3) the left partial indexes of M.F. $A_q(t)$ are equal to zero and right partial indexes of M.F. B_q are equal to q;

4) the operator $M : [\overset{o}{H}^{(q)}(\omega_2)]_m \to [H(\omega_2)]_m$.

5) $\Phi(\delta) = \dfrac{\omega_1(\delta)}{\omega_2(\delta)}$ is nondecreasing on $(0; h]$.

If

$$\lim_{\delta\to+0} \Phi(\delta)\ln^2(\delta) = 0$$

then starting from indices $n \geq n_1$ the SLAE (8) of reduction method is uniquely solvable. The approximate solutions $x_n(t)$ given by formula (7) converge in the norm of space $[H^{(q)}(\omega_2)]_m$ to the exact solution of problem (4)-(5). The following estimation is true:

$$||x - x_n||^m_{\omega_2,q} = O\left(\Phi\left(\frac{1}{n}\right)\ln^2 n\right). \quad (9)$$

4 Auxiliary Results

The vector functions $\dfrac{d^q(Px)(t)}{dt^q}$ and $\dfrac{d^q(Qx)(t)}{dt^q}$ can be represented by integrals of Cauchy type with the same density $v(t)$:

$$\left.\begin{aligned} \frac{d^q(Px)(t)}{dt^q} &= \frac{1}{2\pi i}\int_\Gamma \frac{v(\tau)}{\tau - t}d\tau, \quad t \in F^+, \\ \frac{d^q(Qx)(t)}{dt^q} &= \frac{t^{-q}}{2\pi i}\int_\Gamma \frac{v(\tau)}{\tau - t}d\tau, \quad t \in F^-. \end{aligned}\right\} \quad (10)$$

Using the integral representation (10) we reduce the problem (4)-(5) to an equivalent system of SIE (in terms of solvability):

$$(\Upsilon v \equiv) C(t)v(t) + \frac{D(t)}{\pi i}\int_\Gamma \frac{v(\tau)}{\tau - t}d\tau + \frac{1}{2\pi i}\int_\Gamma h(t,\tau)v(\tau)d\tau = f(t), \quad t \in \Gamma, \quad (11)$$

for unknowns $v(t)$ where

$$C(t) = \frac{1}{2}[A_q(t) + t^{-q}B_q(t)],$$

$$D(t) = \frac{1}{2}[A_q(t) - t^{-q}B_q(t)], \quad (12)$$

$$\begin{aligned} h(t,\tau) = {} & \frac{1}{2}\left[K_q(t,\tau) + K_q(t,\tau)\tau^{-n}\right] - \\ & \frac{1}{2\pi i}\int_\Gamma \left[K_q(t,\bar t) - K_q(t,\bar t)\bar t^{-n}\right]\frac{d\bar t}{\bar t - \tau} \\ & + \sum_{j=0}^{q-1}\left[A_j(t)\tilde M_j(t,\tau) + \int_\Gamma K_j(t,\bar t)\tilde M_j(\bar t,\tau)d\bar t\right] \\ & - \sum_{j=0}^{q-1}\left[B_j(t)\tilde N_j(t,\tau) + \int_\Gamma K_j(t,\bar t)\tilde N_j(\bar t,\tau)d\bar t\right], \end{aligned} \quad (13)$$

where $\tilde M_j(t,\tau), \tilde N_j(t,\tau)$ $j = 0,\ldots,q$ are known Hölder continuous M.F. An explicit form for these functions is given in [15]. By virtue of the properties of the m.f. $\tilde M_j(t,\tau)$, $\tilde N_j(t,\tau)$, $K_j(t,\tau), A_j(t), B_j(t)$, $j = 0,\ldots,q$, we obtain that the m.f. $h(t,\tau)$ is a Hölder continuous M.F. Note that the right hand sides in (11) and (4) coincide by conditions (5).

Lemma 2 *The system of SIE (11) and problem (4)-(5) are equivalent in terms of solvability. That is, for each solution*

$v(t)$ of system of SIE (11), there is a solution of problem (4)-(5), determined by the formulae

$$(Px)(t) = \frac{(-1)^q}{2\pi i(q-1)!} \int_\Gamma v(\tau)[(\tau - t)^{q-1}$$

$$\log\left(1 - \frac{t}{\tau}\right) + \sum_{k=1}^{q-1} \alpha_k \tau^{q-k-1} t^k] d\tau,$$

$$(Qx)(t) = \frac{(-1)^q}{2\pi i(q-1)!} \int_\Gamma v(\tau)\tau^{-q}[(\tau - t)^{q-1}$$

$$\log\left(1 - \frac{\tau}{t}\right) + \sum_{k=1}^{q-2} \beta_k \tau^{q-k-1} t^k] d\tau, \quad (14)$$

where $\alpha_k, k = 1, \ldots, q-1$, and $\beta_k, k = 1, \ldots, q-2$ are vector numbers. On the other hand, for each solution $x(t)$ of the problem (4)-(5) there is a solution $v(t)$

$$v(t) = \frac{d^q(Px)(t)}{dt^q} + t^q \frac{d^q(Qx)(t)}{dt^q},$$

of system of SIE (11). Furthermore, for linearly independent solutions of (11), there are corresponding linearly independent solutions of problem (4)-(5) from (14) and vice versa.

In formula (14), both $\log(1 - t/\tau)$ and $\log(1 - \tau/t)$, for given τ, there are branches that vanish at $t = 0$ and $t = \infty$, respectively. We formulate the theorems about the theoretical background of numerical schemes of the reduction for system of SIE

Theorem 3 *Let the following conditions be satisfied:*

a) M.F. $C(t), D(t)$ and $h(t,\tau) \in [H(\omega_1)]_m$;

b) $det(C(t)) \neq 0$, $det(D(t)) \neq 0$, $t \in \Gamma$;

c) the left partial indexes of m.f. $C(t)$ right partial indexes of m.f.$D(t)$ are equal to zero;

d) operator $\Upsilon = aP + bQ + H$ be invertible in $[H(\omega_2)]_m$, H is integral operator with kernel $h(t,\tau)$, P and Q are Riesz projectors $P = \frac{1}{2}(I + S)$, $Q = \frac{1}{2}(I - S)$, S is a singular operator with Cauchy kernel.

e) $\Phi(\delta) = \dfrac{\omega_1(\delta)}{\omega_2(\delta)}$ is nondecreasing on $(0; h]$.

If

$$\lim_{\delta \to +0} \Phi(\delta) \ln^2(\delta) = 0$$

then the operator of the reduction method

$$\Upsilon_n = S_n[aP + bQ + H]S_n,$$

of operator $\Upsilon v = f$ for large enough numbers $(n \geq n_0)$ is invertible in the space $[H(\omega_2)]_m$ and the approximate solutions $v_n(t) = \Upsilon_n^{-1} S_n f$ converges to the function $v = \Upsilon^{-1} f$. The following estimation is true:

$$||v - v_n||^m_{\omega_2,q} = O\left(\Phi\left(\frac{1}{n}\right)\ln^2\right). \quad (15)$$

5 Proof of convergence theorem

In this section we prove the Theorem 1.
Proof We should show that for numbers $n \geq n_0$ large enough the operator is invertible. The operator M acts from the

subspace $[\overset{o}{X}_n]_m = t^q P[X_n]_m + Q[X_n]_m$ (the norm defined as in $[H(\omega_2)^{(q)}]_m$) to the space $[X_n]_m = S_n[H_{\omega_2}]_m$ of m dimensional polynomials of the form $\sum\limits_{k=-n}^{n} r_k t^k$ (the norm as in $[H(\omega_2)]_m$).

In a similar way, by using formulas (10), we represent the v.f.

$$d^q(P(x_n)(t))/dt^q, \quad d^q(Q(x_n)(t))/dt^q$$

by Cauchy type integrals with the same density $v_n(t)$:

$$\frac{d^q(P(x_n)(t))}{dt^q} = \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{v_n(\tau)}{\tau - t} d\tau,\ t \in F^+,$$

$$\frac{d^q(Q(x_n)(t))}{dt^q} = \frac{t^{-q}}{2\pi i}\int\limits_{\Gamma} \frac{v_n(\tau)}{\tau - t} d\tau,\ t \in F^-. \tag{16}$$

By taking account of the formulas $(Px)^{(r)}(t) = P(x^{(r)})(t)$ and $(Qx)^{(r)}(t) = Q(x^{(r)})(t)$, $r = 1, 2, \ldots$, and the relations

$$(t^{k+q})^{(r)} = \frac{(k+q)!}{(k+q-r)!} t^{k+q-r},\ k = 0, \ldots, n,$$

$$(t^{-k})^{(r)} = (-1)^r \frac{(k+r-1)!}{(k-1)!} t^{-k-r},\ k = 1, \ldots, n,$$

from (16), we obtain

$$v_n(t) = \sum_{k=0}^{n} \frac{(k+q)!}{k!} t^k \xi_k + (-1)^q \sum_{k=1}^{n} \frac{(k+q-1)!}{(k-1)!} t^{-k} \xi_{-k}$$

Consequently $v_n(t) \in [X_n]_m$; here we have used the fact that the polynomials $x_n(t)$, given by (7) can be represented uniquely in the form

$$t^q \sum_{k=0}^{n} \xi_k t^k + \sum_{k=-n}^{-1} \xi_k t^k.$$

Using of the representations (16), Eq. (8), as well as the problem (4)-(5) can be reduced to an equivalent equation (in the sense solvability)

$$S_n R S_n v_n = S_n f, \tag{17}$$

Treated as an equation in the subspace $[X_n]_m$. Obviously, Eq. (17) is the equation of the method of reduction over Faber-Laurent polynomials for the singular integral equation (11), and for singular integral equations, the method of reduction over Faber-Laurent polynomials was considered in [17], where sufficient conditions for the solvability and convergence of this method were obtained. Assumptions in Theorem 3 provide the validity of all assumptions in Theorem 1. We have that the Eq. (17) with $n \geq n_1$ is uniquely solvable; moreover, the approximate solutions $v_n(t)$ of this equation converge to the exact solution $v(t)$ of the system of singular integral equation (11) in the norm of the space $[H(\omega_2)]_m$ as $n \to \infty$:

$$\|v_n - v\|^m_{\omega_2,q} = O\left(\Phi\left(\frac{1}{n}\right)\ln^2(\delta)\right). \tag{18}$$

v.f. $x_n(t)$ can be expressed via the v.f. $v_n(t)$ by formulas (14). From definition of the norm in the space $[H^{(q)}_{\omega_2}]_m$ together with (18) implies estimate (9).

The proof of Theorem 1 is complete.

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