Classical theorems for a Gould type integral

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Abstract: In this paper, we continue the study of the Gould type integral introduced in [30] which generalizes the results of [12, 13, 17, 28] and [29]. We obtain various classical properties, such as a mean type theorem, a Lebesgue (Fatou respectively) type theorem, Hölder and Minkowski inequalities etc. Other results concerning measurability, semi-convexity, diffusion and atoms are also established.

Key–Words: (multi)(sub)measure, semi-convex, Darboux property, diffused, atom, totally-measurable, Gould integral, Lebesgue theorem, Fatou lemma.

1 Introduction

In [20] G. G. Gould introduced an integral for bounded real functions with respect to finitely additive set functions taking values in a Banach space, integral which is more general that the Lebesgue one.

In the last years, the non-additive case and the set-valued case received a special attention because of their applications in mathematical economics, decision theory, artificial intelligence, statistics or theory of games.

A. Precupanu and A. Croitoru generalized Gould’s results [20], studying in [28] a Gould type integral for multimeasures with values in $P_{bc}(X)$, the family of all compact convex nonempty subsets of a real Banach space $X$. Also, Gould type integrals with respect to a (multi)submeasure were studied in [12]–[19]. In [30], A. Precupanu, A. Gavriluț and A. Croitoru introduced and studied a Gould type integral for bounded real functions with respect to a set multifunction of finite variation with values in $P_{bf}(X)$, the family of all bounded closed nonempty subsets of a real Banach space $X$.

On the other hand, notions as atoms, pseudo-atoms, Darboux property, non-atomicity (with different nonequivalent variants - see, for instance, [8, 9]), (finitely) purely atomicity, semi-convexity, diffusion were extensively studied in recent years, due to their applications in many classical measure theory problems, physics and convex analysis (see [1, 3, 4, 5, 6, 8, 9, 10, 11, 21, 23, 24, 25, 26]).

That is why, in this paper, we study these notions for the Gould type integral introduced in [30]. We prove that the Lebesgue theorem, Hölder and Minkowski inequalities, Fatou lemma have here a correspondent and our integral preserves properties like semi-convexity or diffusion. Results regarding measurability are also established.

2 Basic notions

Let $(X, \| \cdot \|)$ be a real normed space, $P_{0}(X)$ the family of all nonvoid subsets of $X$, $P_{b}(X)$ the family of all nonvoid bounded subsets of $X$, $P_{f}(X)$ the family of all nonvoid closed subsets of $X$, $P_{bf}(X)$ the family of all nonvoid compact convex subsets of $X$ and $h$ the Hausdorff pseudometric on $P_{f}(X)$, which becomes a metric on $P_{bf}(X)$.

It is known that $h(M, N) = \max\{e(M, N), e(N, M)\}$, where $e(M, N) = \sup_{d(x, N)} d(x, N)$, for every $M, N \in P_{f}(X)$ is the excess of $M$ over $N$ and $d(x, N)$ is the distance from $x$ to $N$ with respect to the distance induced by the norm of $X$.

We denote $|M| = h(M, \{0\}) = \sup_{x \in M} d(x, N)$, for every $M \in P_{0}(X)$, where $0$ is the origin of $X$.

For every $M, N \in P_{0}(X)$ and every $\alpha \in \mathbb{R}$, let $M + N = \{x + y | x \in M, y \in N\}$ and $\alpha M = \{\alpha x | x \in M\}$. We denote by $\mathcal{M}$ the closure of $M$ with respect to the topology induced by the norm of $X$.

On $P_{0}(X)$ we consider the Minkowski addition $\cdot + $ [18], defined by:

$M \cdot + N = \mathcal{M} + \overline{N}$, for every $M, N \in P_{0}(X)$. 

Let $T$ be an abstract nonvoid set, $\mathcal{P}(T)$ the family of all subsets of $T$ and $C$ a ring of subsets of $T$.

By $i = \frac{\pi}{\sum_n}$ we mean $i \in \{1, 2, \ldots, n\}$, for $n \in \mathbb{N}^*$, where $\mathbb{N}$ is the set of all naturals and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. We also denote $\mathbb{R}_+ = [0, +\infty)$ and $\mathbb{R}_+ = [0, +\infty]$.

Some properties of $h$ are presented in the following proposition (see Hu and Papageorgiu [22], Petrușel and Moț [27]).

**Proposition 1** Let $A, B, C, D, A_n, B_n \in \mathcal{P}_0(X)$, for every $n \in \mathbb{N}^*$. Then:

I) $(\alpha + \beta)A = \alpha A + \beta A$, for every $\alpha, \beta \in \mathbb{R}_+$ and convex $A$.

II) $(A + B) + C = A + (B + C)$.

III) $(A + B) + (C + D) = (A + C) + (B + D)$.

IV) $h(A, B) = h(A, B)$.

V) $e(A, B) = 0$ if and only if $A \subseteq B$.

VI) $h(A, B) = 0$ if and only if $A = B$.

VII) $h(\alpha A, A) = |\alpha| h(A, B)$, for all $\alpha \in \mathbb{R}$.

VIII) $h(\sum_{i=1}^n A_i, \sum_{i=1}^n B_i) \leq \sum_{i=1}^n h(A_i, B_i)$.

IX) $h(\alpha A, \beta A) \leq |\alpha - \beta| \cdot |A|$, for every $\alpha, \beta \in \mathbb{R}$.

X) $h(\alpha A + \beta B, \gamma A + \delta B) \leq |\alpha - \gamma| \cdot |A| + |\beta - \delta| \cdot |B|$, for all $\alpha, \beta, \gamma, \delta \in \mathbb{R}$.

XI) $h(A + C, B + C) = h(A, B)$, for every $A, B \in \mathcal{P} \mathcal{F}(X)$ and $C \subseteq \mathcal{P} \mathcal{F}(X)$.

XII) If $A, A_n \in \mathcal{P} \mathcal{F}(X)$ and $\alpha, \alpha_n \in \mathbb{R}$, for every $n \in \mathbb{N}^*$, are so that $h(A_n, A) \to 0$ and $\alpha_n \to \alpha$, then $h(\alpha_n A_n, \alpha A) \to 0$.

We now recall some classical notions:

**Definition 2** A set function $m : C \to \mathbb{R}_+$, with $m(\emptyset) = 0$, is said to be:

I) monotone if $m(A) \leq m(B)$, for every $A, B \in C$, with $A \subseteq B$.

II) superadditive if $m(\bigcup_{i \in I} A_i) \geq \sum_{i \in I} m(A_i)$, for every sequence of pairwise disjoint sets $(A_i)_{i \in I} \subseteq C$, with $\bigcup_{i \in I} A_i \subseteq C$, $I \subseteq \mathbb{N}$.

III) additive if $m(A \cup B) = m(A) + m(B)$, for every $A, B \in C$, with $A \cap B = \emptyset$.

IV) a submeasure (in Drewnowski’s sense [7]) if $m$ is monotone and subadditive.

**Example 3** I) If $\nu : C \to \mathbb{R}_+$ is a finitely additive set function, then $m : C \to [0, 1]$ defined for every $A \in C$ by $m(A) = \frac{\nu(A)}{\nu(\mathcal{P}(A))}$ is a submeasure.

II) ([8,9]) Let $m_n : C \to \mathbb{R}_+$ be a submeasure for every $n \in \mathbb{N}$. Then the set function $m : C \to \mathbb{R}_+$ defined by $m(A) = \sup_{n} m_n(A)$, for every $A \in C$, is a submeasure, too.

**Remark 4** Suppose $m : C \to \mathbb{R}_+$ is a submeasure of finite variation. If $\overline{\mu}$ denotes the variation of $m$ on $\mathcal{P}(T)$, then:

I) $\overline{\mu}$ is finitely additive on $C$.

II) The following statements are equivalent:

i) $m$ is $o$-continuous;

ii) $m$ is $\sigma$-subadditive;

iii) $\mu$ is $\sigma$-additive on $C$;

iv) $\mu$ is $o$-continuous on $C$.

**Definition 5** For a set multifunction $\mu : C \to \mathcal{P}(X)$, with $\mu(\emptyset) = \{0\}$, we consider:

I) the extended real valued set function $|\mu|$ defined by $|\mu|(A) = |\mu(A)|$, for every $A \in C$.

II) the variation of $\mu$ defined by $\overline{\mu}(A) = \sup\{\sum_{i=1}^n |\mu(A_i)|\}$, for every $A \in \mathcal{P}(T)$, where the supremum is extended over all finite families of pairwise disjoint sets $\{A_i\}_{i=1}^{\infty} \subseteq A$, with $A_i \subseteq A$, for every $i \in \{1, \ldots, n\}$.

$\mu$ is said to be of finite variation on $C$ if $\overline{\mu}(A) < \infty$, for every $A \in C$.

**Definition 6** Let $\mu : C \to \mathcal{P}(X)$ be a set multifunction, with $\mu(\emptyset) = \{0\}$, $\mu$ is said to be:

I) monotone if $\mu(A) \subseteq \mu(B)$, for every $A, B \in C$, with $A \subseteq B$.

II) a multimeasure if $\mu(A \cup B) = \mu(A) + \mu(B)$, for every $A, B \in C$, with $A \cap B = \emptyset$.

III) a multiasubmeasure if $\mu$ is monotone and $\mu(A \cup B) \leq \mu(A) + \mu(B)$, for every $A, B \in C$, with $A \cap B = \emptyset$.

IV) $h$-$\sigma$-subadditive if $|\mu(\bigcup_{n=1}^{\infty} A_n)| \leq \sum_{n=1}^{\infty} |\mu(A_n)|$, for every sequence of pairwise disjoint sets $(A_n)_{n \in \mathbb{N}^*} \subseteq C$, with $\bigcup_{n=1}^{\infty} A_n \subseteq C$.

V) null-additive if $\mu(A \cup B) = \mu(A)$, for every $A, B \in C$, with $\mu(B) = \{0\}$.

VI) null–null-additive if $\mu(A \cup B) = 0$, for every $A, B \in C$, with $\mu(A) = \mu(B) = \{0\}$.

VII) order-continuous (shortly, $\sigma$-continuous) if $\lim_{n \to \infty} h(\mu(A_n), \mu(A)) = 0$, for every decreasing sequence of sets $(A_n)_{n \in \mathbb{N}^*} \subseteq C$, with $\bigcup_{n=1}^{\infty} A_n = \emptyset$ (denoted by $A_n \searrow \emptyset$).

VIII) increasing convergent if $\lim_{n \to \infty} h(\mu(A_n), \mu(A)) = 0$, for every increasing se-
quence of sets \((A_n)_{n \in \mathbb{N}} \subset C\), with \(\bigcap_{n=1}^{\infty} A_n = A \in C\) (denoted by \(A_n \nearrow A\)).

**Remark 7** If \(\mu\) is \(\mathcal{P}_f(X)\)-valued, then in Definition 6-II, III it usually appears the Minkowski addition instead of the classical addition because the sum of two closed sets is not, generally, a closed set.

**Remark 8**. I) \(\overline{\mu}\) is monotone and superadditive on \(\mathcal{P}(T)\). Also (see [12]), if \(\mu : C \rightarrow \mathcal{P}_f(X)\) is a multi(sub)measure, then \(\overline{\mu}\) is finitely additive on \(C\) and \(|\mu|\) is a submeasure.

II) Every monotone multimeasure is, particularly, a multisubmeasure. Also, any multisubmeasure is null-additive. Any null-additive set multifunction is null-null-additive. The converses are not valid.

III) Let \(\mu : A \rightarrow \mathcal{P}_f(X)\) be a multisubmeasure of finite variation. The following statements are equivalent:

i) \(\mu\) is \(h\)-\(\sigma\)-subadditive;

ii) \(\mu\) is order-continuous;

iii) \(\overline{\mu}\) is \(\sigma\)-additive on \(C\).

3 Semi-convexity, Darboux property, diffusion and atoms of set multifunctions

We present some properties regarding semi-convexity, Darboux property, diffusion and atoms for set multifunctions. These properties will be discussed in section 5 in relation with the Gould type set-valued integral.

The following notions are classical in measure theory, but they are extended to the set valued case (see for instance [2, 3, 4, 15, 16]).

**Definition 9** Let \(\mu : C \rightarrow \mathcal{P}_0(X)\) be a set multifunction, with \(\mu(\emptyset) = \{0\}\).

I) We say that \(\mu\)

i) is semi-convex if for every \(A \in C\), with \(\mu(A) \supseteq \{0\}\), there is a set \(B \in C\) such that \(B \subseteq A\) and \(\mu(B) = \frac{1}{2} \mu(A)\).

ii) has the Darboux property if for every \(A \in C\), with \(\mu(A) \supseteq \{0\}\) and every \(p \in (0, 1)\), there exists a set \(B \in C\) such that \(B \subseteq A\) and \(\mu(B) = p \mu(A)\).

iii) is diffused if for every \(t \in T\), with \(\{t\} \in C\), we have \(\mu(\{t\}) = \{0\}\).

II) A set \(A \in C\) is said to be an atom of \(\mu\) if \(\mu(A) \supseteq \{0\}\) and for every \(B \in C\), with \(B \subseteq A\), we have \(\mu(B) = \{0\}\) or \(\mu(A \setminus B) = \{0\}\).

III) We say that \(\mu\)

i) finitely purely atomic if there is a finite disjoint family \((A_i)_{i=1}^{n} \subset C\) of atoms of \(\mu\) so that \(T = \bigcup_{i=1}^{n} A_i\).

ii) purely atomic if there is at most a countable number of atoms \((A_n)_n \subset C\) of \(\mu\) so that \((T \setminus \bigcup_{n=1}^{\infty} A_n) = \{0\}\) (evidently, here \(C\) must be a \(\sigma\)-algebra).

iii) non-atomic if it has no atoms.

IV) We say that \(\mu : C \rightarrow \mathcal{P}_{bc}(\mathbb{R})\) is induced by a set function \(m : C \rightarrow \mathbb{R}_{+}\), with \(m(\emptyset) = 0\), if \(\mu(A) = [0, m(A)]\), for every \(A \in C\).

**Remark 10** I) The Lebesgue measure \(\mu\) is diffused. Also, the set functions \(m_1, m_2 : C \rightarrow \mathbb{R}_{+}\) defined for every \(A \in C\) by \(m_1(A) = \sqrt{\mu(A)}\) and \(m_2(A) = \frac{\mu(A)}{1+\mu(A)}\) are diffused submeasures. The same are the multisubmeasures induced by them.

II) If \(\mu_1, \mu_2 : C \rightarrow \mathcal{P}_0(X)\) are diffused submeasures, then the same is the multimeasure \(\mu_1 + \mu_2\) defined by \((\mu_1 + \mu_2)(A) = \mu_1(A) + \mu_2(A)\), for every \(A \in C\).

III) Let \(\mu : C \rightarrow \mathcal{P}_0(X)\) be a set multifunction, with \(\mu(\emptyset) = \{0\}\). Then the following statements are equivalent:

a) \(\mu\) is diffused;

b) \(|\mu|\) is diffused;

c) \(\overline{\mu}\) is diffused on \(C\).

The following result is obviously true.

**Proposition 11** If the set multifunction \(\mu : C \rightarrow \mathcal{P}_0(X)\), with \(\mu(\emptyset) = \{0\}\), has the Darboux property, then it is semi-convex.

Under some assumptions, the converse of Proposition 11 is also valid, as shown below:

**Theorem 12** Let \(C\) be a \(\sigma\)-ring and \(\mu : C \rightarrow \mathcal{P}_{bc}(X)\) a monotone increasing convergent multimeasure. Then \(\mu\) has the Darboux property if and only if \(\mu\) is semi-convex.

**Proof**. The "only if" part results from Proposition 11. The "if" part. Every \(p \in (0, 1)\) has an expansion \(p = \sum_{n=1}^{\infty} a_n p^n\), where \(a_n \in \{0, 1\}\), for every \(n \in \mathbb{N}\). Let \(A \in C\), so that \(\mu(A) \supseteq \{0\}\) and let \(p \in (0, 1)\).

By the semi-convexity of \(\mu\), there is \(B_1 \subset C\) so that \(B_1 \subseteq A\) and \(\mu(B_1) = \frac{a_1}{p} \mu(A)\).

Analogously, there is \(B_2 \subset C\) so that \(B_2 \subseteq A \setminus B_1\) and \(\mu(B_2) = \frac{a_2}{p^2} \mu(A)\) and so on. Consider \(B = \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n} B_k \subset C\). We have
\( \mu(B) = \lim_{n \to \infty} \sum_{k=1}^{n} \mu(B_k) = \lim_{n \to \infty} \sum_{k=1}^{n} a_k \mu(A) \) (with respect to \( h \)). By Proposition 1-I and XII, it follows \( \mu(B) = p \mu(A) \), as claimed. \( \square \)

**Remark 13** I) If \( \mu \) is monotone, then \( \mu \) is non-atomic if and only if for every \( A \in \mathcal{C} \), with \( \mu(A) \geq \{0\} \), there exists \( B \in \mathcal{C} \), with \( B \subseteq A \), \( \mu(B) \geq \{0\} \) and \( \mu(A \setminus B) \geq \{0\} \).

II) Let \( \nu : \mathcal{C} \to \mathbb{R}_+ \) be a set function, with \( \nu(\emptyset) = 0 \) and \( \mu \) the set multifunction induced by \( \nu \). Then \( \mu \) has the Darboux property if and only if \( \nu \) has it.

III) [15] Suppose \( T \) is a locally compact Hausdorff space. \( B \) is the Borel \( \delta \)-ring generated by the compact subsets of \( T \) and \( \mu : \mathcal{B} \to \mathcal{P}(X) \) is a multisubmeasure. Then \( \mu \) is non-atomic if and only if it is diffused.

4 \( \tilde{\mu} \)-totally-measurability

In this section we present some properties of \( \tilde{\mu} \)-totally-measurable functions. In the sequel, \( \mathcal{A} \) is an algebra of subsets of \( T \), \( \mu : \mathcal{A} \to \mathcal{P}(X) \) is a set multifunction so that \( \mu(\emptyset) = \{0\} \) and \( f : T \to \mathbb{R} \) an arbitrary function.

**Definition 14** I) A partition of a set \( A \in \mathcal{A} \) is a finite family \( P = \{A_i\}_{i=1}^{n} \) of pairwise disjoint sets of \( \mathcal{A} \) such that \( \bigcup_{i=1}^{n} A_i = A \).

We denote by \( \mathcal{P} \) the class of all partitions of \( T \) and if \( A \in \mathcal{A} \) is fixed, by \( \mathcal{P}_A \), the class of all partitions of \( A \).

II) For a set multifunction \( \mu : \mathcal{A} \to \mathcal{P}(X) \), we consider the extended real valued set function \( \tilde{\mu} \) defined by \( \tilde{\mu}(A) = \inf \{\pi(B) ; A \subseteq B, B \in \mathcal{A} \} \) for every \( A \in \mathcal{P}(T) \).

**Remark 15** I) \( \tilde{\mu}(A) = \mu(A) \), for every \( A \in \mathcal{A} \), \( \tilde{\mu} \) is monotone and if \( \pi \) is subadditive, then \( \tilde{\mu} \) is also subadditive.

II) Suppose \( \mu : \mathcal{A} \to \mathcal{P}(X) \) is a multisubmeasure of finite variation. Then:

i) \( \tilde{\mu} \) is a submeasure.

ii) If, moreover, \( \mu \) is \( h \)-\( \sigma \)-subadditive, then \( \tilde{\mu} \) is \( \sigma \)-subadditive.

**Definition 16** I) \( f \) is said to be \( \tilde{\mu} \)-totally-measurable on \( (T, \mathcal{A}, \mu) \) if for every \( \varepsilon > 0 \) there exists a partition \( P_{\varepsilon} = \{A_i\}_{i=1}^{n} \) of \( T \) such that:

\[
\begin{align*}
(\ast) & \quad \tilde{\mu}(A_0) < \varepsilon \quad \text{and} \\
& \quad \sup_{t, s \in A_i} |f(t) - f(s)| = \text{osc}(f, A_i) < \varepsilon, \\
& \quad \text{for every } i = 1, \ldots, n.
\end{align*}
\]

II) \( f \) is said to be \( \tilde{\mu} \)-totally-measurable on \( B \in \mathcal{A} \) if the restriction \( f|_B \) of \( f \) to \( B \) is \( \tilde{\mu} \)-totally measurable on \( (B, \mathcal{A}_B, \mu_B) \), where \( \mathcal{A}_B = \{A \cap B ; A \in \mathcal{A} \} \) and \( \mu_B = \mu|_{\mathcal{A}_B} \).

One can easily observe that if \( f \) is \( \tilde{\mu} \)-totally-measurable on \( T \), then \( f \) is \( \tilde{\mu} \)-totally-measurable on every \( A \in \mathcal{A} \).

**Definition 17** We say that a property \((P)\) holds \( \mu \)-almost everywhere (shortly, \( \mu \)-ae) if there is \( A \in \mathcal{P}(T) \), with \( \mu(A) = 0 \), so that the property \((P)\) is valid on \( T \setminus A \).

**Definition 18** Let \( f_n : T \to \mathbb{R} \) be a real function for every \( n \in \mathbb{N} \). One says that the sequence \( (f_n) \)

I) converges in submeasure to \( f \) (denoted by \( f_n \xrightarrow{\mu} f \)) if for every \( \varepsilon > 0 \), \( \lim_{n \to \infty} \mu(B_n(\varepsilon)) = 0 \), where

\( B_n(\varepsilon) = \{t \in T ; |f_n(t) - f(t)| \geq \varepsilon \}. \)

II) converges almost everywhere to \( f \) (denoted by \( f_n \xrightarrow{\text{a.e.}} f \)) if there is \( A \in \mathcal{P}(T) \) so that \( \mu(A) = 0 \) and \( (f_n) \) pointwise converges to \( f \) on \( T \setminus A \).

III) \((\text{Li}[23, 24])\) is almost uniformly convergent on \( T \) (with respect to \( \tilde{\mu} \)), denoted by \( f_n \xrightarrow{\text{au}} f \), if there exists \( (A_k)_{k \in \mathbb{N}^*} \subseteq \mathcal{A} \), with \( \lim_{k \to \infty} \mu(A_k) = 0 \), such that \( f_n \) converges to \( f \) on \( T \setminus A_k \) uniformly for any fixed \( k \in \mathbb{N}^* \).

From now on, \( \mu \) is supposed to be of finite variation.

**Theorem 19** Let \( \mu : \mathcal{A} \to \mathcal{P}(X) \) be a multisubmeasure.

I) \((\text{Li})\) If \( f, g : T \to \mathbb{R} \) are bounded \( \tilde{\mu} \)-totally-measurable functions, then:

i) \( f + g \) is \( \tilde{\mu} \)-totally-measurable;

ii) \( \lambda f \) is \( \tilde{\mu} \)-totally-measurable, for every \( \lambda \in \mathbb{R} \);

iii) \( f^2 \) and \( f g \) are \( \tilde{\mu} \)-totally-measurable;

iv) \( |f|^p \) is \( \tilde{\mu} \)-totally-measurable, for every \( p \in [1, +\infty] \);

v) If \( \text{inf}_{t \in T} f(t) > 0 \), then \( \frac{1}{f} \) is \( \tilde{\mu} \)-totally-measurable.

II) Suppose \( f, g : T \to \mathbb{R} \) are bounded functions. If \( |f|^p \) and \( |g|^p \) are \( \tilde{\mu} \)-totally-measurable for an arbitrary \( p \in [1, +\infty] \), then \( |f + g|^p \) is \( \tilde{\mu} \)-totally-measurable.

III) \((\text{Li}[13])\) If for every \( n \in \mathbb{N} \), \( f_n : T \to \mathbb{R} \) is bounded \( \tilde{\mu} \)-totally-measurable and \( (f_n) \) is convergent in submeasure to a bounded function \( f : T \to \mathbb{R} \), then \( f \) is \( \tilde{\mu} \)-totally-measurable.
Remark 20 If \( \varphi : \mathbb{R} \to \mathbb{R} \) is Lipschitz, then \( \varphi \circ f \) is \( \tilde{\mu} \)-totally-measurable.

Proposition 21 Let \( \mu : \mathcal{A} \to \mathcal{P}_{f}(X) \) be a (multi)(sub)measure, \( f : T \to \mathbb{R} \) a bounded function and \( A, B \in \mathcal{A} \), with \( A \cap B = \emptyset \). Then \( f \) is \( \tilde{\mu} \)-totally-measurable on \( A \cup B \) if and only if it is \( \tilde{\mu} \)-totally-measurable on \( A \) and \( \tilde{\mu} \)-totally-measurable on \( B \).

Proof. The if part is straightforward. For the only if part, by the \( \tilde{\mu} \)-totally-measurability of \( f \) on \( A \) and \( B \), there are \( P_{\varepsilon}^{A} = \{ A_{i} \}_{i=0}^{\infty} \in \mathcal{P}_{A} \) and \( P_{\varepsilon}^{B} = \{ B_{j} \}_{j=0}^{\infty} \in \mathcal{P}_{B} \) satisfying the condition (*). Since \( \tilde{\mu} \) is additive on \( A \), then \( P_{\varepsilon}^{A \cup B} = \{ A_{0} \cup B_{0}, A_{1}, \ldots, A_{n}, B_{1}, \ldots, B_{q} \} \in \mathcal{P}_{A \cup B} \) also satisfies condition (*), so \( f \) is \( \tilde{\mu} \)-totally-measurable on \( A \cup B \). \( \Box \)

Remark 22 I) In the above proposition, \( A \) and \( B \) need not to be disjoint. Indeed, if we take arbitrary \( A, B \in \mathcal{A} \), since \( A \cup B = (A \setminus B) \cup B \) and \( \tilde{\mu} \)-totally-measurability is hereditary, the statement follows.

II) Under the assumptions of the above proposition, if \( \{ A_{i} \}_{i=1}^{n} \subseteq \mathcal{A} \), then \( f \) is \( \tilde{\mu} \)-totally-measurable on \( \bigcup_{i=1}^{n} A_{i} \), if and only if the same is \( f \) on every \( A_{i} \), \( i = 1, \ldots, n \).

Proposition 23 If \( \mathcal{A} \) is a \( \sigma \)-algebra, \( \mu : \mathcal{A} \to \mathcal{P}_{f}(X) \) is an \( \sigma \)-continuous (multi)(sub)measure, \( f : T \to \mathbb{R} \) is a bounded function and \((A_{n})_{n} \subseteq \mathcal{A} \) are pairwise disjoint, then \( f \) is \( \tilde{\mu} \)-totally-measurable on every \( A_{n}, n \in \mathbb{N} \) if and only if the same is \( f \) on \( A = \bigcup_{n=1}^{\infty} A_{n} \).

Proof. The only if part immediately follows. The if part: Since \( \mu \) is an \( \sigma \)-continuous (multi)(sub)measure of finite variation, then \( \tilde{\mu} \) is additive on \( \mathcal{A} \), so \( \tilde{\mu} \) is also \( \sigma \)-continuous on \( \mathcal{A} \). We observe that \( A \setminus \bigcup_{k=1}^{n} A_{k} \neq \emptyset \), so for every \( \varepsilon > 0 \), there is \( n_{0} \in \mathbb{N} \), with \( \tilde{\mu}(A \setminus \bigcup_{k=1}^{n_{0}} A_{k}) < \varepsilon \).

Since for every \( l = \bigcup_{k=1}^{n_{0}} A_{k} \), \( f \) is \( \tilde{\mu} \)-totally-measurable on \( A_{l} \), let \( \{ B_{j}^{l} \}_{i=1}^{\infty} \subseteq \mathcal{P}_{A_{l}}(\mathbb{R}) \) be the corresponding partitions satisfying (*).

The partition \( P_{\varepsilon}^{A_{l}} = \{ (A \setminus \bigcup_{k=1}^{n_{0}} A_{k}), \{ B_{j}^{l} \}_{j=1}^{\infty} \subseteq \mathcal{P}_{A_{l}}(\mathbb{R}) \} \subseteq \mathcal{P}_{A} \) satisfies (*), so \( f \) is \( \tilde{\mu} \)-totally-measurable on \( A = \bigcup_{n=1}^{\infty} A_{n} \). \( \Box \)

Theorem 24 Suppose \( \mathcal{A} \) is a \( \sigma \)-algebra, \( \mu : \mathcal{A} \to \mathbb{R}_{+} \) is an \( \sigma \)-continuous submeasure of finite variation and \((f_{n})_{n \in \mathbb{N}^{+}} \) is a sequence of uniformly bounded \( \tilde{\mu} \)-totally-measurable functions \( f_{n} : T \to \mathbb{R} \). Then \( g \) defined for every \( t \in T \) by \( g(t) = \inf_{n \in \mathbb{N}^{+}} f_{n}(t) \), is \( \tilde{\mu} \)-totally-measurable.

Proof. One can easily check that for every \( t, s \in T \), the following inequality holds:

\[
\left| g(t) - g(s) \right| \leq \sup_{n \in \mathbb{N}^{+}} \left| f_{n}(t) - f_{n}(s) \right|.
\]

Since for every \( n \in \mathbb{N}^{+} \), \( f_{n} \) is \( \tilde{\mu} \)-totally-measurable, then for every \( \varepsilon > 0 \), there is a partition \( P_{\varepsilon}^{n} = \{ A_{j}^{n} \}_{j=0}^{\infty} \subseteq \mathcal{P} \) so that \( \tilde{\mu}(A_{0}^{n}) < \frac{\varepsilon}{2n+1} \) and

\[
\sup_{t,s \in A_{j}^{n}} \left| f_{n}(t) - f_{n}(s) \right| < \frac{\varepsilon}{2n+1}, \text{ for every } j = 1, 2, \ldots, p_{n}.
\]

Let \( A_{0} = \bigcup_{n=1}^{\infty} A_{n}^{n} \in \mathcal{A} \). Because \( \mu \) is an \( \sigma \)-continuous submeasure of finite variation, then, by Remark 4-II, \( \tilde{\mu} \) is \( \sigma \)-additive on \( \mathcal{A} \), so,

\[
\tilde{\mu}(A_{0}) \leq \sum_{n=1}^{\infty} \tilde{\mu}(A_{n}^{n}) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2n+1} < \frac{\varepsilon}{2}.
\]

On the other hand,

\[
c_{A_{0}} = \sum_{n=1}^{\infty} c_{A_{n}^{n}} = \sum_{n=1}^{\infty} (A_{1}^{n} \cup A_{2}^{n} \cup \ldots \cup A_{p_{n}}^{n}) = (A_{1}^{1} \cup A_{2}^{1} \cup \ldots \cup A_{p_{1}}^{1}) \cap (A_{1}^{2} \cup A_{2}^{2} \cup \ldots \cup A_{p_{2}}^{2}) \cap \ldots = \bigcup_{(i_{n}) \in \prod_{n=1}^{\infty} I_{n}} (A_{i_{1}}^{1} \cap A_{i_{2}}^{2} \cap \ldots \cap A_{i_{n}}^{n}, \ldots),
\]

where \( I_{n} = \{1, 2, \ldots, p_{n}\} \), for every \( n \in \mathbb{N}^{+} \). Denote the last reunion by \( \bigcup_{n=1}^{\infty} B_{n} \). Now let \( C_{n} = \bigcup_{k=1}^{n} B_{k} \) and \( D_{n} = c_{A_{0}} \setminus C_{n} \), for every \( n \in \mathbb{N}^{+} \). We observe that \( B_{n} \cap B_{m} = \emptyset \) whenever \( n \neq m \), \( \bigcup_{n=1}^{\infty} C_{n} = \bigcup_{n=1}^{\infty} B_{n} = c_{A_{0}} \) and \( D_{n} \neq \emptyset \).

Since \( \tilde{\mu} \) is \( \sigma \)-continuous, there is \( n_{0}(\varepsilon) = n_{0} \in \mathbb{N}^{+} \) such that \( \tilde{\mu}(c_{A_{0}} \setminus \bigcup_{i=1}^{n_{0}} B_{i}) < \frac{\varepsilon}{2} \). Because \( \tilde{\mu}(A_{0}) < \frac{\varepsilon}{2} \), we get \( \tilde{\mu}(c_{A_{0}} \setminus \bigcup_{i=1}^{n_{0}} B_{i}) < \varepsilon \).

From (1) and (2), we have:

\[
\sup_{t,s \in B_{i}} \left| f_{n}(t) - f_{n}(s) \right| \leq \sup_{t,s \in B_{i}} \left| f_{n}(t) - f_{n}(s) \right| < \frac{\varepsilon}{2}, \quad \forall i \in \{1, \ldots, n_{0}\}.
\]

If we now consider the partition \( P_{\varepsilon} = \{ c_{\bigcup_{i=1}^{n_{0}} B_{i}}, B_{1}, \ldots, B_{n_{0}} \} \), we obtain that \( g \) is \( \tilde{\mu} \)-totally-measurable. \( \Box \)
Corollary 25 Under the assumptions of Theorem 24, the function \( h \) defined for every \( t \in T \) by \( h(t) = \sup_{n \in \mathbb{N}^*} f_n(t) \), is \( \tilde{\mu} \)-totally-measurable. Moreover, supposing there exists \( \lim_{n \to \infty} f_n(t) = f(t) \), for every \( t \in T \), then \( f \) is \( \tilde{\mu} \)-totally-measurable on \( T \).

Theorem 26 Suppose \( (T, \rho) \) is a compact metric space, \( B \) is the Borel \( \sigma \)-ring generated by the compact subsets of \( T \), \( f : T \to \mathbb{R} \) is continuous on \( T \) and \( \mu : B \to \mathcal{P}_f(X) \) is a finitely purely atomic multimeasure. Then \( f \) is \( \tilde{\mu} \)-totally-measurable on \( T \).

Proof. According to Remark 22, it is sufficient to establish the \( \tilde{\mu} \)-totally-measurability of \( f \) on an arbitrary, fixed atom \( A_0 \) of \( \mu \). Since \( \mu \) is a multimeasure, by [15], there is an unique \( a_0 \in A_0 \) so that \( \mu(A_0 \setminus \{a_0\}) = 0 \).

Let \( \varepsilon > 0 \). Since \( f \) is continuous in \( a_0 \), there is \( \delta_\varepsilon > 0 \) so that for every \( t \in A_0 \), with \( \rho(t, a_0) < \delta_\varepsilon \), we have \( |f(t) - f(a_0)| < \frac{\varepsilon}{2} \).

Let \( B_\varepsilon = \{ t \in A_0 \mid \rho(t, a_0) < \delta_\varepsilon \} \) and \( B_0(a_0, \delta_\varepsilon) \) be the open ball of center \( a_0 \) and radius \( \delta_\varepsilon \). It results \( B_\varepsilon \in \mathcal{B} \). Since \( A_0 \) is an atom, we have \( \mu(B_\varepsilon) = \{0\} \) or \( \mu(A_0 \setminus B_\varepsilon) = \{0\} \).

If \( \mu(B_\varepsilon) = \{0\} \), then \( a_0 \in B_\varepsilon \), we get \( \mu(A_0 \setminus \{a_0\}) = 0 \). But \( \mu(A_0 \setminus \{a_0\}) = 0 \), so \( \mu(A_0) = \{0\} \), a contradiction. So, we have \( \mu(A_0 \setminus B_\varepsilon) = \{0\} \).

Now, one can easily observe that the partition \( P_{A_0} = A_0 \setminus B_\varepsilon \), assures the \( \tilde{\mu} \)-totally-measurability of \( f \).

5 Semi-convexity, diffusion, atoms and purely atomicity for a Gould type set-valued integral

In this section, we establish results concerning semi-convexity, diffusion, atoms and purely atomicity for the Gould type set-valued integral introduced and studied in [30].

In what follows, without any special assumptions, we suppose \( A \) is an algebra of subsets of \( T \), \( X \) is a Banach space, \( \mu : A \to \mathcal{P}_f(X) \) is a set multifunction of finite variation, with \( \mu(\emptyset) = \{0\} \) and \( f : T \to \mathbb{R} \) is a bounded function. We now recall the following notions and results (see [12, 13, 28, 29]).

Remark 27 If \( \mu : A \to \mathcal{P}_f(X) \) is of finite variation, then \( \mu \) takes its values in \( \mathcal{P}_0f(X) \).

Definition 28 I) Let \( P = \{A_i\}_{i=1}^{\infty} \) and \( P' = \{B_j\}_{j=1}^{\infty} \) be two partitions of \( T \). \( P' \) is said to be finer than \( P \), denoted \( P \leq P' \) (or \( P' \geq P \)) if for every \( j = 1, \ldots, m \), there exists \( i_j = 1, \ldots, m \) so that \( B_j \subseteq A_{i_j} \).

II) The common refinement of two partitions \( P = \{A_i\}_{i=1}^{\infty} \) and \( P' = \{B_j\}_{j=1}^{\infty} \) is the partition \( P \land P' = \{A_i \cap B_j\}_{i=1}^{\infty} \).

Definition 29 ([30]) For every partition \( P = \{A_i\}_{i=1}^{\infty} \) of \( T \) and every \( t_i \in A_i \), let \( \sigma_{t_i}(P) \) (or, if there is no doubt, \( \sigma_f(P), \sigma_{\mu}(P), \sigma(P) \)) be:

\[
\sigma_f(P) = \sum_{i=1}^n f(t_i)\mu(A_i) = f(t_1)\mu(A_1) + \ldots + f(t_n)\mu(A_n).
\]

I) \( f \) is said to be \( \mu \)-integrable on \( (T, A, \mu) \) if the net \( \sigma_f(P) \) is convergent in \( \mathcal{P}_f(X) \), where \( P \) is ordered by the relation \( \leq \) given in Definition 4.2.

II) For an arbitrary \( B \subseteq A \), \( f \) is said to be \( \mu \)-integrable on \( B \) if the restriction \( f |_B \) is \( \mu \)-integrable on \( (B, A_B, \mu_B) \).

Remark 30 I) \( f \) is \( \mu \)-integrable on \( T \) if and only if there exists a set \( I \subseteq \mathcal{P}_f(X) \) such that for every \( \varepsilon \), there exists a partition \( P_{\varepsilon} \) of \( T \), so that for every other partition of \( T \), \( P = \{A_i\}_{i=1}^{\infty} \), with \( P \geq P_{\varepsilon} \) and every choice of points \( t_i \in A_i \), we have \( h(\sigma_f(P), I) < \varepsilon \).

II) If \( \mu \) is a measure (multimeasure, submeasure, monotone set multifunction, respectively), we obtain the corresponding definitions of [28, 12, 17, 29], respectively.

III) If \( \mu \) is a multimeasure and \( f = 1 \), then \( \int_T f \, d\mu = \mu(T) \).

IV) If \( \mu : A \to \mathcal{P}_{kc}(X) \), then \( \int_T f \, d\mu \in \mathcal{P}_{kc}(X) \).

V) Suppose \( m : A \to \mathbb{R}_+ \) is an arbitrary set function of finite variation with \( m(\emptyset) = 0 \) and consider the set multifunction \( \mu : A \to \mathcal{P}_f(\mathbb{R}) \) defined by \( \mu(A) = \{m(A)\} \) for every \( A \subseteq A \). Then, by II), \( f \) is \( m \)-integrable on \( T \) if and only if there is \( I \subseteq \mathbb{R} \) such that for every \( \varepsilon \), there exists a partition \( P_{\varepsilon} \) of \( T \), so that for every other partition of \( T \), \( P = \{A_i\}_{i=1}^{\infty} \), with \( P \geq P_{\varepsilon} \) and every choice of points \( t_i \in A_i \), we have \( |\sigma_f(P) - I| = |\sum_{i=1}^n f(t_i)m(A_i) - I| < \varepsilon \).

Here, \( I = \int_T f \, dm \).

Moreover, if \( m \) is finitely additive and \( f = 1 \), then \( \int_T f \, dm = m(T) \).

VI) Our integral, if it exists, is unique and has the following properties: homogeneity and additivity with
respect to the function \( f \) and the set multifunction \( \mu \), additivity with respect to the set, monotonicity with respect to the function \( f \), to the set multifunction \( \mu \), and to the set (see [28]–[30] for details). The assumption of monotonicity is not necessary in [29], as observed in [30]).

**VII** Let \( m : A \to [0, 1] \) be a submeasure of finite variation. One can easily check that the set function \( m_1 : A \to [0, 1] \) defined for every \( A \in \mathcal{A} \) by \( m_1(A) = \sin m(A) \) is also a submeasure of finite variation (since \( \overline{\mu}(A) \leq \mu(A) \), for every \( A \subseteq T \)).

Suppose \( f : T \to \mathbb{R} \) is bounded. Since, according to [17], \( \mu \)-integrability of \( f \) is equivalent to its \( \overline{m} \)-totally-measurability and because \( \frac{2}{\pi} t \leq \sin t \leq t \), for every \( t \in [0, \frac{\pi}{2}] \), then \( f \) is \( m \)-integrable if and only if \( f \) is \( m_1 \)-integrable.

**Theorem 31**  
**I** Let \( f : T \to \mathbb{R} \) be a \( \mu \)-integrable function. Then

\[
\left| \int_T f \, d\mu \right| \leq \sup_{t \in T} (f(t)) \cdot \overline{\mu}(T).
\]

**II** Let \( f : T \to \mathbb{R} \) and \( A, B \in \mathcal{A} \), with \( A \cap B = \emptyset \). If \( f \) is \( \mu \)-integrable on \( A \) and \( \mu \)-integrable on \( B \), then \( f \) is \( \mu \)-integrable on \( A \cup B \) and \( \int_{A \cup B} f \, d\mu = \int_A f \, d\mu + \int_B f \, d\mu \).

**III** Suppose \( \mu : A \to \mathcal{P}_{\mathcal{K}}(X) \). If \( f : T \to \mathbb{R} \) is \( \mu \)-integrable on \( T \), then \( f \) is \( \mu \)-integrable on every \( B \in \mathcal{A} \).

**IV** If \( f : T \to \mathbb{R} \) is \( \mu \)-integrable on every \( A \in \mathcal{A} \), then the set multifunction \( M : A \to \mathcal{P}(X) \), defined by

\[
(*) M(A) = \int_A f \, d\mu, \text{ for every } A \in \mathcal{A},
\]

is a monotone multimeasure, \( M \ll \mu \) and \( M \) is strongly absolutely continuous with respect to \( \mu \).

**V** If \( f, g : T \to \mathbb{R} \) are bounded functions so that \( f \) is \( \mu \)-integrable on \( T \) and \( f = g \) \( \mu \)-a.e., then \( g \) is \( \mu \)-integrable on \( T \) and \( \int_T f \, d\mu = \int_T g \, d\mu \).

**Remark 32** By Theorem 31-1 and Remark 10-III, we immediately get that if \( \mu : A \to \mathcal{P}_{\mathcal{K}}(X) \) is diffused, then the same is \( M \) defined in (**). Also, by Remark 30-I, if \( \inf_{t \in T} f(t) > 0 \), then the converse is also valid.

So, in this case, \( \mu \) is diffused if and only if the same is \( M \).

**Proposition 33** Let \( m_1, m_2 : A \to \mathbb{R}_+ \) be set functions of finite variation, so that \( m_1 \leq m_2 \) and \( m_1(\emptyset) = m_2(\emptyset) = 0 \), \( f : T \to \mathbb{R} \) and \( \mu : A \to \mathcal{P}_{\mathcal{K}}(\mathbb{R}) \) the set multifunction defined by \( \mu(A) = [m_1(A), m_2(A)] \), for every \( A \in \mathcal{A} \). Then \( f \) is \( \mu \)-integrable on \( T \) if and only if \( f \) is \( m_1 \)-integrable on \( T \) and \( m_2 \)-integrable on \( T \) and, in this case,

\[
\int_T f \, d\mu = \left[ \int_T f \, dm_1, \int_T f \, dm_2 \right].
\]

**Proof.** \( f \) is \( m_1 \)-integrable on \( T \) and \( m_2 \)-integrable on \( T \) if and only if for every \( \varepsilon > 0 \), there exists a partition \( P_\varepsilon \) of \( T \) so that for every other partitions of \( T \), \( P' = \{ A_i \}_{i=\overline{1,n}}, P'' = \{ B_j \}_{j=\overline{1,p}} \), so that \( P' \geq P_\varepsilon \), \( P'' \geq P_\varepsilon \) and every \( t_i \in A_i, i = \overline{1,n}, s_j \in B_j, j = \overline{1,p} \), we have

\[
\left| \sum_{i=1}^n f(t_i)m_k(A_i) - \sum_{j=1}^p f(s_j)m_k(B_j) \right| < \varepsilon, \quad k = 1, 2.
\]

Since

\[
h\left( \sum_{i=1}^n f(t_i)m_1(A_i), \sum_{j=1}^p f(s_j)m_1(B_j) \right) = h\left( \sum_{i=1}^n f(t_i)m_1(A_i), \sum_{i=1}^n f(t_i)m_2(A_i) \right),
\]

\[
\left\| \sum_{j=1}^p f(s_j)m_1(B_j) \cdot \sum_{j=1}^p f(s_j)m_2(B_j) \right\| = \max \left\{ \left| \sum_{i=1}^n f(t_i)m_1(A_i) - \sum_{j=1}^p f(s_j)m_1(B_j) \right|, \right.
\]

\[
\left. \left| \sum_{i=1}^n f(t_i)m_2(A_i) - \sum_{j=1}^p f(s_j)m_2(B_j) \right| \right\}
\]

it follows that for every \( \varepsilon > 0 \), there exists a partition \( P_\varepsilon \) of \( T \) so that for every other partitions of \( T \), \( P' = \{ A_i \}_{i=\overline{1,n}}, P'' = \{ B_j \}_{j=\overline{1,p}} \), so that \( P' \geq P_\varepsilon \), \( P'' \geq P_\varepsilon \) and every \( t_i \in A_i, i = \overline{1,n}, s_j \in B_j, j = \overline{1,p} \), we have

\[
h\left( \sum_{i=1}^n f(t_i)m(A_i), \sum_{j=1}^p f(s_j)m(B_j) \right) < \varepsilon,
\]

which means that \( f \) is \( \mu \)-integrable on \( T \).

Now, let us prove that \( \int_T f \, d\mu = \left[ \int_T f \, dm_1, \int_T f \, dm_2 \right] \).

Since \( f \) is \( \mu \)-integrable on \( T \), \( m_1 \)-integrable on \( T \) and \( m_2 \)-integrable on \( T \), it results that for every \( \varepsilon > 0 \), there exists a partition \( \{ C_k \}_{k=\overline{1,T}} \) of \( T \) so that for every...
\[ s_k \in C_k, k = 1, \ldots, n, \text{ we have} \]

\[ h(\int_T f \, d\mu, \sum_{k=1}^l f(s_k)\mu(C_k)) < \frac{\varepsilon}{2} \quad \text{and} \]

\[ \left| \int_T f \, d\mu_m - \sum_{k=1}^l f(s_k)m_i(C_k) \right| < \frac{\varepsilon}{2}, \quad i = 1, 2. \]

Then

\[ h(\int_T f \, d\mu, [\int_T f \, d\mu_1, \int_T f \, d\mu_2]) \leq \]

\[ \leq h(\int_T f \, d\mu, \sum_{k=1}^l f(s_k)\mu(C_k)) \]

\[ + h(\sum_{k=1}^l f(s_k)\mu(C_k), [\int_T f \, d\mu_1, \int_T f \, d\mu_2]) = \]

\[ = h(\int_T f \, d\mu, \sum_{k=1}^l f(s_k)\mu(C_k)) \]

\[ + \max \left\{ \left| \int_T f \, d\mu_1 - \sum_{k=1}^l f(s_k)m_1(C_k) \right|, \right. \]

\[ \left. \left| \int_T f \, d\mu_2 - \sum_{k=1}^l f(s_k)m_2(C_k) \right| \right\} < \varepsilon, \]

for every \( \varepsilon > 0 \) and this implies \( \int_T f \, d\mu = [\int_T f \, d\mu_1, \int_T f \, d\mu_2] \).

Taking \( m_1 = 0 \) in Proposition 33, we obtain the following result.

**Corollary 34** Let \( m : A \to \mathbb{R}_+ \) be a set function of finite variation with \( m(\emptyset) = 0 \), \( \mu : C \to \mathcal{P}_{kc}(\mathbb{R}) \) the set multifunction defined by \( \mu(A) = [0, m(A)] \), for every \( A \in C \) and \( f : T \to \mathbb{R} \). Then \( f \) is \( \mu \)-integrable on \( T \) if and only if \( f \) is \( m \)-integrable on \( T \) and, in this case,

\[ \int_T f \, d\mu = [0, \int_T f \, d\mu]. \]

**Theorem 35** Let \( \mu : A \to \mathcal{P}_{kc}(X) \) be a semi-convex multifunction and \( f : T \to \mathbb{R} \) a \( \tilde{\mu} \)-totally-measurable bounded function on \( T \). Then \( M \) defined in (**) is also semi-convex.

**Proof.** The following statements, even they are established for \( T \), remain valid for any arbitrary set \( A \in \mathcal{A} \). Also, according to [28], \( f \) is \( \mu \)-integrable on \( T \) and on every \( A \in \mathcal{A} \). Consider arbitrary \( \varepsilon > 0 \) and let

\[ M = \max_{T \in \mathcal{T}} |\tilde{\mu}(T)|, \quad \text{sup}_{T \in \mathcal{T}} |f(t)|. \]

By the \( \mu \)-integrability of \( f \) on \( T \), there is a partition \( \{A_i\}_{i=1}^{n, q} \) of \( T \) such that for every \( s_i \in A_i \),

\[ i = 1, n, \text{ we have} \]

\[ h(\int_T f \, d\mu, \sum_{i=1}^n f(s_i)\mu(A_i)) < \frac{2\varepsilon}{3}, \]

so \( h(\frac{1}{2} \int_T f \, d\mu, \sum_{i=1}^n f(s_i)\frac{1}{2}\mu(A_i)) < \frac{\varepsilon}{3} \).

Because \( \mu \) is semi-convex, for every \( i = 1, n \), there is \( B_i \subset A_i \) so that \( B_i \in A \) and \( \mu(B_i) = \frac{1}{2}\mu(A_i) \), which implies

\[ h(\frac{1}{2} \int_T f \, d\mu, \sum_{i=1}^n f(s_i)\mu(B_i)) < \frac{\varepsilon}{3}. \]

Since \( f \) is \( \mu \)-integrable on \( B = \bigcup_{i=1}^n B_i \), there exists a partition \( \tilde{\mu} = \{D_k\}_{k=1}^\infty \in \mathcal{P}_B \) so that for every partition \( P \in \mathcal{P}_B \), with \( P \geq \tilde{\mu} \), we have

\[ h(f \, d\mu, \sigma(P)) < \frac{\varepsilon}{3}. \]

On the other hand, because \( f \) is \( \tilde{\mu} \)-totally-measurable on \( B \), there is a partition \( \tilde{\mu} = \{E_i\}_{i=0}^\infty \in \mathcal{P}_B \) such that \( |\tilde{\mu}(E_0)| < \frac{\varepsilon}{2|\mathcal{M}|} \) and

\[ \sup_{t \in \mathcal{T}, s \in E_l} |f(t) - f(s)| < \frac{\varepsilon}{6|\mathcal{M}|}, \text{ for every } l = 1, m. \]

Consider \( \{D_k \cap E_i\}_{k=1}^\infty \), \( \{l, m\} \in \mathcal{P}_B \) and denote it by \( \{C_j\}_{j=1}^{12} \). For instance, \( C_1 = D_1 \cap E_0, C_2 = D_2 \cap E_0, \ldots, C_{12} = D_1 \cap E_0, C_{13} = D_1 \cap E_1 \) etc. We observe that

\[ \tilde{\mu}(\bigcup_{j=1}^{12} C_j) = \tilde{\mu}(E_0) < \frac{\varepsilon}{12|\mathcal{M}|} \quad \text{and} \]

\[ \sup_{t \in \mathcal{T}, s \in C_j} |f(t) - f(s)| < \frac{\varepsilon}{6|\mathcal{M}|}, \text{ for every } j = s + 1, q. \]

Let \( \mu_B = \{B_i \cap C_j\}_{i=1}^{1, n}, j=1, m \in \mathcal{P}_B \). Since \( \mu_B \geq \tilde{\mu} \), then \( h(f \, d\mu, \sigma(\mu_B)) < \frac{\varepsilon}{3}. \)

Now, we have:

\[ h(\frac{1}{2} \int_T f \, d\mu, \int_B f \, d\mu) \leq h(\frac{1}{2} \int_T f \, d\mu, \sum_{i=1}^n f(s_i)\mu(B_i)) \]

\[ + h(\int_B f \, d\mu, \sigma(\mu_B)) \]

\[ + h(\sigma(\mu_B), \sum_{i=1}^n f(s_i)\mu(B_i)) < \frac{2\varepsilon}{3}. \]

It only remains to prove that for every \( \theta_{ij} \in B_i \cap C_j \), \( i = 1, n, j = 1, q. \)

\[ h(\sigma(\mu_B), \sum_{i=1}^n f(s_i)\mu(B_i)) \]

\[ = h(\sum_{i=1}^n f(\theta_{ij})\mu(B_i \cap C_j), \sum_{i=1}^n f(s_i)\mu(B_i)) < \frac{\varepsilon}{3}. \]
Indeed, we have:

\[ h(\sum_{i=1}^{n} \sum_{j=1}^{q} f(\theta_{ij}) \mu(B_i \cap C_j), \sum_{i=1}^{n} f(s_i) \mu(B_i)) = \]

\[ = h(\sum_{i=1}^{n} \sum_{j=1}^{q} f(\theta_{ij}) \mu(B_i \cap C_j), \sum_{i=1}^{n} f(s_i) \mu(B_i)) \leq \]

\[ \leq \sum_{i=1}^{n} \sum_{j=1}^{q} |f(s_i) - f(\theta_{ij})| \cdot |\mu(B_i \cap C_j)| = \]

\[ = \sum_{i=1}^{n} \sum_{j=1}^{q} |f(s_i) - f(\theta_{ij})| \cdot |\mu(B_i \cap C_j)| + \]

\[ + \sum_{i=1}^{n} \sum_{j=s+1}^{q} |f(s_i) - f(\theta_{ij})| \cdot |\mu(B_i \cap C_j)| \leq \]

\[ \leq 2M \sum_{j=1}^{s} \overline{P}(C_j) + \sum_{j=s+1}^{q} |f(s_i) - f(\theta_{ij})| \cdot \overline{P}(C_j) < \]

\[ < 2M \overline{P}(\bigcup_{j=1}^{s} C_j) + \frac{\varepsilon}{6M} \overline{P}(\bigcup_{j=s+1}^{q} C_j) \]

\[ < 2M \frac{\varepsilon}{12M} + \frac{\varepsilon}{6M} = \frac{\varepsilon}{3}. \]

Consequently, \( h(\frac{1}{2} \int_{T} f \, d\mu, \int_{B} f \, d\mu) < \varepsilon, \) for every \( \varepsilon > 0, \) so \( \frac{1}{2} \int_{T} f \, d\mu = \int_{B} f \, d\mu. \) Therefore, \( M \) is semi-convex. \( \square \)

**Theorem 36** Suppose \( \mu : A \to \mathcal{P}_f(X) \) is monotone, null-additive and finitely purely atomic. If \( f \) is \( \mu \) -totally-measurable on \( T, \) then \( f \) is \( \mu \) -integrable on \( T. \)

**Proof.** According to Theorem 31-II, it will be sufficient to prove that \( f \) is \( \mu \) -integrable on every atom \( A \) of \( \mu. \) First, we observe that, if \( A \) is an atom of \( \mu \) and if \( \{A_i\}_{i=1}^{n} \in \mathcal{P}_A, \) then, there exists only one set, for instance, without any loss of generality, \( A_1, \) so that \( \mu(A_1) \geq \{0\} \) and \( \mu(A_2) = \ldots = \mu(A_n) = \{0\}. \)

Let \( A \in \mathcal{A} \) be an atom of \( \mu. \)

Since \( f \) is \( \mu \) -totally-measurable on \( A, \) then for every \( \varepsilon > 0 \) there exists a partition \( P_{\varepsilon} = \{A_i\}_{i=1}^{n} \) of \( A \) such that:

\[ (*) \]

\[ i) \ \mu(A_0) < \frac{\varepsilon}{2M} \text{ (where } M = \sup_{t \in T} |f(t)| \text{ and } \)

\[ ii) \sup_{t \in A_i} |f(t) - f(s)| < \frac{\varepsilon}{M \overline{P}(T)}, \text{ for every } i = 1, n. \]

Let \( \{B_j\}_{j=1}^{k}, \{C_p\}_{p=1}^{s} \in \mathcal{P}_A \) be two arbitrary partitions which are finer than \( P_{\varepsilon} \) and consider \( s_j \in B_j, j = 1, k, \theta_p \in C_p, p = 1, s. \)

We prove that

\[ h(\sum_{j=1}^{k} f(s_j) \mu(B_j), \sum_{p=1}^{s} f(\theta_p) \mu(C_p)) < \varepsilon. \]

We have two cases:

I. \( \mu(A_0) \geq \{0\}. \) Then \( \mu(A_1) = \ldots = \mu(A_n) = \{0\}. \)

Suppose, without any loss of generality that \( \mu(B_1) \geq \{0\}, \mu(C_1) \geq \{0\} \) and \( \mu(B_2) = \ldots = \mu(B_k) = \{0\}, \mu(C_2) = \ldots = \mu(C_s) = \{0\}. \) Then \( B_1 \subset A_0 \) and \( C_1 \subset A_0. \) Consequently,

\[ h(\sum_{j=1}^{k} f(s_j) \mu(B_j), \sum_{p=1}^{s} f(\theta_p) \mu(C_p)) \]

\[ = h(f(s_1) \mu(B_1), f(\theta_1) \mu(C_1)) \leq \]

\[ \leq |f(s_1)| |\mu(B_1)| + |f(\theta_1)| |\mu(C_1)| \leq \]

\[ \leq 2M \overline{P}(A_0) < \varepsilon. \]

II. \( \mu(A_0) = \{0\}. \) Then, without any loss of generality, \( \mu(A_1) \geq \{0\} \) and \( \mu(A_i) = \{0\}, \) for every \( i = 2, n. \) Suppose that \( \mu(B_1) \geq \{0\}, \mu(C_1) \geq \{0\} \) and \( \mu(B_2) = \ldots = \mu(B_k) = \{0\}, \mu(C_2) = \ldots = \mu(C_s) = \{0\}. \) Then \( B_1 \subset A_1 \) and \( C_1 \subset A_1, \) and, therefore,

\[ h(\sum_{j=1}^{k} f(s_j) \mu(B_j), \sum_{p=1}^{s} f(\theta_p) \mu(C_p)) \]

\[ = h(f(s_1) \mu(B_1), f(\theta_1) \mu(C_1)). \]

Since \( A \) is an atom of \( \mu \) and \( \mu(B_1) \geq \{0\}, \) then \( \mu(A \setminus B_1) = \{0\}, \) so \( \mu(C_1 \setminus B_1) = \{0\}. \) By the null-additivity of \( \mu, \) we get \( \mu(C_1) = \mu(B_1). \) Then

\[ h(\sum_{j=1}^{k} f(s_j) \mu(B_j), \sum_{p=1}^{s} f(\theta_p) \mu(C_p)) \]

\[ = h(f(s_1) \mu(B_1), f(\theta_1) \mu(B_1)). \]

By Proposition 1, we have

\[ h(\sum_{j=1}^{k} f(s_j) \mu(B_j), \sum_{p=1}^{s} f(\theta_p) \mu(C_p)) \]

\[ \leq |\mu(B_1)| |f(s_1) - f(\theta_1)| \leq \overline{P}(T) \frac{\varepsilon}{\overline{P}(T)} = \varepsilon. \]

Therefore, the net \( (\sigma(P))_{P \in \mathcal{P}_A} \) is a Cauchy one in the complete metric space \( (\mathcal{P}_f(X), h), \) hence \( f \) is \( \mu \) -integrable on \( A. \) \( \square \)

In [8, 9], submeasures of the following type are studied. Here, we investigate the relationship between their Gould integrals.
Theorem 37 Let \((m_n)_{n \in \mathbb{N}}\) be an uniformly bounded sequence of submeasures of finite variation, \(m_n : \mathcal{A} \to \mathbb{R}_+\), \(\forall n \in \mathbb{N}\) and \(m : \mathcal{A} \to \mathbb{R}_+\) defined by \(m(A) = \sup_n m_n(A)\), for every \(A \in \mathcal{A}\).

Suppose \(A_0 \in \mathcal{A}\) is an atom of \(m\) and \(f : T \to \mathbb{R}\) is \(\bar{m}\)-totally-measurable on \(T\). Then \(\int_{A_0} f dm = \sup_n \int_{A_0} f dm_n\).

**Proof.** By Example 3-II), \(m\) is a submeasure too. Since \(m_n(A) \leq m(A)\), for every \(A \in \mathcal{A}\), then for every \(n \in \mathbb{N}\), \(f\) is \(m_n\)-totally-measurable on \(T\). According to [17], \(f\) is \(m\)-integrable and \(m_n\)-integrable on \(T\) and on every \(A \in \mathcal{A}\). By [17], \(\int_{A_0} f dm_n \leq \int_{A_0} f dm\), for every \(n \in \mathbb{N}\).

Since \(m(A_0) = \sup_n m_n(A_0)\), we get that for every \(\varepsilon > 0\), there is \(n_0(\varepsilon, A_0) = n_0\) so that \(m(A_0) < m_{n_0}(A_0) + \frac{\varepsilon}{m(A_0)}\), where \(M = \sup\{f(t)\}\).

Because \(f\) is \(m\)-integrable and \(m_{n_0}\)-integrable on \(A_0\), we have that for every \(\varepsilon > 0\), there is a common partition \(\{B_j\}_{j=1}^{k} \in \mathcal{P}_{A_0}\) so that for every \(t_j \in B_j\),

\[
\left| \int_{A_0} f dm - \sum_{j=1}^{k} f(t_j) m(B_j) \right| < \frac{\varepsilon}{k}.
\]

Since \(\{B_j\}_{j=1}^{k} \in \mathcal{P}_{A_0}\), we observe that there can exist only one set, for instance, \(B_1\), so that \(m(B_1) > 0\) and \(m(B_j) = 0\), for every \(j = 2, \ldots, k\). Then \(m_{n_0}(B_j) = 0\), for every \(j = 2, \ldots, k\).

Consequently, because \(m(B_1) = m(A_0)\) and \(m_{n_0}(B_1) = m_{n_0}(A_0)\), we have

\[
\int_{A_0} f dm \leq \left| \int_{A_0} f dm - \sum_{j=1}^{k} f(t_j) m(B_j) \right| + \left| \int_{A_0} f dm_{n_0} - \sum_{j=1}^{k} f(t_j) m_{n_0}(B_j) \right| + |f(t_1)| \cdot |m(B_1) - m_{n_0}(B_1)| + \int_{A_0} f dm_{n_0} < \varepsilon + \frac{\varepsilon}{M} + \int_{A_0} f dm_{n_0} = \varepsilon + \int_{A_0} f dm_{n_0},
\]

so \(\int_{A_0} f dm = \sup_n \int_{A_0} f dm_n\), as claimed.

\[\Box\]

6 Classical results for the Gould type set-valued integral

In this section we obtain some classical theorems (such as Hölder inequality, Minkowski inequality, mean convergence theorem, Lebesgue theorem, Fatou lemma) for the Gould type set-valued integral introduced in [30].

**Theorem 38 (Hölder Inequality)** Let \(m : \mathcal{A} \to \mathbb{R}_+\) be a submeasure of finite variation and \(f, g : T \to \mathbb{R}\) \(m\)-integrable bounded functions on \(T\). Then

\[
\int_T |fg| dm \leq \left( \int_T |f|^p dm \right)^{\frac{1}{p}} \cdot \left( \int_T |g|^q dm \right)^{\frac{1}{q}},
\]

for every \(p, q \in (1, \infty)\), with \(\frac{1}{p} + \frac{1}{q} = 1\).

**Proof.** Since (see [17]) for submeasures, \(m\)-integrability is equivalent to \(\bar{m}\)-totally-measurability, then by Theorem 19-I and Theorem 2.17 [17], \(|f|, |g|, |fg|, |f|^p, |g|^q,\) are also \(m\)-integrable, so, for every \(\varepsilon > 0\), there is a common partition \(P_\varepsilon = \{A_i\}_{i=1}^{n}\) such that for every \(t_i \in A_i, i = 1, \ldots, n\), we have:

\[
\left| \int_T |fg| dm - \sum_{i=1}^{n} f(t_i) g(t_i) m(A_i) \right| < \frac{\varepsilon}{3},
\]

\[
\left| \int_T |f| dm - \sum_{i=1}^{n} f(t_i) m(A_i) \right| < \frac{\varepsilon}{3}
\]

\[
\left| \int_T |g| dm - \sum_{i=1}^{n} g(t_i) m(A_i) \right| < \frac{\varepsilon}{3}.
\]

Since

\[
\sum_{i=1}^{n} |f(t_i) g(t_i)| m(A_i)
\]

\[
= \sum_{i=1}^{n} \left[ |f(t_i)| m(A_i) \right]^{\frac{1}{p}} \cdot \left[ |g(t_i)| m(A_i) \right]^{\frac{1}{q}}
\]

\[
\leq \left( \sum_{i=1}^{n} |f(t_i)|^p m(A_i) \right)^{\frac{1}{p}} \cdot \left( \sum_{i=1}^{n} |g(t_i)|^q m(A_i) \right)^{\frac{1}{q}},
\]

we immediately have the conclusion. \(\Box\)

Using the above theorem, we obtain the Minkowski inequality, by a classical proof.

**Theorem 39 (Minkowski inequality)** Let \(m : \mathcal{A} \to \mathbb{R}_+\) be a submeasure of finite variation and \(f, g : T \to \mathbb{R}\) \(m\)-integrable bounded functions on \(T\). Then

\[
\left( \int_T |f + g|^p dm \right)^{\frac{1}{p}} \leq \left( \int_T |f|^p dm \right)^{\frac{1}{p}} + \left( \int_T |g|^p dm \right)^{\frac{1}{p}},
\]

for every \(p \in [1, +\infty)\).

If \(m : \mathcal{A} \to \mathbb{R}_+\) is a submeasure of finite variation, we consider the space \(\mathbb{L}^p = \{ f : T \to \mathbb{R}; f\) is bounded on \(T\) and \(|f|^p\) is \(m\)-integrable on \(T)\).
Remark 40 From Theorem 19-II, it results that if $f, g \in L^p$, then $f + g \in L^p$. So, $L^p$ is a linear space.

Corollary 41 Let $m: A \rightarrow \mathbb{R}_+$ be a submeasure of finite variation and $p \in [1, +\infty)$. Then the function \( \| \cdot \|: L^p \rightarrow \mathbb{R}_+ \), defined for every $f \in L^p$ by \( \| f \| = (\int_T |f|^p \, dm)^{\frac{1}{p}} \), is a semi-norm.

Definition 42 Let $\mu: A \rightarrow \mathcal{P}(X)$ be a set multifunction with $\mu(\emptyset) = \{0\}$. If for every $n \in \mathbb{N}$, $f_n: T \rightarrow \mathbb{R}$ is $\mu$-integrable on $T$, then the sequence $(f_n)$ is said to be mean convergent to $f$ on $T$ if \( \lim_{n \to \infty} \int_T (f_n - f) \, d\mu = \{0\} \) (with respect to $h$).

Theorem 43 (Mean Convergence Theorem) Let $\mu: A \rightarrow \mathcal{P}(X)$ be a set multifunction of finite variation, with $\mu(\emptyset) = \{0\}$ and $f_n: T \rightarrow \mathbb{R}$, for every $n \in \mathbb{N}$. Suppose $(f_n)$ is an uniformly bounded sequence of $\mu$-integrable functions such that $(f_n)$ is convergent in submeasure to a bounded function $f: T \rightarrow \mathbb{R}$. Then $f$ is $\mu$-integrable on $T$ and on every $A \in \mathcal{A}$,

\[
\lim_{n \to \infty} \int_A (f_n - f) \, d\mu = \{0\}
\]

(with respect to $h$)

Proof. Let $M' = \pi(T)$, $M_1 = \sup_{t \in T} |f(t)|$, $M_2 = \sup_{t \in T, n \in \mathbb{N}} |f_n(t)|$ and $M = \max\{M_1, M_2\}$.

Since $f_n \overset{\mu}{\rightarrow} f$, it results that for every $\varepsilon > 0$, there exists $n_0(\varepsilon) \in \mathbb{N}$ so that $\mu(B_n\left(\frac{\varepsilon}{4M}\right)) < \frac{\varepsilon}{4M}$, for every $n \geq n_0$.

Particularly, $\mu(B_{n_0}\left(\frac{\varepsilon}{6M}\right)) < \frac{\varepsilon}{12M}$. By the definition of $\mu$, there is $C_{n_0} \in \mathcal{A}$ so that $B_{n_0}\left(\frac{\varepsilon}{6M}\right) \subseteq C_{n_0}$ and $\mu(C_{n_0}) = \pi(C_{n_0}) < \frac{\varepsilon}{12M}$.

First, we prove that $f$ is $\mu$-integrable on $C_{n_0}$. Indeed, for every $\varepsilon > 0$, there is a partition $P_\varepsilon = \{C_{n_0}\} \in \mathcal{P}C_{n_0}$ so that, for every other partition $P = \{D_j\}_{j=1}^{q} \in \mathcal{P}C_{n_0}$, with $P \geq P_\varepsilon$ and every $t_i \in D_i, i = 1, q$, and $c \in C_{n_0}$, we have:

\[
h\left(\sum_{i=1}^{q} f(t_i) \mu(D_i), f(c) \mu(C_{n_0})\right) \leq \sum_{i=1}^{q} |f(t)| \cdot |\mu(D_i)| + \frac{\varepsilon}{3M} \cdot |\mu(D_i)| + \frac{\varepsilon}{4M} \cdot |\mu(C_{n_0})| < \frac{\varepsilon}{2}.
\]

Consider another partition $P' = \{E_s\}_{s=\Gamma}^{\Pi} \in \mathcal{P}C_{n_0}$, with $P' \geq P_\varepsilon$ and $r_s \in E_s, s = 1, q$, arbitrarily.

In a similar way we get

\[
h\left(\sum_{s=1}^{q} f(r_s) \mu(E_s), f(c) \mu(C_{n_0})\right) < \frac{\varepsilon}{2}, \quad \text{whence,}
\]

\[
h\left(\sum_{i=1}^{p} f(t_i) \mu(D_i), \sum_{s=1}^{q} f(r_s) \mu(E_s)\right) < \varepsilon. \quad \text{Then $f$ is $\mu$-integrable on $C_{n_0}$.}
\]

Consequently, according to Theorem 31-II, in order to prove that $f$ is $\mu$-integrable on $T$, it is sufficient to establish the $\mu$-integrability of $f$ on $T \setminus C_{n_0}$.

Since for every $n \in \mathbb{N}$ $f_n$ is $\mu$-integrable on $T$, then $f_{n_0}$ is $\mu$-integrable on $T \setminus C_{n_0}$. Consequently, there is a partition $P_{n_0} = \{A_i\}_{i=\Gamma}^{\Pi} \in \mathcal{P}C_{n_0}$ so that, for every other partition $P \in \mathcal{P}T \setminus C_{n_0}$, with $P \geq P_{n_0}$, we have $h(\sigma(P), \sigma(P_{n_0})) < \frac{\varepsilon}{2}$.

Let $P = \{D_j\}_{j=1}^{\Gamma} \in \mathcal{P}T \setminus C_{n_0}$, with $P \geq P_{n_0}$ be arbitrarily, but fixed. For every $t_j \in D_j, j = 1, l$ and every $c_i \in A_i, i = 1, m_{n_0}$, we have:

\[
h\left(\sum_{j=1}^{l} f(t_j) \mu(D_j), \sum_{i=1}^{m_{n_0}} f(c_i) \mu(A_i)\right) < \varepsilon.
\]

A similar inequality for every other partition $P' \in \mathcal{P}T \setminus C_{n_0}$, with $P' \geq P_{n_0}$, may analogously be obtained. Then, by the triangular inequality, $f$ is $\mu$-integrable on $T \setminus C_{n_0}$ and, according to Theorem 45-II, $f$ is $\mu$-integrable on $T$.

Now, we prove that $\lim_{n \to \infty} \int_T (f_n - f) \, d\mu = \{0\}$ with respect to $h$. According to Theorem 31-III, there exist $\int_A f \, d\mu$ and $\int_A f_n \, d\mu$, for every $n \in \mathbb{N}$ and every $A \in \mathcal{A}$.

We shall use the same $B_n\left(\frac{\varepsilon}{6M}\right)$, with $n \geq n_0$, as before. By the definition of $\mu$, we get that for every $n \geq n_0$, there exists $C_n \in \mathcal{A}$ so that $B_n\left(\frac{\varepsilon}{6M}\right) \subseteq C_n$ and $\mu(C_n) = \pi(C_n) < \frac{\varepsilon}{12M}$.
Then, for every \( n \geq n_0 \), we have:
\[
\left| \int_{A} (f_n - f) \, d\mu \right| = \left| \int_{A \cap C_n} (f_n - f) \, d\mu \right| + \int_{A\setminus C_n} (f_n - f) \, d\mu \leq \\
\leq \sup_{t \in A \cap C_n} |f_n(t) - f(t)| \cdot \overline{\mu}(A \setminus C_n) + \sup_{t \in A \cap C_n} |f_n(t) - f(t)| \cdot \overline{\mu}(A \cap C_n) < \\
< \frac{\varepsilon}{6M'} \cdot M' + 2M \cdot \overline{\mu}(C_n) < \frac{\varepsilon}{2} + 2M \cdot \frac{\varepsilon}{4M} = \varepsilon,
\]
so
\[
\lim_{n \to \infty} \int_{A} (f_n - f) \, d\mu = \{0\} \text{ (with respect to } h)\), for every \( A \in \mathcal{A} \).
\]

**Theorem 44 (Lebesgue type Theorem)** Let \( \mu : \mathcal{A} \to \mathcal{P}_{\mathcal{F}}(X) \) be a set multifunction of finite variation, with \( \mu(\emptyset) = \{0\} \) and \( f_n : T \to \mathbb{R} \), for every \( n \in \mathbb{N} \). Suppose \((f_n)_n\) is an uniformly bounded sequence of \( \mu\)-integrable functions such that \((f_n)_n\) is convergent in submeasure to a bounded function \( f : T \to \mathbb{R} \). Then, \( f \) is \( \mu \)-integrable on every \( A \in \mathcal{A} \) and
\[
\lim_{n \to \infty} \int_{A} f_n \, d\mu = \int_{A} f \, d\mu \text{ (with respect to } h)\).

**Proof.** By the proof of Theorem 43, it results that \( f \) is \( \mu \)-integrable on every \( A \in \mathcal{A} \). Using the same sets as before, we have for every \( n \geq n_0 \) and every \( A \in \mathcal{A} \):
\[
\begin{align*}
\int_{A} f_n \, d\mu &= \int_{A \cap C_n} f_n \, d\mu + \int_{A \cap C_n} f_n \, d\mu + \int_{A \setminus C_n} f_n \, d\mu \\
&= h(\int_{A \cap C_n} f_n \, d\mu, \int_{A \setminus C_n} f_n \, d\mu) \leq h(\int_{A \cap C_n} f_n \, d\mu, \int_{A \setminus C_n} f_n \, d\mu) \\
&\leq \sup_{t \in A \cap C_n} |f_n(t) - f(t)| \cdot \overline{\mu}(A \setminus C_n) + \sup_{t \in A \cap C_n} |f_n(t) - f(t)| \cdot \overline{\mu}(A \cap C_n) < \\
&< \frac{\varepsilon}{6M'} \cdot M' + 2M \cdot \overline{\mu}(C_n) < \frac{\varepsilon}{2} + 2M \cdot \frac{\varepsilon}{4M} = \varepsilon,
\end{align*}
\]
and the conclusion follows.

**Theorem 45 (Fatou Lemma)** Suppose \( \mathcal{A} \) is a \( \sigma \)-algebra, \( \mu : \mathcal{A} \to \mathbb{R}_+ \) is a submeasure of finite variation so that \( \tilde{\mu} \) is \( o \)-continuous and \((f_n)_n \in \mathbb{N} \) is a sequence of uniformly bounded, \( \tilde{\mu} \)-totally-measurable functions \( f_n : T \to \mathbb{R} \). Then
\[
\int_{T} \liminf \limits_{n} f_n \, d\mu \leq \int_{T} \limsup \limits_{n} \int_{T} f_n \, d\mu.
\]

**Proof.** For every \( n \in \mathbb{N} \), consider \( g_n \) defined for every \( t \in T \) by \( g_n(t) = \inf_{k \geq n} f_k(t) \). Let also be \( f : T \to \mathbb{R} \), \( f(t) = \lim \limits_{n \to \infty} g_n(t) \), for every \( t \in T \). We observe that \( g_n \overset{ae}{\to} f \) and \( g_n \leq f_n \), for every \( n \in \mathbb{N} \).

According to Theorem 24, \((g_n)_n\) is also a sequence of uniformly bounded, \( \tilde{\mu} \)-totally-measurable functions, so, by Corollary 25, \( f \) is \( \tilde{\mu} \)-totally-measurable on \( T \).

By [17], \( f_n \) and \( f \) are \( \mu \)-integrable on \( T \), for every \( n \in \mathbb{N} \).

Since \( g_n \overset{ae}{\to} f \) and \( \tilde{\mu} \) is an \( o \)-continuous submeasure on \( \mathcal{P}(T) \), then, according to Li [23], \( g_n \overset{\tilde{\mu}}{\to} f \), so, by [13],
\[
\int_{T} \liminf \limits_{n} f_n \, d\mu = \int_{T} f \, d\mu = \lim_{n \to \infty} \int_{T} g_n \, d\mu.
\]

Consequently,
\[
\int_{T} \liminf \limits_{n} f_n \, d\mu = \liminf \limits_{n} \int_{T} f_n \, d\mu \\
\leq \liminf \limits_{n} \int_{T} f_n \, d\mu.
\]

This completes the proof.

**References:**


