by applying the Noether theorem. All first integrals of vector fields Z, T are quite simple: the zeroth order functions x, y^2+z^2 and their prolongations

$$\mathscr{D}^r x,\ \mathscr{D}^r(y^2+z^2)\quad (r=0,1,\ldots;$$
$$\mathscr{D}=D=\frac{\partial}{\partial t}+\dot{x}\frac{\partial}{\partial x}+\dot{y}\frac{\partial}{\partial y}+\dot{z}\frac{\partial}{\partial z}+\cdots)$$

and moreover all functions

$$\mathscr{D}^r W\quad \left(W=\arctan\frac{z}{y};\ r=1,2,\ldots\right)$$

as follows from $ZW=1$, $TW=0$ and the commutativity $[\mathscr{D},Z]=[\mathscr{D},T]=0$. Recall that

$$\mathscr{D}^r\breve{\varphi}(Z),\ \mathscr{D}^r\breve{\varphi}(T)\quad (r=0,1,\ldots)$$

are first integrals, too. If they may be included into coordinates on the orbit space, we have the normal case. We will not state the (rather clumsy) normality requirement (20) here. (Roughly speaking, it is satisfied on an open dense set for all nonconstant functions F.)

It follows that Theorem 20 with $K=2$ may be applied. We may choose

$$W^{m-K+1}:=\arctan\frac{z}{y},\ W^{m-K+2}:=t$$
$$(m=3, K=2)$$

for the functions (39) and then the form

$$\tilde{\varphi}=(F-c(1)\mathscr{D}\arctan\frac{z}{y}-c(2)\mathscr{D}t)dt$$
$$=\left(F-c(1)\frac{y\dot{z}-z\dot{y}}{y^2+z^2}-c(2)\right)dt$$

determines the Routh integral (40). Recall that it is considered on the orbit space, i.e., under the restriction (41).

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