An operator preserving inequalities between polynomials

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Abstract: For the B-operator $B[P(z)]$ where P(z) is a polynomial of degree n, a problem has been considered of investigating the dependence of $|B[P(Rz)] - \alpha B[P(rz)]|$ on the maximum modulus of $P(z)$ on $|z| = 1$ for every real or complex number α with $|\alpha| \le 1$, $R > r \ge 1$ in order to establish some new operator preserving inequalities between polynomials.

Key–Words: Polynomials, B-operator, Inequalities in the complex domain.

1 Introduction

Let $P_n(z)$ denote the space of all complex polynomials of degree n. If $P \in P_n$, then

$$\max_{|z|=1} |P'(z)| \le n \max_{|z|=1} |P(z)| \quad (1)$$

and

$$\max_{|z|=R>1} |P(z)| \le R^n \max_{|z|=1} |P(z)| \quad (2)$$

Inequality (1) is an immediate consequence of S. Bernstein's Theorem (see [11,16]) on the derivative of a trigonometric polynomial. Inequality (2) is a simple deduction from the maximum modulus principle (see [11] or [13]). If we restrict ourselves to the class of polynomials $P \in P_n$ having no zero in $|z| < 1$, then inequalities (1) and (2) can be respectively replaced by

$$\max_{|z|=1} |P'(z)| \le \frac{n}{2} \max_{|z|=1} |P(z)| \quad (3)$$

and

$$\max_{|z|=R>1} |P(z)| \le \frac{R^n+1}{2} \max_{|z|=1} |P(z)| . \quad (4)$$

Inequality (3) was conjectured by P. Erdös and later verified by P.D.Lax [9] (see also [2]). Ankeny and Rivilin [1] used (3) to prove inequality (4). As a compact generalization of inequalities (1) and (2), Aziz and Rather [7] have shown that, if $P \in P_n$,then for every real or complex α with $|\alpha| \le 1$, $R > 1$ and $|z| \ge 1$,

$$|P(Rz) - \alpha P(z)| \le (R^n - 1)|z|^n \max_{|z|=1} |P(z)| . \quad (5)$$

The result is sharp and equality in (5) holds for $P(z) = \lambda z^n, \lambda \ne 0$

As a corresponding compact generalization of inequalities (3) and (4), they [7] have also shown that if $P \in P_n$ and $P(z)$ does not vanish in $|z| < 1$, then for every real or complex number α with $|\alpha| \le 1$, $R > 1$ and $|z| \ge 1$,

$$|P(Rz) - \alpha P(z)| \le \frac{|R^n-\alpha||z|^n + |1-\alpha|}{2} \max_{|z|=1} |P(z)| . \quad (6)$$

Equality in (6) holds for $P(z) = az^n + b$, and $|a| = |b| = 1$.

Inequalities of the type (3)and (4) were further generalized among others by jain [8], Aziz and Dawood [3], Aziz and Rather [4] and extended to L_p norm by Aziz and Rather [5,6].

Consider a class B_n of operator B that carries polynomial $P \in P_n$ into

$$B[P(z)] = \lambda_0 P(z) + \lambda_1 \left(\frac{nz}{2}\right) \frac{P'(z)}{1!} + \lambda_2 \left(\frac{nz}{2}\right)^2 \frac{P''(z)}{2!} \quad (7)$$

where λ_0, λ_1 and λ_2 are real or complex numbers such that all the zeros of

$$u(z) = \lambda_0 + \lambda_1 C(n,1)z + \lambda_2 C(n,2)z^2, \quad (8)$$

lie in half the plane

$$|z| \le |z - n/2|. \quad (9)$$

Note that for $0 \le r \le n$,

$$C(n,r) = n!/r!(n-r)!.$$

It was proposed by Q.I. Rahman to study inequalities concerning the maximum modulus of $B[P](z)$ for $P \in P_n$. As an attempt to this, Q.I.Rahman [14] (see also [15,16]) extended inequalities (1), (2), (3) and (4) to the class of operators $B \in B_n$ by showing that that if $P \in P_n$, then

$$|P(z)| \leq \max_{|z|=1} |P(z)| \;\; for \;\; |z| = 1$$

implies

$$|B[P](z)| \leq |B[z^n]| \max_{|z|=1} |P(z)| \;\; for \;\; |z| \geq 1 \quad (10)$$

and if $P(z) \neq 0$ in $|z| < 1$, then

$$|B[P](z)| \leq \frac{|B[z^n]| + |\lambda_0|}{2} \max_{|z|=1} |P(z)| \quad (11)$$

for $|z| \geq 1$.

In this paper an attempt has been made to investigate the dependence of $|B[P(Rz)] - \alpha B[P(rz)]|$ on the maximum modulus of $P(z)$ on $|z| = 1$ for every real or complex number α with $|\alpha| \leq 1$, $R > r \geq 1$ and develop a unified method for arriving at various results simultaneously. In this direction, we first present the following interesting result which is a compact generalization of the inequalities (1), (2), (5) and (10).

Theorem 1 *If $P \in P_n$, then for every real or complex number α with $|\alpha| \leq 1$, $R > r \geq 1$ and for $|z| \geq 1$,*

$$\begin{aligned} &|B[P(Rz)] - \alpha B[P(rz)]| \\ &\leq |R^n - \alpha r^n||B[z^n]| \max_{|z|=1} |P(z)| . \end{aligned} \quad (12)$$

where $B \in B_n$. The result is best possible and equality in (12) holds for $P(z) = \lambda z^n, \lambda \neq 0$.

Substituting for $B[P](z)$, one gets from (12) for every real or complex α with $|\alpha| \leq 1$, $R > r \geq 1$ and for $|z| \geq 1$,

$$\begin{aligned} &\left| \sum_{j=0}^{2} \lambda_j \left(\frac{nz}{2}\right)^j \frac{\left(P^{(j)}(Rz) - \alpha P^{(j)}(rz)\right)}{j!} \right| \\ \leq \; & |R^n - \alpha r^n| \, |z|^n \times \\ &\left| \sum_{j=0}^{2} \lambda_j \left(\frac{n}{2}\right)^j C(n,j) \right| \max_{|z|=1} |P(z)| \end{aligned} \quad (13)$$

where λ_0, λ_1 and λ_2 are such that all the zeros of $u(z)$ defined by (8) lie in the half plane (9).

Remark 2 *For $\alpha = 0$, from inequality (12), we obtain for $|z| \geq 1$ and $R > 1$,*

$$|B[P](Rz)| \leq |B[R^n z^n]| \max_{|z|=1} |P(z)| \quad (14)$$

where $B \in B_n$, which contains inequality (10) as a special case.

By taking $\lambda_0 = \lambda_2 = 0$ in (13) and noting that in this case all the zeros of $u(z)$ defined by (8) lie in the half-plane (9), we get:

Corollary 3 *If $P \in P_n$, then for every real or complex number α with $|\alpha| \leq 1$, $R > r \geq 1$ and $|z| \geq 1$,*

$$\begin{aligned} &|RP'(Rz) - \alpha r P'(rz)| \\ &\leq n \left| R^n - \alpha r^n \right| |z|^{n-1} \max_{|z|=1} |P(z)| . \end{aligned} \quad (15)$$

The result is sharp and equality in (15) holds for $P(z) = \lambda z^n, \lambda \neq 0$.

If we divide the two sides of (15) by $R - r$ with $\alpha = 1$ and let $R \to r$, we get for $r \geq 1$ and $|z| \geq 1$,

$$\begin{aligned} &|P'(rz) + rzP''(rz)| \\ &\leq n^2 r^{n-1} |z|^{n-1} \max_{|z|=1} |P(z)| \end{aligned}$$

The result is best possible.

By setting $\lambda_1 = \lambda_2 = 0$ in (13), it follows that if $P \in P_n$, then for every real or complex number α with $|\alpha| \leq 1$,$R > r \geq 1$ and $|z| \geq 1$,

$$\begin{aligned} &|P(Rz) - \alpha P(rz)| \\ &\leq \left| R^n - \alpha r^n \right| |z|^n \max_{|z|=1} |P(z)| . \end{aligned} \quad (16)$$

Equality in (16) holds for $P(z) = \lambda z^n$, $\lambda \neq 0$.

Inequality (16) is equivalent to inequality (5) for $r = 1$. For $\alpha = 0$, inequality (16) includes inequality (2) as a special case. If we divide the two sides of the inequality (17) by $R - r$ with $\alpha = 1$ and make $R \to r$, we get for $r \geq 1$, $|z| \geq 1$,

$$\left|P'(rz)\right| \leq n r^{n-1} |z|^{n-1} \max_{|z|=1} |P(z)| ,$$

which, in particular, yields inequality (1) as a special case.

Next we use Theorem 1 to prove the following result.

Theorem 4 *If $P \in P_n$, then for every real or complex number α with $|\alpha| \leq 1$, $R > r \geq 1$ and for $|z| \geq 1$,*

$$\begin{aligned} &|B[P(Rz)] - \alpha B[P(rz)]| + |B[Q(Rz)] - \alpha B[Q(rz)]| \\ &\leq (|R^n - \alpha r^n||B[z^n]| \\ &\quad + |1 - \alpha||\lambda_0|) Max_{|z|=1} |P(z)| \end{aligned} \quad (17)$$

where $Q(z) = z^n\overline{P(1/\bar{z})}$ and $B \in B_n$.The result is best possible and equality in (17) holds for $P(z) = \lambda z^n$, $\lambda \neq 0$.

Remark 5 *Theorem 4 includes some well known polynomial inequalities as special cases. For example, inequality (17) reduces to a result due to Q. I. Rahman (see [14, inequality (5.2)]) for $\alpha = 0$.*

If we choose $\lambda_0 = \lambda_2 = 0$ in (17), we obtain:

Corollary 6 *If $P \in P_n$, then for every real or complex number α with $|\alpha| \leq 1$, $R > r \geq 1$ and $|z| \geq 1$,*

$$\left|RP'(Rz) - \alpha rP'(rz)\right| + \left|RQ'(Rz) - \alpha rQ'(rz)\right| \leq n\left|R^n - \alpha r^n\right| |z|^{n-1} \max_{|z|=1} |P(z)|. \quad (18)$$

Equality in (18) holds for $P(z) = \lambda z^n, \lambda \neq 0$.

If we divide the two sides of (18) by $R - r$ with $\alpha = 1$ and let $R \to r$, we get:

Corollary 7 *If $P \in P_n$, then for every α with $|\alpha| \leq 1$, $R > r \geq 1$ and $|z| \geq 1$,*

$$|P'(rz) + rzP''(rz)| + |Q'(rz) + rzQ''(rz)| \leq n^2 r^{n-1} |z|^{n-1} Max_{|z|=1} |P(z)|$$

where $Q(z) = z^n\overline{P(1/\bar{z})}$.

For $\lambda_1 = \lambda_2 = 0$ and $\alpha = 1$, Theorem 4 includes a result due to A. Aziz and Rather [7] as a special case.

For the class of polynomials $P \in P_n$ having no zero in $|z| < 1$, inequality (12) can be improved. In this direction, we present the following result which is a compact generalization of the inequalities (3), (4), (6) and (12).

Theorem 8 *If $P \in P_n$ and $P(z)$ does not vanish in $|z| < 1$, then for every real or complex number α with $|\alpha| \leq 1$, $R > r \geq 1$ and $|z| \geq 1$,*

$$|B[P(Rz)] - \alpha B[P(rz)]| \leq \frac{|R^n - \alpha r^n||B[z^n]| + |1-\alpha||\lambda_0|}{2} \max_{|z|=1} |P(z)| \quad (19)$$

where $B \in B_n$. The result is best possible and equality in (19) holds for $P(z) = az^n + b, |a| = |b| = 1$.

Substituting for $B[P(z)]$ (19), we get for every real or complex α with $|\alpha| \leq 1$, $R > r \geq 1$ and for $|z| \geq 1$,

$$\left|\sum_{j=0}^{2} \lambda_j \left(\frac{nz}{2}\right)^j \frac{\left(P^{(j)}(Rz) - \alpha P^{(j)}(rz)\right)}{j!}\right| \leq \frac{1}{2}[|R^n - \alpha r^n|\,|\sum_{j=0}^{2} \lambda_j \left(\frac{n}{2}\right)^j C(n,j)||z|^n + |1-\alpha||\lambda_0|] \max_{|z|=1} |P(z)|, \quad (20)$$

where λ_0, λ_1 and λ_2 are such that all the zeros of $u(z)$ defined by (8) lie in the half-plane (9).

Remark 9 *For $\alpha = 0$, inequality (11) is a special case of inequality (20). If we choose $\lambda_0 = \lambda_2 = 0$ in (20) and note that in this case all the zeros of $u(z)$ defined by (8) lie in the half-plane defined by (9), it follows that if $P(z) \neq 0$ in $|z| < 1$, then for $R > r \geq 1$ and $|z| \geq 1$,*

$$|RP'(Rz) - \alpha rP'(rz)| \leq n\frac{|R^n - \alpha r^n|}{2} |z|^{n-1} \max_{|z|=1} |P(z)| \quad (21)$$

Setting $\alpha = 0$ in (21), we obtain for $|z| \geq 1$ and $R > 1$,

$$\left|P'(Rz)\right| \leq \frac{n}{2} R^{n-1} |z|^{n-1} \max_{|z|=1} |P(z)|$$

which, in particular, gives inequality (3).

Next choosing $\lambda_1 = \lambda_2 = 0$ in (7), we get

$$|P(Rz) - \alpha P(rz)| \leq \frac{|R^n - \alpha r^n||z|^n + |1-\alpha|}{2} \max_{|z|=1} |P(z)|. \quad (22)$$

for $R > r \geq 1$ and $|z| \geq 1$. The result is sharp and equality in (22) holds for $P(z) = az^n + b, |a| = |b| = 1$.

Inequality (22) is a compact generalization of the inequalities (3), (4) and (6).

A polynomial $P \in P_n$ is said to be self- inversive if $P(z) = Q(z)$ where $Q(z) = n^n\overline{P(1/\bar{z})}$. It is known [12,20] that if $P \in P_n$ is a self-inversive polynomial, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|. \quad (23)$$

Here we also establish the following result for self-inversive polynomials.

Theorem 10 *If $P \in P_n$ is a self-inversive polynomial, then for every real or complex number α with $|\alpha| \leq 1$, $R > r \geq 1$ and $|z| \geq 1$,*

$$|B[P(Rz)] - \alpha B[P(rz)]| \leq \frac{|R^n - \alpha r^n||B[z^n]| + |1-\alpha||\lambda_0|}{2} \max_{|z|=1} |P(z)| \quad (24)$$

where $B \in B_n$. The result is best possible and equality in (24) holds for $P(z) = z^n + 1$.

The following result immediately follows from of Theorem 10 by taking $\alpha = 0$.

Corollary 11 *If $P \in P_n$ is a self-inversive polynomial, then for $R > 1$ and $|z| \geq 1$*

$$|B[P](Rz)| \leq \frac{|B[R^n z^n]| + |\lambda_0|}{2} \max_{|z|=1} |P(z)| \quad (25)$$

where $B \in B_n$. The result is sharp as shown by the polynomial $P(z) = z^n + 1$.

Corollary 11 includes a result due to Shah and Liman [21] as a special case.

Next choosing $\lambda_1 = \lambda_2 = 0$ in (24), we immediately get

Corollary 12 *If $P \in P_n$ is a self-inversive polynomial, then for every real or complex number α with $|\alpha| \leq 1$, $R > r \geq 1$ and $|z| \geq 1$,*

$$\begin{aligned} &|P(Rz) - \alpha P(rz)| \\ &\leq \tfrac{|R^n - \alpha r^n||z|^n + |1-\alpha|}{2} \max_{|z|=1} |P(z)| \end{aligned} \quad (26)$$

The result is sharp and equality in (26) holds for $P(z) = az^n + b, |a| = |b| = 1$.

Inequality (26) contains inequality (23) as special case. If we divide the two sides of (26) by $R - r$ with $\alpha = 1$ and let $R \to r$, we get

$$|P'(rz)| \leq \frac{n}{2} r^{n-1} |z|^{n-1} \max_{|z|=1} |P(z)|$$

for $r \geq 1$ and $|z| \geq 1$.

Above inequality reduces to inequality (23) for $r = 1$. Further for $\alpha = 0$, inequality (26) gives

$$\max_{|z|=R>1} |P(z)| \leq \frac{R^n + 1}{2} \max_{|z|=1} |P(z)|.$$

Setting $\lambda_0 = \lambda_1 = 0$ in (24) and note that in this case all the zeros of $u(z)$ defined by (8) lie in the half-plane $|z| < |z - z/n|$, it follows that if $P \in P_n$ is a self-inversive polynomial, then for every real or complex number α with $|\alpha| \leq 1$, $R > r \geq 1$ *and* $|z| \geq 1$,

$$\begin{aligned} &|R^2 P''(Rz) - \alpha r^2 P''(rz)| \\ &\leq \tfrac{n(n-1)}{2} |R^n - \alpha r^n||z|^{n-2} \max_{|z|=1} |P(z)|. \end{aligned}$$

For $\alpha = 0$, this inequality gives, for self-inversive polynomials $P \in P_n$,

$$|P''(Rz)| \leq \frac{n(n-1)}{2} R^{n-2} |z|^{n-2} \max_{|z|=1} |P(z)|$$

for $R \geq 1$ and $|z| \geq 1$. The result is best possible and equality holds for $P(z) = z^n + 1$.

Remark 13 *Many other interesting results can be deduced from Theorem 10 in the same way as have been deduced from Theorem 1 and Theorem 4.*

For the class of polynomials $P \in P_n$, having all their zeros in $|z| \leq 1$, we have

$$\min_{|z|=1} |P'(z)| \geq n \min_{|z|=1} |P(z)| \quad (27)$$

and

$$\min_{|z|=R>1} |P(z)| \geq R^n \min_{|z|=1} |P(z)|. \quad (28)$$

Inequalities (27) and (28) are due to A. Aziz and Q. M. Dawood [3]. Both the results are sharp and equality in (27) and (28) holds for $P(z) = \lambda z^n, \lambda \neq 0$.

As a compact generalization of inequalities (27) and (28), Rather [17] proved that if P(z) is a polynomial of degree n having all its zeros in $|z| \leq 1$, then for every real or complex number α with $|\alpha| \leq 1, R > 1$ and $|z| \geq 1$,

$$|P(Rz) - \alpha P(z)| \geq |R^n - \alpha| \min_{|z|=1} |P(z)|. \quad (29)$$

The result is sharp and equality in (29) holds for $P(z) = \lambda z^n, \lambda \neq 0$.

Finally in this paper we present the following result.

Theorem 14 *If $P \in P_n$ and P(z) has all its zeros in $|z| \leq 1$, then for every real or complex number α with $|\alpha| \leq 1, R > r \geq 1$ and $|z| \geq 1$,*

$$\begin{aligned} &|B[P(Rz)] - \alpha B[P(rz)]| \\ &\geq |R^n - \alpha r^n||B[z^n]| \min_{|z|=1} |P(z)|. \end{aligned} \quad (30)$$

where $B \in B_n$.The result is best possible and equality in (30) holds for $P(z) = \lambda z^n, \lambda \neq 0$.

Substituting for $B[P](z)$, we get, from (30), for every real or complex α with $|\alpha| \leq 1$, $R > r \geq 1$ and for $|z| \geq 1$,

$$\begin{aligned} &\left| \sum_{j=0}^{2} \lambda_j \left(\frac{nz}{2}\right)^j \frac{\left(P^{(j)}(Rz) - \alpha P^{(j)}(rz)\right)}{j!} \right| \\ \geq\; & |R^n - \alpha r^n| \, |z|^n \times \\ &\left| \sum_{j=0}^{2} \lambda_j \left(\frac{n}{2}\right)^j C(n,j) \right| \min_{|z|=1} |P(z)| \end{aligned} \quad (31)$$

where λ_0, λ_1 and λ_2 are such that all the zeros of $u(z)$ defined by (8) lie in the half plane (9).

Remark 15 *For $\alpha = 0$, from inequality (30), we have for $|z| \geq 1$ and $R > 1$,*

$$|B[P](Rz)| \geq |B[R^n z^n]| \min_{|z|=1} |P(z)| \quad (32)$$

where $B \in B_n$.The result is best possible.

Taking $\lambda_0 = \lambda_2 = 0$ in (31) and noting that all the zeros of u(z) defined by (8) lie in the half plane (9), we get

Corollary 16 *If $P \in P_n$ and $P(z)$ has all its zeros in $|z| \leq 1$, then for every real or complex number α with $|\alpha| \leq 1, R > r \geq 1$ and $|z| \geq 1$,*

$$\begin{aligned} &|RP'(Rz) - \alpha rP'(rz)| \\ &\geq n\,|R^n - \alpha r^n|\,|z|^{n-1} \min_{|z|=1} |P(z)|\,. \end{aligned} \tag{33}$$

The result is sharp and extremal polynomial is $P(z) = \lambda z^n, \lambda \neq 0$.

If we divide the two sides of (33) by $R - r$ with $\alpha = 1$ and let $R \to r$, we get for $r \geq 1$ and $|z| \geq 1$,

$$\begin{aligned} &|P'(rz) + rzP''(rz)| \\ &\geq n^2 r^{n-1} |z|^{n-1} \min_{|z|=1} |P(z)|\,. \end{aligned}$$

The result is best possible.

Next setting $\lambda_1 = \lambda_2 = 0$ in (31), we obtain

Corollary 17 *If $P \in P_n$ and $P(z)$ has all its zeros in $|z| \leq 1$, then for every real or complex number α with $|\alpha| \leq 1, R > r \geq 1$ and $|z| \geq 1$,*

$$\begin{aligned} &|P(Rz) - \alpha P(rz)| \\ &\geq |R^n - \alpha r^n|\,|z|^n \min_{|z|=1} |P(z)|\,. \end{aligned} \tag{34}$$

The result is best possible and equality in (34) holds for $P(z) = \lambda z^n$, $\lambda \neq 0$.

Inequality (34) includes inequality (29) as a special case.

2 Lemmas

For the proofs of these theorems, we need the following lemmas.

Lemma 18 *If $P \in P_n$ and $P(z)$ has all its zeros in $|z| \leq 1$, then for $R > r \geq 1$ and $|z| = 1$,*

$$|P(Rz)| > |P(rz)|\,.$$

Proof: Since all the zeros of $P(z)$ lie in $|z| \leq 1$, we can write

$$P(z) = C \prod_{j=1}^{n} (z - r_j e^{i\theta_j})$$

where $r_j \leq 1$. Now for $0 \leq \theta < 2\pi$ and $R \geq r \geq 1$, we have

$$\begin{aligned} &\left|\frac{Re^{i\theta} - r_j e^{i\theta_j}}{re^{i\theta} - r_j e^{i\theta_j}}\right|^2 \\ &= \left\{\frac{R^2 + r_j^2 - 2Rr_j Cos(\theta - \theta_j)}{r^2 + r_j^2 - 2rr_j Cos(\theta - \theta_j)}\right\} \geq \left(\frac{R + r_j}{r + r_j}\right)^2 \end{aligned}$$

if

$$\frac{R^2 + r_j^2 - 2Rr_j \cos(\theta - \theta_j)}{r^2 + r_j^2 - 2rr_j \cos(\theta - \theta_j)} \geq \frac{R^2 + r_j^2 - 2Rr_j}{r^2 + r_j^2 - 2rr_j},$$

or, if

$$\begin{aligned} &\left(R^2 + r_j^2 - 2Rr_j \cos(\theta - \theta_j)\right)\left(r^2 + r_j^2 - 2rr_j\right) \\ &\geq \left(r^2 + r_j^2 - 2rr_j \cos(\theta - \theta_j)\right)\left(R^2 + r_j^2 - 2Rr_j\right), \end{aligned}$$

that is ,if

$$\begin{aligned} &\left\{2rr_j(R^2 + r_j^2) - 2Rr_j(r^2 + r_j^2)\right\}\cos(\theta - \theta_j) \\ &\geq 2Rr_j(r^2 + r_j^2) - 2rr_j(R^2 + r_j^2). \end{aligned}$$

Equivalently,if

$$(R-r)(R^2 - rr_j)\cos(\theta - \theta_j) \geq -(R-r)(R^2 - rr_j).$$

That is, if

$$\cos(\theta - \theta_j) \geq -1,$$

which is true. Hence for $0 \leq \theta < 2\pi$ and $R > r \geq 1$,

$$\begin{aligned} \left|\frac{P(Re^{i\theta})}{P(re^{i\theta})}\right| &= \prod_{j=1}^{n} \left|\frac{Re^{i\theta} - r_j e^{i\theta_j}}{re^{i\theta} - r_j e^{i\theta_j}}\right| \\ \geq \quad &\left(\frac{R + r_j}{r + r_j}\right)^n \geq \left(\frac{R+1}{r+1}\right)^n, \end{aligned}$$

which implies

$$\left|P(Re^{i\theta})\right| \geq \left(\frac{R+1}{r+1}\right)^n \left|P(re^{i\theta})\right| \tag{35}$$

for $0 \leq \theta < 2\pi$ and $R > r \geq 1$. Since $f(Re^{i\theta}) \neq 0$ for $R > r \geq 1$ and $R + 1 > r + 1$, it follows from (35) that

$$\left|P(Re^{i\theta})\right| > \left(\frac{r+1}{R+1}\right)^n \left|P(Re^{i\theta})\right| \geq \left|P(re^{i\theta})\right|$$

for $0 \leq \theta < 2\pi$ and $R > r \geq 1$. This implies

$$|P(Rz)| > |P(rz)|$$

for every $R > r \geq 1$ and $|z| = 1$, which completes the proof of the Lemma 18.

The next lemma follows from Corollary 18.3 of [10, p. 86].

Lemma 19 *If $P \in P_n$ and $P(z)$ has all its zeros in $|z| \leq 1$, then all the zeros of $B[P(z)]$ also lie in $|z| \leq 1$.*

Lemma 20 *If $P \in P_n$ and $P(z)$ does not vanish in $|z| < 1$, then for every real or complex number α with $|\alpha| \leq 1$ and $R > r \geq 1$,.*

$$|B[P(Rz)] - \alpha B[P(rz)]| \leq |B[Q(Rz)] - \alpha B[Q(rz)]| \tag{36}$$

where $Q(z) = z^n \overline{P(1/\bar{z})}$. The result is sharp and equality in (36) holds for $P(z) = z^n + 1$.

Proof: Since the polynomial $P(z)$ has all its zeros in $|z| \geq 1$, therefore, for every real or complex number β with $|\beta| > 1$, the polynomial $f(z) = P(z) - \beta Q(z)$ where $Q(z) = z^n \overline{P(1/\bar{z})}$, has all its zeros in $|z| \leq 1$. Applying Lemma 18 to the polynomial $f(z)$, we obtain for every $R > r \geq 1$,

$$|f(rz)| < |f(Rz)| \quad for \quad |z| = 1.$$

Using Rouche's theorem and noting that all the zeros of $f(Rz)$ lie in $|z| \leq (1/R) < 1$, we conclude that the polynomial

$$g(z) = f(Rz) - \alpha f(rz)$$

has all its zeros in $|z| < 1$ for every real or complex number α with $|\alpha| \leq 1$. Applying Lemma 19 to the polynomial $g(z)$ and noting that B is a linear operator, it follows that all the zeros of polynomial

$$\begin{aligned} T(z) &= B[g](z) \\ &= (B[P(Rz)] - \alpha B[P(rz)]) \\ &\quad -\beta(B[Q(Rz)] - \alpha B[Q(rz)]) \end{aligned} \tag{37}$$

lie in $|z| < 1$ for all real or complex numbers α, β with $|\alpha| \leq 1, |\beta| > 1$ and $R > r \geq 1$. This implies

$$|B[P(Rz)] - \alpha B[P(rz)]| \leq |B[Q(Rz)] - \alpha B[Q(rz)]| \tag{38}$$

for $|z| \geq 1$. If inequality (38) is not true, then there is a point $z = w$ with $|w| \geq 1$ such that

$$|\{B[P(Rz)] - \alpha B[P(rz)]\}_{z=w}| > |\{B[Q(Rz)] - \alpha B[Q(rz)]\}_{z=w}|.$$

But all the zeros of $Q(z)$ lie in $|z| \leq 1$, therefore, it follows (as in case of $f(z)$) that all the zeros of $Q(Rz) - \alpha Q(rz)$ lie in $|z| < 1$. Hence by Lemma 19, all the zeros of $B[Q(Rz)] - \alpha B[Q(rz)]$ lie in $|z| < 1$ so that $B[Q(Rz)] - \alpha B[Q(rz)]_{z=w} \neq 0$. We take

$$\beta = \frac{\{B[P(Rz)] - \alpha B[P(rz)]\}_{z=w}}{\{B[Q(Rz)] - \alpha B[Q(rz)]\}_{z=w}},$$

then β is a well defined real or complex number with $|\beta| > 1$ and with this choice of β, from (37), we obtain $T(w) = 0$ where $|w| \geq 1$. This contradicts the fact that all the zeros of $T(z)$ lie in $|z| < 1$. Thus

$$|B[P(Rz)] - \alpha B[P(rz)]| \leq |B[Q(Rz)] - \alpha B[Q(rz)]|$$

for every α with $|\alpha| \leq 1$ and $R > r \geq 1$. This proves Lemma 20.

3 Proofs of the Theorems

Proof of Theorem 1: Let $M = \max_{|z|=1} |P(z)|$, then

$$|P(z)| \leq M \quad for \quad |z| = 1.$$

By Rouche's theorem, it follows that all the zeros of polynomial $F(z) = P(z) - \lambda z^n M$ lie in $|z| < 1$ for every real or complex number λ with $|\lambda| > 1$. Therefore, by Lemma 18, we have for $R > r \geq 1$,

$$|F(rz)| < |F(Rz)| \quad for \quad |z| = 1.$$

Since all the zeros of polynomial $F(Rz)$ lie in $|z| \leq (1/R) < 1$, applying Rouche's theorem again, we conclude that all the zeros of polynomial $G(z) = F(Rz) - \alpha F(rz)$ lie in $|z| < 1$ for every real or complex α with $|\alpha| \leq 1$. Hence by Lemma 19, the polynomial

$$\begin{aligned} L(z) &= B[G(z)] \\ &= B[F(Rz)] - \alpha B[F(rz)] \\ &= (B[P(Rz)] - \alpha B[P(rz)]) \\ &- \lambda(R^n - \alpha r^n) B[z^n] M \end{aligned} \tag{39}$$

has all its zeros in $|z| < 1$ for every real or complex number λ with $|\lambda| > 1$. This implies

$$|B[P(Rz)] - \alpha B[P(rz)]| \leq |R^n - \alpha r^n||B[z^n]|M \tag{40}$$

for $|z| \geq 1$ and $R > r \geq 1$. If inequality (40) is not true, then there is a point $z = w$ with $|w| \geq 1$ such that

$$|\{B[P(Rz)] - \alpha B[P(rz)]\}_{z=w}| > |R^n - \alpha r^n||\{B[z^n]\}_{z=w}|M$$

for $|z| \geq 1$. Since $(B[z^n])_{z=w} \neq 0$. we take

$$\lambda = \frac{\{B[P(Rz)] - \alpha B[P(rz)]\}_{z=w}}{(R^n - \alpha r^n)\{B[z^n]\}_{z=w}},$$

so that λ is a well defined real or complex number with $|\lambda| > 1$ and with this choice of λ, from (39), we get $L(w) = 0$ where $|w| \geq 1$, which is clearly a contradictions to the fact that all the zeros of $L(z)$ lie in $|z| < 1$. Thus for every real or complex number α with $|\alpha| \leq 1$, $R > r \geq 1$ and $|z| \geq 1$,

$$|B[P(Rz)] - \alpha B[P(rz)]| \leq |R^n - \alpha r^n||B[z^n]|M$$

This completes the proof of Theorem 1.

Proof of Theorem 4: Let $M = \max_{|z|=1} |P(z)|$, then

$$|P(z)| \leq M \;\; for \;\; |z| = 1.$$

If μ is any real or complex number with $|\mu| > 1$, then by Rouche's theorem, the polynomial

$$F(z) = P(z) - \mu M$$

does not vanish in $|z| < 1$. Applying Lemma 20 to the polynomial $F(z)$ and noting the fact that B is a linear operator, it follows that for every real or complex number α with $|\alpha| \leq 1, R > r \geq 1$,

$$|B[F(Rz)] - \alpha B[F(rz)]| \leq |B[H(Rz)] - \alpha B[H(rz)]|$$

for $|z| \geq 1$ where

$$\begin{aligned} H(z) &= z^n\overline{F(1/\bar{z})} = z^n\overline{P(1/\bar{z})} - \bar{\mu}z^n M \\ &= Q(z) - \bar{\mu}z^n M. \end{aligned}$$

Using the fact that $B[1] = \lambda_0$, we obtain

$$|B[P(Rz)] - \alpha B[P(rz)] - \mu(1-\alpha)\lambda_0 M| \leq |B[Q(Rz)] - \alpha B[Q(rz)] - \bar{\mu}(R^n - \alpha r^n)B[z^n]M| \tag{41}$$

for all real or complex numbers α, μ with $|\alpha| \leq 1, |\mu| > 1, R > r \geq 1$ and $|z| \geq 1$. Now choosing the argument of μ such that

$$|B[Q(Rz)] - \alpha B[Q(rz)] - \bar{\mu}(R^n - \alpha r^n)B[z^n]M| = |\mu||R^n - \alpha r^n||B[z^n]|M - |B[Q(Rz)] - \alpha B[Q(rz)]|,$$

which is possible by Theorem 1, we get from (41), for $|\mu| > 1$, and $|z| \geq 1$.

$$|B[P(Rz)] - \alpha B[P(rz)]| + |B[Q(Rz)] - \alpha B[Q(rz)]| \leq |\mu|(|R^n - \alpha r^n||B[z^n]| + |1-\alpha||\lambda_0|)\max_{|z|=1}|P(z)|.$$

Letting $|\mu| \to 1$, we obtain

$$|B[P(Rz)] - \alpha B[P(rz)]| + |B[Q(Rz)] - \alpha B[Q(rz)]| \leq (|R^n - \alpha r^n||B[z^n]| + |1-\alpha||\lambda_0|)\max_{|z|=1}|P(z)|.$$

This proves of Theorem 4.

Proof of Theorem 8: Lemma 20 and Theorem 4 together yields, for every real or complex number α with $|\alpha| \leq 1$, $R > r \geq 1$ and $|z| \geq 1$,

$$\begin{aligned} &2|B[P(Rz)] - \alpha B[P(rz)]| \\ &\leq |B[P(Rz)] - \alpha B[P(rz)]| + |B[Q(Rz)] - \alpha B[Q(rz)]| \\ &\leq (|R^n - \alpha r^n||B[z^n]| + |1-\alpha||\lambda_0|)\max_{|z|=1}|P(z)|, \end{aligned}$$

which is equivalent to (19) and this completes the proof of Theorem 8.

Proof of Theorem 10: By hypothesis $P \in P_n$ is a self-inversive polynomial, therefore, for all $z \in C$.

$$|B[P(Rz)] - \alpha B[P(rz)]| = |B[Q(Rz)] - \alpha B[Q(rz)]|.$$

Combining this with Theorem 4, we get for every real or complex number α with $|\alpha| \leq 1, R > r \geq 1$ and $|z| \geq 1$,

$$\begin{aligned} &2|B[P(Rz)] - \alpha B[P(rz)]| \\ &= |B[P(Rz)] - \alpha B[P(rz)]| + |B[Q(Rz)] - \alpha B[Q(rz)]| \\ &\leq (|R^n - \alpha r^n||B[z^n]| + |1-\alpha||\lambda_0|)\max_{|z|=1}|P(z)|, \end{aligned}$$

which immediately leads to the desired result and this completes the proof of Theorem 10.

Proof of Theorem 14: Let $m = \min_{|z|=1} |P(z)|$, then

$$m|z|^n \leq |P(z)| \;\; for \;\; |z| = 1.$$

We first show that the polynomial $F(z) = P(z) - \delta m z^n$ has all its zeros in $|z| \leq 1$ for every real or complex number δ with $|\delta| < 1$. This is clear if $m = 0$. Henceforth we assume that all the zeros of $P(z)$ lie in $|z| < 1$, then $m > 0$ and it follows by Rouche's theorem that the polynomial $F(z) = P(z) - \beta m z^n$ has all its zeros in $|z| < 1$ for every real or complex number δ with $|\delta| < 1$. Applying Lemma 18 to the polynomial $F(z)$, we get

$$|F(rz)| < |F(Rz)|$$

for $|z| = 1$ and $R > r \geq 1$. Using Rouche's theorem, we conclude that all the zeros of polynomial

$$G(z) = F(Rz) - \alpha F(rz)$$

lie in $|z| < 1$ for every real or complex number α with $|\alpha| \leq 1$ and $R > r \geq 1$. Applying Lemma 19 to the polynomial $G(z)$ and noting that B is a linear operator, it follows that all the zeros of the polynomial

$$\begin{aligned} T(z) &= B[G(z)] = B[F(Rz)] - \alpha B[F(rz)] \\ &= B[P(Rz)] - \alpha B[P(rz)] - \delta\left(R^n - \alpha r^n\right) B[z^n] \end{aligned} \tag{42}$$

lie in $|z| < 1$ for every real or complex number δ with $|\delta| < 1$ and $R > r \geq 1$, which implies

$$|B[P(Rz)] - \alpha B[P(rz)]| \geq m\,|R^n - \alpha r^n|\,|B[z^n]|.$$

for $|z| \geq 1$. If above inequality is not true, then there is a point $z = w$ with $|w| \geq 1$ such that

$$\begin{aligned} &|\{B[P(Rz)] - \alpha B[P(rz)]\}_{z=w}| \\ &< m\,|R^n - \alpha r^n|\,|\{B[z^n]\}_{z=w}|. \end{aligned}$$

Since all the zeros of $B[z^n]$ lie in $|z| < 1$, therefore, $\{B[z^n]\}_{z=w} \neq 0$. We take

$$\delta = \frac{\{B[P(Rz)] - \alpha B[P(rz)]\}_{z=w}}{m(R^n - \alpha r^n)\,\{B[z^n]\}_{z=w}},$$

then δ is well defined real or complex number with $|\delta| < 1$ and with choice of δ, from (42) we get, $T(w) = 0$ with $|w| \geq 1$, which contradicts the fact that all the zeros of $T(z)$ lie in $|z| < 1$. Thus

$$|B[P(Rz)] - \alpha B[P(rz)]| \geq m\,|R^n - \alpha r^n|\,|B[z^n]|$$

for every real or complex number α with $|\alpha| \leq 1$, $R > r \geq 1$ and $|z| \geq 1$. This completes the proof of Theorem 14.

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