} \omega_2 & & & & & \\ & \ddots & & & & \\ & & \omega_n & & & \\ & & & \omega_2^{-1} & & \\ & & & & \ddots & \\ & & & & & \omega_n^{-1} \end{pmatrix} \widetilde{V}_1^T.$$

Let $r = 1 : n-1$, $k = n : 2(n-1)$ and

$$\widetilde{U} = \begin{pmatrix} \mathbf{1} & & 0 & \\ 0_{n-1,1} & \widetilde{U}_1(r,r) & & \widetilde{U}_1(r, n:k \\ 0 & & \mathbf{1} & \\ 0 & \widetilde{U}_1(k,r) & & \widetilde{U}_1(k,k) \end{pmatrix}$$

and

$$\widetilde{V} = \begin{pmatrix} \mathbf{1} & & 0 & \\ 0_{n-1,1} & \widetilde{V}_1(r,r) & & \widetilde{V}_1(r,k) \\ 0 & & \mathbf{1} & \\ 0 & \widetilde{V}_1(k,r) & & \widetilde{V}_1(k,k) \end{pmatrix}$$

Setting $U = U_1^T \widetilde{U}$ and $V = V_1^T \widetilde{V}$ we obtain then the desired decomposition

$$U^T S V = \begin{pmatrix} \omega_1 & & & & & \\ & \ddots & & & & \\ & & \omega_n & & & \\ & & & \omega_1^{-1} & & \\ & & & & \ddots & \\ & & & & & \omega_n^{-1} \end{pmatrix}$$

The proof is then complete. □

Algorithm 4.1: Symplectic SVD-like algorithm of symplectic matrix
Input : A symplectic matrix $S \in \mathbb{R}^{2n\times 2n}$
Output $U, V \in \mathbb{R}^{2n\times 2n}$ orthogonal and symplectic matrices such that

$$S = U \begin{pmatrix} \omega_1 & & & & & \\ & \ddots & & & & \\ & & \omega_n & & & \\ & & & \omega_1^{-1} & & \\ & & & & \ddots & \\ & & & & & \omega_n^{-1} \end{pmatrix} V^T$$

1. **Initialization** $U = I_{2n}$**;** $V = I_{2n}$**;** $D = S$

2. By theorem 8 given above, there exists two orgthogonal matrices P**,** Q and a positive real numbers $w_1, w_2, \cdots, w_n$ such that

$$S = P \begin{pmatrix} \omega_1 & & & & & \\ & \ddots & & & & \\ & & \omega_n & & & \\ & & & \omega_1^{-1} & & \\ & & & & \ddots & \\ & & & & & \omega_n^{-1} \end{pmatrix} Q^T$$

Set $p_i = Pe_i$ and $q_i = Qe_i$ for $i = 1, \cdots, n$ $\widetilde{P} = [p_1\, p_2\, \ldots\, p_n \quad -Jp_1\, -Jp_2\, \ldots\, -Jp_n]$ and $\widetilde{Q} = [q_1\, q_2\, \ldots\, q_n \quad -Jq_1\, -Jq_2\, \ldots\, -Jq_n]$

3. **For** $k = 1, \cdots, n$, **do** $u = \widetilde{P}e_k$ and $v = \widetilde{Q}e_k$

For $j = 1, \cdots, k-1$ **do** $u(j) \leftarrow 0$, $u(n+j) \leftarrow 0$ and $v(j) \leftarrow 0$, $v(n+j) \leftarrow 0$

End(j)

If $\|u\| \neq 0$ and $\|v\| \neq 0$ **then** $u \longleftarrow \frac{u}{\|u\|}$ and $v \longleftarrow \frac{v}{\|v\|}$

We put $X = [u \quad -Ju]$, $Y = [v \quad -Jv]$ and $E_k = [I_{2n}(:,k), I_{2n}(:,n+k)]$
We construct a symplectic reflectors R_X, R_Y which transform respectively X, Y to E_k

$$R_X = (X + E_1)(I_2 + E_1^J X)^{-1}(X + E_1)^J - I_{2(n-k)}$$

$$R_Y = (Y + E_1)(I_2 + E_1^J Y)^{-1}(Y + E_1)^J - I_{2(n-k)}$$

Set $U := UU_kU$, $PT := UU_kPT$, $V := VV_kV$ and $QT =:= VV_kQT$

EndIf
EndFor
End.

5 Numerical examples

In this section, we compared and tested the numerical methods given above (algorithm 3.1 and algorithm 4.1) with Xu method given in [13]. Our numerical experiments were carried out with Matlab 7.8.0 (R2009a) and run it on a Core Duo Pentium processor.

Example 1: Let B be a rectangular matrix defined as follows, (see, [13]),

$$B = Q \left(\begin{array}{ccc|ccc} \Sigma & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & I & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \Sigma & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) U^T$$

where Q is a random orthogonal matrix and U is a 14×14 random orthogonal symplectic matrix. We compute in this example the error occurred on the computed symplectic SVD-like decomposition. Let

$$\Sigma_1 = diag(4, 3, 2, 1)$$
$$\Sigma_2 = diag(10^{-4}, 10^{-2}, 1, 10^2)$$
$$\Sigma_3 = diag(10^2, 2, 1, 10^{-2})$$

Then we obtain the following results,

	Alg 3.1	Xu method [13]
Σ_1	$8.0883e-015$	$9.0858e-015$
Σ_2	$5.2122e-009$	141.6480
Σ_3	$1.6946e-010$	141.7189

Example 2: Let the symplectic matrix A obtained from the diagonal matrix $\begin{pmatrix} B & 0_n \\ 0_n & B^{-T} \end{pmatrix}$ where B is defined as follows, (see, [4]),

$$\begin{pmatrix} 4/5 & -3/5 & 0 & 0 & 0 & 0 \\ 3/5 & 4/5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & -1 & 3 \end{pmatrix}$$

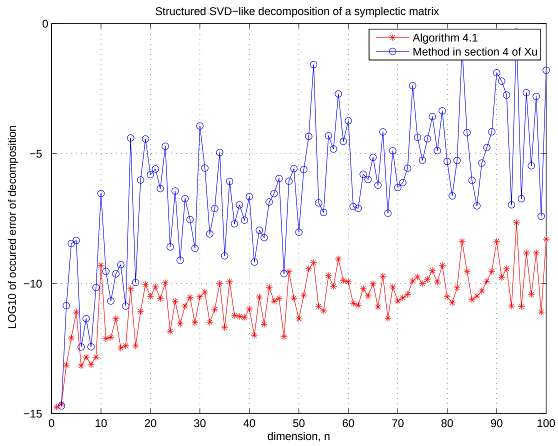
by using symplectic similarity transformations randomly generated by symplectic reflectors. We then obtained the following error occurred on the computed symplectic SVD-like decomposition

	Alg 4.1	Xu method [13]
	$2.4428e-013$	$1.1946e-010$

Example 3: Let the symplectic matrix A obtained from the diagonal matrix $\begin{pmatrix} D & 0_n \\ 0_n & D^{-1} \end{pmatrix}$ by using symplectic similarity transformations randomly generated by symplectic reflectors, where $D = diag(v)$ such that $v_{2i-1} = 10^{log(i)}, v_{2i} = 10^{-log(i)}$ for $i = 1 \dots \frac{n}{2}$ and $v(n) = 10^{-n}$. We obtained the following error occurred on the computed symplectic *SVD*-like decomposition

	Alg 4.1	Xu method [13]
$n = 5$	$1.2909e-009$	$7.5570e-004$
$n = 10$	$9.4812e-006$	$1.7747e+005$

Now, let's get $D = diag[1, 2, \dots, n]$. The error occurred on the computed symplectic SVD-like decomposition is represented bellow for n from 1 to 100.



6 Conclusion

We have developed a new way to compute a symplectic SVD-like decomposition of both real rectangular and symplectic matrices. Numerical results given above show the efficiency of our approach in computing the decomposition especially in case of ill-conditioned matrices.

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