

Dynamic Analysis of a System with Warm Standby and Common-Cause Failure

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Abstract: In this paper we analyze the dynamic behavior of a two unit parallel system with warm standby and common-cause failure. By the semigroup theory of linear operators on the Banach space, we give the wellposedness of the system and then prove the existence of the nonnegative dynamic solution and the steady solution of system. By spectral analysis of the system operator, we show that all the spectrum points of system operator besides 0 are in the left half-plane, hence we obtain the asymptotic stability of the system. Further we prove that 0 is a dominant eigenvalue of the system. Especially we discuss the essential spectral bound of the system operator and the radius of the essential spectrum of the semigroup associated with the system. Those results show that the dynamic solution of the system converges exponentially to the steady solution. Finally, we analyze the some reliability indic of the system.

Key-Words: a two unit parallel system; steady solution; Asymptotic stability; Exponential stability; Reliability

1 Introduction

One of the methods of increasing reliability of an item is to introduce redundancy. Several types of redundant configurations are used to increase reliability of engineering systems. In the usual analysis of such redundant configurations, the occurrence of common-cause failures is not considered. This may or may not represent the real life conditions under which such systems have to operate. In [1], by supplementary variables, the author had established the mathematic model of two units parallel system with warm standby and common-cause failure (see the system described in section 2), and by using Laplace transforms, the steady solution of system was studied. However, the results are based on following two assumptions:

- 1) The system has unique dynamic solution;
- 2) The limit of the dynamic solution exists, and converges to steady solution.

Whether these assumptions is true or not, i.e. whether the dynamic solution is unique and the dynamic solution does converge to steady solution? So far, these problems are not solved. In [2], the authors proved that the solution of the system exists and is unique.

In this paper, under more normal assumptions, we will prove the existence of rigorous dominant eigenvalue. Further we analyze the essential spectrum of

the semigroup associated with the system. We will prove the dynamic solution of the system converges exponentially to the steady solution.

The rest is organized as follows. In section 2, we recall the mathematical model and basic assumptions of the system under consideration. In section 3, we study the steady solution of the system. In section 4, we discuss the asymptotic convergence of the solution. We will prove that the dynamic solution of the system converges asymptotically to the steady solution. In section 5, we investigate the exponential convergence. by some technique we prove that the dynamic solution converges exponentially to the steady solution.

2 The basic assumptions and mathematic model of the system

Suppose that a system consists of working part and the service part in where the failed system will be repaired.

The following assumptions were made on this system:

- i) The system is composed of three identical units(two in parallel and one on standby).
- ii) The standby and or its switching mechanisms

may fail.

iii) The units are operating and one unit is on warm standby (it means the standby unit is active to a certain degree).

As soon as both the operating units fail, the standby unit goes into operation.

iv) All failure rates associated with the system are constant.

v) Common-cause and other failures are statistically independent.

vi) The occurrence of a common-cause failure causes the total system failure.

vii) A common-cause failure may occur when only one unit is operating normally. For example fire in a room containing the system will cause the total system failure irrespective of whether one or all units are good.

viii) At least one of the three units must operate normally for the system success.

ix) The repaired system is as good as new.

x) The failed system repair times are arbitrarily distributed.

xi) The system fails either due to a common-cause failure or when all of its units fail.

This system undergoes seven states 0, 1, 2, 3, 4, 5, 6 which stand for the following meaning

- 0: Two units working, one on standby;
- 1: One unit working, other failed, one on standby;
- 2: Two units working, standby failed;
- 3: One unit working, other failed, standby failed;
- 4: Two units failed, standby working;
- 5: System failed other than due to common cause failures;
- 6: System failed due to common cause failures.

The diagram of the system state transition is shown as in Figure 1.

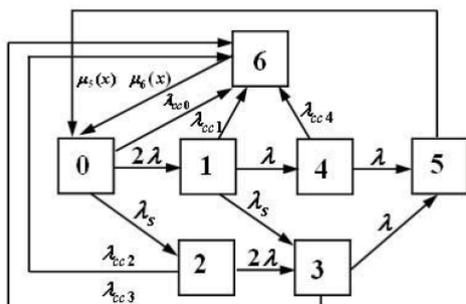


Figure 1 System state-space diagram

The following symbols are used in this article:
 t time.

$P_i(x, t)$ Probability density (with respect to repair time) that the failed system is in state i and has an elapsed repair time of $x; i=5,6$.

$\mu_i(x)$ repair rate and probability density function of repair time, respectively, when the failed system is in state i and has an elapsed repair time of x ; for $i=5,6$.

$P_i(t)$ The probability that the system is in state i at time t ; for $i=0,1,2,3,4,5,6$.

λ Constant failure rate of a unit.

λ_s Constant failure rate of the warm standby and/or switching mechanism.

λ_{cci} Constant common cause failure rate of the system form state i , for $i=0,1,2,3,4$.

Base on the preceding assumptions, by the method of supplementary variables, the dynamic behavior of the system can be expressed as the following the integral and differential equation groups [1].

$$\begin{cases} \frac{dp_0(t)}{dt} = -(2\lambda + \lambda_s + \lambda_{cc0})p_0(t) + \int_0^\infty \mu_5(x)p_5(x, t)dx + \int_0^\infty \mu_6(x)p_6(x, t)dx, \\ \frac{dp_1(t)}{dt} = -(\lambda + \lambda_s + \lambda_{cc1})p_1(t) + 2\lambda p_0(t), \\ \frac{dp_2(t)}{dt} = -(2\lambda + \lambda_{cc2})p_2(t) + (\lambda_s)p_0(t), \\ \frac{dp_3(t)}{dt} = -(\lambda + \lambda_{cc3})p_3(t) + \lambda_s p_1(t), \\ \frac{dp_4(t)}{dt} = -(\lambda + \lambda_{cc4})p_4(t) + \lambda p_1(t), \\ \frac{\partial p_5(x, t)}{\partial x} + \frac{\partial p_5(x, t)}{\partial t} = -\mu_5(x)p_5(x, t), \\ \frac{\partial p_6(x, t)}{\partial x} + \frac{\partial p_6(x, t)}{\partial t} = -\mu_6(x)p_6(x, t) \end{cases} \quad (1)$$

with the boundary conditions

$$\begin{cases} p_5(0, t) = \lambda(p_3(t) + p_4(t)), \\ p_6(0, t) = \sum_{i=0}^4 \lambda_{cci}p_i(t) \end{cases} \quad (2)$$

and the initial condition

$$(p_0(0), p_1(0), p_2(0), p_3(0), p_4(0), p_5(x, 0), p_6(x, 0)) = (1, 0, 0, 0, 0, 0, 0) \quad (3)$$

In the present paper, we will consider problem under the following conditions

$$\begin{aligned} 0 < c_i &= \inf_{x \in \mathbb{R}^+} \mu_i(x), \\ M &= \sup_{x \in \mathbb{R}^+} \mu_i(x) < \infty, \\ \sup_{\xi \geq 0} \int_{\xi}^{\infty} e^{-\int_{\xi}^x \mu_i(s)ds} dx &< \infty, i = 5, 6. \end{aligned} \quad (4)$$

Let state space \mathbb{X} be

$$\mathbb{X} = \mathbb{C}^5 \times (L^1(\mathbb{R}^+))^2$$

For $y = (y_0, y_1, y_2, y_3, y_4, y_5(x), y_6(x)) \in \mathbb{X}$, the norm of y is defined

$$\|y\| = |y_0| + |y_1| + |y_2| + |y_3| + |y_4| + \|y_5\|_{L^1} + \|y_6\|_{L^1}.$$

Obviously, $(\mathbb{X}, \|\cdot\|)$ is a Banach space.

Set

$$\begin{aligned} a_0 &= 2\lambda + \lambda_s + \lambda_{cc0}, \\ a_1 &= \lambda + \lambda_s + \lambda_{cc1}, \\ a_2 &= 2\lambda + \lambda_{cc2}, \\ a_3 &= \lambda + \lambda_{cc3}, \\ a_4 &= \lambda + \lambda_{cc4}. \end{aligned}$$

In space \mathbb{X} , we define the operator A by

$$A = -diag(a_0, a_1, a_2, a_3, a_4, \frac{d}{dx} + \mu_5(x), \frac{d}{dx} + \mu_6(x))$$

with domain

$$D(A) = \left\{ p \in \mathbb{X} \mid \begin{aligned} &\frac{dp_i(x)}{dx} \in L^1(\mathbb{R}^+), i = 5, 6 \\ &p_5(0) = \lambda(p_3 + p_4), \\ &p_6(0) = \sum_{i=0}^4 \lambda_{cci} p_i \end{aligned} \right\}$$

and define operator $B : \mathbb{X} \rightarrow \mathbb{X}$

$$B = \begin{pmatrix} 0 & 0 & 0 & B_1 \\ B_2 & B_3 & B_4 & 0 \end{pmatrix}$$

where

$$\begin{aligned} B_1 &= (0, 0, \int_0^\infty \mu_5(x) \cdot dx, \int_0^\infty \mu_6(x) \cdot dx), \\ B_2 &= (2\lambda, \lambda_s, 0, 0, 0, 0)^T, \\ B_3 &= (0, 0, \lambda_s, \lambda, 0, 0)^T, \\ B_4 &= (0, 0, 2\lambda, 0, 0, 0)^T. \end{aligned}$$

By the definition of the operators A and B , the system (1) can be written as an abstract Cauchy problem in \mathbb{X} :

$$\begin{aligned} \frac{dp(t)}{dt} &= (A + B)p(t), \\ p(0) &= (1, 0, 0, 0, 0, 0), \\ p(t) &= (p_0(t), p_1(t), p_2(t), p_3(t), p_4(t), p_5(x, t), p_6(x, t)) \end{aligned} \tag{5}$$

3 The steady solution of the system

In this section we will discuss the steady state of the system (1). Similar to [2]-[5] we can prove that the system (1) is well posed. We begin with studying the eigenvalue problem of system operator.

Suppose that $\gamma \in \mathbb{C}$ is an eigenvalue of the operator $A + B$, and P is an eigenvector associated with γ , i.e. $(\gamma I - A - B)P = 0$, where $P = (p_0, p_1, p_2, p_3, p_4, p_5(x), p_6(x))$. The eigenvalue problem is equivalent to existence of nonzero of

the following equations

$$\begin{cases} (\gamma + a_0)p_0 - \int_0^\infty p_5(x)\mu_5(x)dx \\ - \int_0^\infty p_6(x)\mu_6(x)dx = 0, \\ 2\lambda p_0 - (\gamma + a_1)p_1 = 0, \\ \lambda_s p_0 - (\gamma + a_2)p_2 = 0, \\ \lambda_s p_1 + 2\lambda p_2 - (\gamma + a_3)p_3 = 0, \\ \lambda p_1 - (\gamma + a_4)p_4 = 0, \\ \frac{dp_i(x)}{dx} + (\gamma + \mu_i(x))p_i(x) = 0, \\ \quad i = 5, 6 \\ p_5(0) = \lambda(p_3 + p_4), \\ p_6(0) = \sum_{i=0}^4 \lambda_{cci} p_i. \end{cases} \tag{6}$$

Solving the differential equation in (6) yields

$$p_i(x) = p_i(0)e^{-\int_0^x (\gamma + \mu_i(\xi))d\xi}, i = 5, 6. \tag{7}$$

Inserting (7) into the first equation in (6) yields

$$(\gamma + a_0)p_0 - p_5(0)\mu_{5,\gamma} - p_6(0)\mu_{6,\gamma} = 0 \tag{8}$$

where

$$\mu_{i,\gamma} = \int_0^\infty \mu_i(x)e^{-\int_0^x (\gamma + \mu_i(\xi))d\xi} dx, i = 5, 6$$

Combining the equation (8) and the other equation in (6), we get an algebraic equations

$$\begin{cases} (\gamma + a_0)p_0 - p_5(0)\mu_{5,\gamma} \\ - p_6(0)\mu_{6,\gamma} = 0, \\ 2\lambda p_0 - (\gamma + a_1)p_1 = 0, \\ \lambda_s p_0 - (\gamma + a_2)p_2 = 0, \\ \lambda_s p_1 + 2\lambda p_2 - (\gamma + a_3)p_3 = 0, \\ \lambda p_1 - (\gamma + a_4)p_4 = 0, \\ \lambda(p_3 + p_4) - p_5(0) = 0, \\ \sum_{j=0}^4 \lambda_{ccj} p_j - p_6(0) = 0, \end{cases} \tag{9}$$

Let $\det D(\gamma)$ be the determinant of coefficient matrix in (9), i.e.,

$$\det D(\gamma) = \begin{vmatrix} a_{11} & 0 & 0 & 0 & 0 & a_{16} & a_{17} \\ 2\lambda & a_{22} & 0 & 0 & 0 & 0 & 0 \\ \lambda_s & 0 & a_{33} & 0 & 0 & 0 & 0 \\ 0 & \lambda_s & 2\lambda & a_{44} & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & a_{55} & 0 & 0 \\ 0 & 0 & 0 & \lambda & \lambda & -1 & 0 \\ \lambda_{cc0} & \lambda_{cc1} & \lambda_{cc2} & \lambda_{cc3} & \lambda_{cc4} & 0 & -1 \end{vmatrix}$$

where $a_{11} = \gamma + a_0, a_{22} = \gamma + a_1, a_{33} = \gamma + a_2, a_{44} = \gamma + a_3, a_{55} = \gamma + a_4, a_{16} = -\mu_{5,\gamma}, a_{17} = -\mu_{6,\gamma}$.

Clearly, $\gamma \in \mathbb{C}$ is an eigenvalue of $A + B$ in and only if the algebraic equations (9) has nonzero solution. Therefore, γ is an eigenvalue of $A + B$, then $\det D(\gamma) = 0$. Conversely, if $\gamma \in \mathbb{C}$ makes $\det D(\gamma) = 0$, the algebraic equations (9) has at least one nonzero solution $(p_0, p_1, p_2, p_3, p_4, p_5(0), p_6(0))$. Then we can define functions $p_i(x)$ according to (7) and hence $(p_0, p_1, p_2, p_3, p_4, P_5, P_6,) \in D(A + B)$ and it also is a solution of (6). In particular, when $\gamma = 0$, we have

$$\mu_{i,\gamma} = \int_0^\infty \mu_i(x) e^{-\int_0^x \mu_i(\xi) d\xi} dx = 1, \quad i = 5, 6$$

A direct computation gives $\det D(\gamma) = 0$. So $\gamma = 0$ is an eigenvalue of $A + B$, and corresponding an eigenvector P is given by

$$P = (p_0, p_1, p_2, p_3, p_4, p_5(x), p_6(x)),$$

where

$$\begin{cases} p_1 = \frac{2\lambda}{a_1} p_0, \\ p_2 = \frac{\lambda_s}{a_2} p_0, \\ p_3 = 2\lambda\lambda_s \left(\frac{1}{a_1 a_3} + \frac{1}{a_2 a_3} \right) p_0, \\ p_4 = \frac{2\lambda^2}{a_1 a_4} p_0, \\ p_5(x) = \left[2\lambda^2 \lambda_s \left(\frac{a_2 + a_1}{a_1 a_3 a_2} \right) + \frac{2\lambda^3}{a_1 a_4} \right] \\ \quad \times p_0 e^{-\int_0^x \mu_5(\xi) d\xi} \\ p_6(x) = \left[\lambda_{cc0} + \frac{2\lambda\lambda_{cc1}}{a_1} + \frac{\lambda_s \lambda_{cc2}}{a_2} \right. \\ \quad \left. + 2\lambda\lambda_s \lambda_{cc3} \left(\frac{a_2 + a_1}{a_1 a_3 a_2} \right) + 2\lambda^2 \lambda_{cc4} \right] \\ \quad \times p_0 e^{-\int_0^x \mu_6(\xi) d\xi} \end{cases} \quad (10)$$

Let $Q = (1, 1, 1, 1, 1, 1, 1)$. For $p_0 > 0$, we have

$$\langle P, Q \rangle = \sum_{j=0}^4 p_j + \sum_{j=5}^6 \int_0^\infty p_j(x) dx > 0.$$

A straightforward calculation gives that for any $P \in D(A + B)$,

$$\langle (A + B)P, Q \rangle = \langle P, (A + B)^*Q \rangle = 0,$$

which implies that $(A + B)^*Q = 0$, this means that 0 also is an eigenvalue of $(A + B)^*$ and Q is a corresponding eigenvector. Note that the fact $\langle P, Q \rangle \neq 0$, so 0 is a simple eigenvalue of $A + B$.

Now we take \hat{P}_0 :

$$\hat{P}_0 = \frac{P}{\|P\|} = \frac{1}{\|P\|} (p_0, p_1, p_2, p_3, p_4, p_5(x), p_6(x)) \quad (11)$$

where $p_1, p_2, p_3, p_4, p_i(x) (i = 5, 6)$ are given as in (10). Then \hat{P}_0 is the steady positive solution of the system (1) and satisfies $\langle \hat{P}_0, Q \rangle = 1$.

4 The asymptotic stability of the system

In this section we will study the asymptotic stability of the system (1). Note that 0 is a simple eigenvalue of the operator $A + B$, \hat{P}_0 is corresponding an eigenvector, which is the steady solution of the system. We will prove that the dynamic solution of the system (1) converges the steady solution under the condition (4).

It is well known that if the condition (4) is fulfilled, for any $y \in L^1(\mathbb{R}_+)$, it holds that $\int_0^x e^{-\int_s^x \mu_i(s) ds} y(\xi) d\xi \in L^1(\mathbb{R}_+)$ and

$$\int_0^\infty e^{\int_0^\xi \mu_i(s) ds} |y(\xi)| d\xi \int_\xi^\infty e^{-\int_0^x \mu_i(s) ds} dx < \infty, \quad (12)$$

So, the functions defined by

$$p_i(x) = p_i(0) e^{-\int_0^x \mu_i(s) ds} + \int_0^x e^{-\int_0^\xi \mu_i(s) ds} y_i(\xi) d\xi \quad (13)$$

also satisfy $p_i(x) \in L^1(\mathbb{R}_+)$, $i = 5, 6$.

Using this fact, we can prove the following theorem.

Theorem 1 *The spectra of $A + B$ are in the left half-plane, and all points but 0 on the imaginary axis are in resolvent set.*

Proof. For any $P \in D(A + B) = D(A)$, taking $Q = (\text{sgn}p_0, \text{sgn}p_1, \text{sgn}p_2, \text{sgn}p_3, \text{sgn}p_4, \text{sgn}p_5(x), \text{sgn}p_6(x)) \in \mathbb{X}^*$, a straightforward computation gives $\Re\langle AP, Q \rangle \leq 0$, and for any $\gamma \in C$ with $\Re\gamma > 0$, we have $\gamma \in \rho(A)$. Therefore, we only need to prove the imaginary axis but 0 are in the resolvent set $\rho(A)$. Let $\gamma \in \mathbb{R}$ with $\gamma \neq 0$. For any $y = (y_0, y_1, y_2, y_3, y_4, y_5(x), y_6(x)) \in \mathbb{X}$, we consider the resolvent equation $(i\gamma - A - B)p = y$, namely

$$\begin{cases} (i\gamma + a_0)p_0 - \int_0^\infty p_5(x)\mu_5(x)dx \\ \quad - \int_0^\infty p_6(x)\mu_6(x)dx = y_0, \\ -2\lambda p_0 + (i\gamma + a_1)p_1 = y_1, \\ -\lambda_s p_0 + (i\gamma + a_2)p_2 = y_2, \\ -\lambda_s p_1 - 2\lambda p_2 + (i\gamma + a_3)p_3 = y_3, \\ -\lambda p_1 + (i\gamma + a_4)p_4 = y_4, \\ \frac{dp_i(x)}{dx} + (i\gamma + \mu_i(x))p_i(x) = y_i(x), \quad i = 5, 6 \\ p_5(0) = \lambda(p_3 + p_4), \\ p_6(0) = \sum_{i=0}^4 \lambda_{cci} p_i. \end{cases} \quad (14)$$

Solving the differential equation in (14), we get

$$p_i(x) = p_i(0) e^{-\int_0^x (i\gamma + \mu_i(\xi)) d\xi} + \int_0^x p_i(\xi) \mu_i(\xi) dx \quad (15)$$

where $i = 5, 6$. From (13), we know that $p_i(x) \in L^1(\mathbb{R}_+)$, $i = 5, 6$. Inserting (15) in (14) we get an algebraic equations about $(p_0, p_1, p_2, p_3, p_4, p_5(0), p_6(0))$:

$$\begin{cases} (i\gamma + a_0)p_0 - p_5(0)\mu_{5,i\gamma} - p_6(0)\mu_{6,i\gamma} \\ = y_0 + G_5 + G_6, \\ -2\lambda p_0 + (i\gamma + a_1)p_1 = y_1, \\ -2\lambda p_0 + (i\gamma + a_2)p_2 = y_2, \\ -\lambda_s p_1 - 2\lambda p_2 + (i\gamma + a_3)p_3 = y_3, \\ -\lambda p_1 + (i\gamma + a_4)p_4 = y_4 \\ \lambda(p_3 + p_4) - p_5(0) = 0, \\ \sum_{i=0}^4 \lambda_{cci} p_i - p_6(0) = 0 \end{cases} \quad (16)$$

where

$$G_i = \int_0^\infty \mu_i(x) dx \int_0^x e^{-\int_i^x (i\gamma + \mu_i(\xi)) d\xi} y_i(t) dt$$

$$\mu_{i,i\gamma} = \int_0^\infty \mu_i(x) e^{-\int_0^x (i\gamma + \mu_i(\xi)) d\xi} dx, \quad i = 5, 6.$$

Let $D(i\gamma)$ be the coefficient matrix in (16), i.e.,

$$D = \begin{bmatrix} b_{11} & 0 & 0 & 0 & 0 & b_{16} & b_{17} \\ 2\lambda & b_{22} & 0 & 0 & 0 & 0 & 0 \\ \lambda_s & 0 & b_{33} & 0 & 0 & 0 & 0 \\ 0 & \lambda_s & 2\lambda & b_{44} & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & b_{55} & 0 & 0 \\ 0 & 0 & 0 & \lambda & \lambda & -1 & 0 \\ \lambda_{cc0} & \lambda_{cc1} & \lambda_{cc2} & \lambda_{cc3} & \lambda_{cc4} & 0 & -1 \end{bmatrix}$$

where $b_{11} = i\gamma + a_0, b_{22} = i\gamma + a_1, b_{33} = i\gamma + a_2, b_{44} = i\gamma + a_3, b_{55} = i\gamma + a_4, b_{16} = i\gamma + a_0, b_{17} = -\mu_{5,\gamma}, b_{17} = -\mu_{6,\gamma}$.

Now, we prove the $D(i\gamma)$ is nonsingular matrix. By definition of $a_i, i = 0, 1, 2, 3, 4$, we have

$$\begin{aligned} a_0 &= 2\lambda + \lambda_s + \lambda_{cc0} < |i\gamma + a_0|, \\ a_1 &= \lambda + \lambda_s + \lambda_{cc1} < |i\gamma + a_1|, \\ a_2 &= 2\lambda + \lambda_{cc2} < |i\gamma + a_2|, \\ a_3 &= \lambda + \lambda_{cc3} < |i\gamma + a_3|, \\ a_4 &= \lambda + \lambda_{cc4} < |i\gamma + a_4| \end{aligned}$$

and

$$|\mu_{i,i\gamma}| = \left| \int_0^\infty \mu_i(x) e^{-\int_0^x (i\gamma + \mu_i(\xi)) d\xi} dx \right| < \int_0^\infty \mu_i(x) e^{-\int_0^x \mu_i(\xi) d\xi} dx = 1, \quad i = 5, 6.$$

Hence the matrix $D(i\gamma)$ is strictly diagonally dominant by columns, which implies $\det D(i\gamma) \neq 0$ (see, [6]). Therefore the algebraic equations (16) has unique a solution for given $y \in \mathbb{X}$, denote by

$$(\widehat{p}_0, \widehat{p}_1, \widehat{p}_2, \widehat{p}_3, \widehat{p}_4, \widehat{p}_5(0), \widehat{p}_6(0)).$$

Define functions by

$$p_i(x) = \widehat{p}_i(0) e^{-\int_0^x (i\gamma + \mu_i(\xi)) d\xi} + \int_0^\infty p_i(x) \mu_i(x) dx, \quad i = 5, 6.$$

Thus the vector

$$(\widehat{p}_0, \widehat{p}_1, \widehat{p}_2, \widehat{p}_3, \widehat{p}_4, p_5(x), p_6(x))$$

belongs to $D(A)$ and satisfies the resolvent equation (14). So $\gamma \in \rho(A + B)$. Therefore, the all point but 0 on the imaginary axis are in the resolvent set of A . The proof is then complete. \square

According to stability theorem of linear operator semigroup (see,[7]), we have the following conclusions.

Theorem 2 Let \mathbb{X} and A, B be defined as before. Let $T(t)$ be the C_0 contraction semigroup generated by $A + B$. Then for any initial value $P(0)$, the dynamic solution of system (5) is given by $P(t) = T(t)P(0)$. In particular, if $P(0)$ is non-negative, the dynamic solution is also negative for all $t \geq 0$.

Let $\widehat{P}_0 = (p_0, p_1, p_2, p_3, p_4, p_5(x), p_6(x))$ be the negative steady state of the system (1) with $\|\widehat{P}_0\| = 1$. Then we have

$$\lim_{t \rightarrow \infty} P(t) = \lim_{t \rightarrow \infty} T(t)P(0) = (P(0), Q)\widehat{P}_0,$$

where $Q = (1, 1, 1, 1, 1, 1, 1)$.

Proof The first assertion can be proved similar to [3] and [4]. The second assertion is a direct consequence of Theorem 1 and stability Theorem of semigroup (see [7]). \square

5 Exponential convergence of the solution of system

In this section, we prove that the system converges exponentially to the steady state of the system (5) or (1) under the condition (4).

Theorem 3 Let A be defined as before and c_i be defined as in (4). Denote

$$c = \min_{0 \leq j \leq 4} \{ \min_{i=5,6} \{ c_i \}, \lambda, \lambda_s, \lambda_{ccj} \}.$$

Then when $\Re\gamma > -c$, we have $\gamma \in \rho(A)$ and

$$\|(\gamma I - A)^{-1}\| \leq \frac{2}{\Re\gamma + c}.$$

Proof Let $\gamma \in \mathbb{C}$ with $\Re\gamma > -c$. For any given $y = (y_0, y_1, y_2, y_3, y_4, y_5(x), y_6(x)) \in \mathbb{X}$, we consider the resolvent equation $(\gamma I - A)P = y$ where $P = (p_0, p_1, p_2, p_3, p_4, p_5(x), p_6(x)) \in D(A)$. That is $(p_0, p_1, p_2, p_3, p_4, p_5(x), p_6(x))$ satisfying the following equations

$$\begin{cases} (\gamma + a_0)p_0 = y_0, \\ (\gamma + \theta)p_1 = y_1, \\ (\gamma + a_2)p_2 = y_2, \\ (\gamma + a_3)p_3 = y_3, \\ (\gamma + a_4)p_4 = y_4, \\ \frac{dp_i(x)}{dx} + (\gamma + \mu_i(x))p_i(x) = y_i(x), \\ \quad i = 5, 6, \\ p_5(0) = \lambda(p_3 + p_4), \\ p_6(0) = \sum_{i=0}^4 \lambda_{cci}p_i. \end{cases} \quad (17)$$

When $\Re\gamma > -c$, we have $\Re\gamma + a_i \neq 0 (i = 0, 1, 2, 3, 4)$. Solving the differential equations in (17), we get

$$\begin{cases} p_i = \frac{y_i}{\gamma + a_i}, i = 0, 1, 2, 3, 4. \\ p_5(x) = \left(\frac{\lambda y_3}{\gamma + a_3} + \frac{\lambda y_4}{\gamma + a_4} \right) e^{-\gamma x - \int_0^x \mu_5(\xi) d\xi} \\ \quad + \int_0^x e^{-\gamma(x-\tau) - \int_\tau^x \mu_5(\xi) d\xi} y_5(\tau) d\tau, \\ p_6(x) = \sum_{i=0}^4 \frac{\lambda_{cci} y_i}{\gamma + a_i} e^{-\gamma x - \int_0^x \mu_6(\xi) d\xi} \\ \quad + \int_0^x e^{-\gamma(x-\tau) - \int_\tau^x \mu_6(\xi) d\xi} y_6(\tau) d\tau. \end{cases} \quad (18)$$

Since

$$\begin{aligned} \|P\| &= \sum_{i=0}^4 |p_i| + \|p_5\| + \|p_6\| \\ &\leq \sum_{i=0}^4 \left| \frac{y_i}{\gamma + a_i} \right| \\ &\quad + \left(\left| \frac{\lambda y_3}{\gamma + a_3} + \frac{\lambda y_4}{\gamma + a_4} \right| \int_0^\infty e^{-\Re\gamma x - \int_0^x \mu_5(\xi) d\xi} dx \right. \\ &\quad + \int_0^\infty dx \int_0^x e^{-\Re\gamma(x-\tau) - \int_\tau^x \mu_5(\xi) d\xi} |y_5(\tau)| d\tau \\ &\quad + \sum_{i=0}^4 \left| \frac{\lambda_{cci} y_i}{\gamma + a_i} \right| \int_0^\infty e^{-\Re\gamma x - \int_0^x \mu_6(\xi) d\xi} dx \\ &\quad + \int_0^\infty dx \int_0^x e^{-\Re\gamma(x-\tau) - \int_\tau^x \mu_6(\xi) d\xi} |y_6(\tau)| d\tau \\ &\leq \sum_{i=0}^4 \left| \frac{y_i}{\gamma + a_i} \right| \\ &\quad + \left| \frac{\lambda y_3}{\gamma + a_3} + \frac{\lambda y_4}{\gamma + a_4} \right| \int_0^\infty e^{-(\Re\gamma + c)x} dx \\ &\quad + \int_0^\infty |y_5(\tau)| d\tau \int_\tau^\infty e^{-(\Re\gamma + c)(x-\tau)} dx \end{aligned}$$

$$\begin{aligned} &+ \sum_{i=0}^4 \left| \frac{\lambda_{cci} y_i}{\gamma + a_i} \right| \int_0^\infty e^{-(\Re\gamma + c)x} dx \\ &+ \int_0^\infty |y_6(\tau)| d\tau \int_\tau^\infty e^{-(\Re\gamma + c)(x-\tau)} dx \\ &= \sum_{i=0}^4 \left| \frac{y_i}{\gamma + a_i} \right| + \frac{1}{\Re\gamma + c} \left| \frac{\lambda y_3}{\gamma + a_3} + \frac{\lambda y_4}{\gamma + a_4} \right| \\ &\quad + \frac{1}{\Re\gamma + c} \sum_{i=0}^4 \left| \frac{\lambda_{cci} y_i}{\gamma + a_i} \right| \\ &\quad + \frac{1}{\Re\gamma + c} (\|y_5\| + \|y_6\|) \\ &\leq \frac{1}{\Re\gamma + c} \frac{(\Re\gamma + c + \lambda_{cc0})}{|\gamma + a_0|} |y_0| \\ &\quad + \frac{1}{\Re\gamma + c} \frac{(\Re\gamma + c + \lambda_{cc1})}{|\gamma + a_1|} |y_1| \\ &\quad + \frac{1}{\Re\gamma + c} \frac{(\Re\gamma + c + \lambda_{cc2})}{|\gamma + a_2|} |y_2| \\ &\quad + \frac{1}{\Re\gamma + c} \frac{(\Re\gamma + c + a_3)}{|\gamma + a_3|} |y_3| \\ &\quad + \frac{1}{\Re\gamma + c} \frac{(\Re\gamma + c + a_4)}{|\gamma + a_4|} |y_4| \\ &\quad + \frac{1}{\Re\gamma + c} (\|y_5\| + \|y_6\|), \end{aligned}$$

while

$$\Re\gamma + c + \lambda_{cci} \leq |\gamma + a_i|, \quad i = 0, 1, 2$$

and

$$|\gamma + a_i| \geq c, \quad i = 3, 4,$$

so we have

$$\|p\| < \frac{2}{\Re\gamma + c} \left(\sum_{i=0}^4 |y_i| + \|y_5\| + \|y_6\| \right) = \frac{2}{\Re\gamma + c} \|y\|.$$

This shows that $\Re\gamma + c > 0$, $(\gamma I - A)^{-1} : \mathbb{X} \rightarrow \mathbb{X}$ are bounded linear operators, hence $\gamma \in \rho(A)$, and

$$\|(\gamma I - A)^{-1}\| \leq \frac{2}{\Re\gamma + c}.$$

This completes the proof. \square

Above Theorem shows that the spectra of A are in the half-plane $\Re\gamma \leq -c$. The following theorem shows that the spectrum of the semigroup $S(t)$ generated by A is in the disc $\{z \in \mathbb{C} \mid |z| \leq e^{-ct}\}$.

Theorem 4 Let A and c be defined as before, and let $S(t)$ be the C_0 semigroup generated by operator A . Then for any $c > \omega > 0$, there is a positive constant M such that $\|S(t)\| \leq M e^{-\omega t}$, $t \geq 0$.

Proof. Firstly let us define a new operator A_0 by

$$A_0 = -diag(a_0, a_1, a_2, a_3, a_4, \frac{d}{dx} + \mu_5(x), \frac{d}{dx} + \mu_6(x))$$

with domain

$$D(A_0) = \left\{ p \in \mathbb{X} \mid \begin{array}{l} \frac{dp_i(x)}{dx} \in L^1(\mathbb{R}^+), i = 5, 6 \\ p_5(0) = 0, p_6(0) = 0 \end{array} \right\}$$

we define a linear operator \mathcal{L} by

$$\mathcal{L}f = f - (0, 0, 0, 0, e^{-x}\lambda(f_3 + f_4), e^{-x} \sum_{j=0}^4 \lambda_{ccj} f_j)$$

for any $f \in \mathbb{X}$. Clearly, \mathcal{L} is bounded invertible.

For any $P = (p_0, p_1, p_2, p_3, p_4, p_5, p_6) \in D(A)$, we have $\mathcal{L}P \in D(A_0)$ and

$$\begin{aligned} A_0 \mathcal{L}P &= -(a_0 p_0, a_1 p_1, a_2 p_2, a_3 p_3, a_4 p_4, \\ &\quad p'_5 + \mu_5 p_5, p'_6 + \mu_6 p_6) \\ &\quad - e^{-x} (0, 0, 0, 0, \\ &\quad (1 - \mu_5(x))\lambda(p_3 + p_4), \\ &\quad (1 - \mu_6(x)) \sum_{j=0}^4 \lambda_{ccj} p_j) \end{aligned}$$

Set

$$\begin{aligned} B_1 P &= -e^{-x} (0, 0, 0, 0, \\ &\quad (1 - \mu_5(x))\lambda(p_3 + p_4), \\ &\quad (1 - \mu_6(x)) \sum_{j=0}^4 \lambda_{ccj} p_j) \end{aligned}$$

Then we have

$$A_0 \mathcal{L} = A - B_1$$

This means that $A_0 \mathcal{L}$ is two rank perturbation of A . We shall prove that the spectrum of A_0 are in the half-plane $\Re \gamma < c$.

In fact, for any $P = (p_0, p_1, p_2, p_3, p_4, p_5, p_6) \in D(A_0)$, and $Q = (\text{sgn} p_0, \text{sgn} p_1, \text{sgn} p_2, \text{sgn} p_3, \text{sgn} p_4, \text{sgn} p_5, \text{sgn} p_6)$,

we have

$$\begin{aligned} \langle A_0 P + cP, Q \rangle &= -(a_0 - c)|p_0| - (a_1 - c)|p_1| \\ &\quad - (a_2 - c)|p_3| - (a_4 - c)|p_4| \\ &\quad - \int_0^\infty (p'_5(x) + (\mu_5(x) - c)p_5(x)) \text{sgn} p_5(x) dx \\ &\quad - \int_0^\infty (p'_6(x) + (\mu_6(x) - c)p_6(x)) \text{sgn} p_6(x) dx \\ &= -a_0|p_0| - a_1|p_1| - a_2|p_3| - a_4|p_4| \\ &\quad - \int_0^\infty (\mu_5(x) - c)|p_5(x)| dx + |p_5(0)| \\ &\quad - \int_0^\infty (\mu_6(x) - c)|p_6(x)| dx + |p_6(0)| \\ &< 0. \end{aligned}$$

This implies that A_0 is strictly dissipative operator in \mathbb{X} . So the semigroup $S_0(t)$ generated by A_0 satisfying $\|S_0(t)\| \leq e^{-ct}$.

Note that A is a two rank perturbation of A_0 . According to Theorem 3, there is no spectral point in $\Re \gamma > -c$, so for any $c > \omega > 0$, there is a positive constant M such that $\|S(t)\| \leq M e^{-\omega t}$, $t \geq 0$. The desired result follows. \square

Remark 5 Usually the perturbation Theorem only ensures that the essential spectrum of $S(t)$ is the same as the one of $S_0(t)$. (e.g. see [9] and [10]). Since we have proved Theorem 3, we can assert that the spectrum of $S(t)$ is in the disc $\sigma(S(t)) \subset \{z \in \mathbb{C} \mid |z| \leq e^{-ct}\}$. Since $S(t)$ need not to be dissipative for $\Re \gamma + c > 0$, so there is a constant M in Theorem 4.

Since B is a finite rank operator, it is a compact operator. According to compact perturbation theorem of operator semigroup, we have the following conclusion.

Theorem 6 Suppose that A and c are defined as before. Then the C_0 semigroup $T(t)$ generated by $A + B$ possesses the following properties:

- 1). when $\gamma \in \mathbb{C}$, $\Re \gamma + c > 0$, $\gamma \in \sigma(A + B) \Leftrightarrow \det D(\gamma) = 0$.
- 2). Denote $\gamma_0 = 0$. For any

$$\gamma_k \in \{\gamma \in \mathbb{C} \mid \Re \gamma > -c, \det D(\gamma) = 0\}, \gamma_k \neq \gamma_0$$

where $\Re \gamma_{k+1} \leq \Re \gamma_k, k = 1, 2, \dots, N$, we have $\Re \gamma_k < \gamma_0$, that is $\gamma_0 = 0$ is a strict dominant eigenvalue.

- 3). Set $\hat{P}_0 = (p_0, p_1, p_2, p_3, p_4, p_5(x), p_6(x))$ be the steady solution of the system with $(\hat{P}_0, Q) = 1$. Taking $\omega > 0$ such that $\Re \gamma_1 < -\omega < \gamma_0$, then for any $P \in \mathbb{X}$, it holds that $\|T(t)P - (P, Q)\hat{P}_0\| \leq 2e^{-\omega t}$, $t \geq 0$, where $Q = (1, 1, 1, 1, 1, 1, 1)$.

Proof 1) When $\Re\gamma > -c$, according to the theorem 3, $\gamma \in \rho(A)$, then

$$(\gamma I - A - B) = (\gamma I - A)(I - R(\gamma, A)B).$$

Since B is a finite rank operator, $R(\gamma, A)B$ is a compact operator, $\gamma \in \rho(A + B)$ if and only if 1 is not an eigenvalue of $R(\gamma, A)B$. So when $\Re\gamma + c > 0$, we have

$$\gamma \in \sigma(A + B) \Leftrightarrow D(\gamma) = 0$$

2) When $\Re\gamma + c > \delta > 0$, $\det D(\gamma)$ is an analytic function in this region, there are at most finite number of zeros of $\det D(\gamma)$ and there is no accumulation point in the finite region. Since 0 is a simple eigenvalue of $A + B$, and it has positive eigenvector. According to the definition, 0 is a strict dominant eigenvalue.

Set $\gamma_0 = 0$, for any

$$\gamma_k \in \{\gamma \in \mathbb{C} \mid \Re\gamma > -c, D(\gamma) = 0\}, \gamma_k \neq \gamma_0.$$

where $\Re\gamma_{k+1} \leq \Re\gamma_k, k = 1, 2, 3 \dots N$, then $\Re\gamma_k < \gamma_0 = 0$ due to it being a strict dominant eigenvalue of $A + B$.

3) Finally, let $T(t)$ be the semigroup generated by $A + B$ and $S(t)$ the semigroup generated by A . Applying the perturbation theorem of bounded linear operator semigroup to the semigroup $T(t)$ and $S(t)$, we have $\omega_{ess}(T(t)) \leq \omega_{ess}(S(t)) \leq \omega_0(T(t))$ (see,[9] and [10]), where $\omega_{ess}(T(t))$ denotes the essential spectrum bound of $T(t)$.

Suppose that \hat{P}_0 is defined as in (11), and $\Re\gamma_1 < -\omega < \gamma_0$, according to the finite expansion theorem of semigroup, for any $P \in X$, we have

$$\|T(t)P - (P, Q)\hat{P}_0\| \leq 2e^{-\omega t}, t \geq 0,$$

where $Q = (1, 1, 1, 1, 1, 1)$. The above conclusion shows that the dynamic solution of the system (5) converges exponentially the steady solution of the system.

□

6 The reliability analysis of the system

In this section we will analyze some indices of reliability of the system (1). Set

$$\Pi_i = \int_0^\infty e^{-\int_0^x \mu_i(\xi)d\xi} dx, \quad i = 5, 6$$

According to the conditions satisfied by c and the initial value $p(0) = (1, 0, 0, 0, 0, 0)$, the dynamic solution of the system is

$$p(t) = (p_0(t), p_1(t), p_2(t), p_3(t), p_4(t), p_5(x, t), p_6(x, t))$$

By (11), the steady solution of system is

$$(P(0), Q)\hat{P}_0 = \hat{P}_0 = (p_0, p_1, p_2, p_3, p_4, p_5(x), p_6(x))$$

By (10), normalizing $\|P\| = 1$ we have $p_0 = \frac{1}{Z}$, where

$$\begin{aligned} Z = & 1 + \frac{2\lambda}{a_1} + \frac{\lambda_s}{a_2} + \frac{2\lambda\lambda_s}{a_1a_3} + \frac{2\lambda\lambda_s}{a_2a_3} + \frac{2\lambda^2}{a_1a_4} \\ & + \left(\frac{2\lambda^2\lambda_s}{a_1a_3} + \frac{2\lambda^2\lambda_s}{a_2a_3} + \frac{2\lambda^3}{a_1a_4}\right)\Pi_5 \\ & + \left(\lambda_{cc0} + \frac{2\lambda\lambda_{cc1}}{a_1} + \frac{\lambda_s\lambda_{cc2}}{a_2} + \frac{2\lambda\lambda_s\lambda_{cc3}}{a_1a_3}\right. \\ & \left. + \frac{2\lambda\lambda_s\lambda_{cc3}}{a_2a_3} + \frac{2\lambda^2\lambda_{cc4}}{a_1a_4}\right)\Pi_6 \end{aligned}$$

So, the asymptotic behavior of system (1) near by the steady solution is given by

$$\|P(t) - \hat{P}_0\| \leq 2e^{-\omega t}, \forall t > 0.$$

Clearly, when $t = \frac{3-\ln 2}{\omega}$, $\|P(t) - \hat{P}_0\| \leq 0.01$.

When $t = \frac{3-\ln 2}{\omega}$, the natural working probability of the system (1) is

$$\begin{aligned} & p_0(t) + p_1(t) + p_2(t) + p_3(t) + p_4(t) \\ & = p_0 + p_1 + p_2 + p_3 + p_4 + (p_0(t) - p_0) \\ & + (p_1(t) - p_1) + (p_2(t) - p_2) + (p_3(t) - p_3) \\ & + (p_4(t) - p_4) \\ & \leq p_0 + p_1 + p_2 + p_3 + p_4 + 0.01 \\ & = \frac{1}{Z} \left(1 + \frac{2\lambda}{a_1} + \frac{\lambda_s}{a_2} + \frac{2\lambda\lambda_s}{a_1a_3} + \frac{2\lambda\lambda_s}{a_2a_3} + \frac{2\lambda^2}{a_1a_4}\right) + 0.01 \end{aligned}$$

The failure rate of the system (1) is

$$\begin{aligned} & \int_0^\infty p_5(x, t)dx + \int_0^\infty p_6(x, t)dx \\ & \leq \int_0^\infty p_5(x)dx + \int_0^\infty p_6(x)dx + 0.01 \\ & = \frac{1}{Z} \left(\frac{2\lambda^2\lambda_s}{a_1a_3} + \frac{2\lambda^2\lambda_s}{a_2a_3} + \frac{2\lambda^3}{a_1a_4}\right)\Pi_5 \\ & + \left(\lambda_{cc0} + \frac{2\lambda\lambda_{cc1}}{a_1} + \frac{\lambda_s\lambda_{cc2}}{a_2} + \frac{2\lambda\lambda_s\lambda_{cc3}}{a_1a_3} + \frac{2\lambda\lambda_s\lambda_{cc3}}{a_2a_3} + \frac{2\lambda^2\lambda_{cc4}}{a_1a_4}\right)\Pi_6 + 0.01 \end{aligned}$$

From the formula

$$\Pi_i = \int_0^\infty e^{-\int_0^x \mu_i(\xi)d\xi} dx, \quad i = 5, 6$$

we can see that the greater μ_i is, the smaller Π_i is, so the failure rate becomes small. Hence, the natural working probability of the system (1) becomes larger. It shows that the reliability of the system (1) is increased.

Acknowledgements: This work is sported by the National Science Foundation of the Education Department of Henan province(122300410116).

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