Positive Solutions of Operator Equations and Nonlinear Beam Equations with a Perturbed Loading Force

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Abstract: In this paper we are concerned with the existence and uniqueness of positive solutions for an operator equation $x = Ax + \lambda Bx$ on an order Banach space, where A and B are nonlinear operators and λ is a parameter. By properties of cones we obtain that there exists a $\lambda^* > 0$ such that the operator equation has a unique positive solution which is increasing in λ for $\lambda \in [0, \lambda^*]$, and further, we give an estimate for λ^*. In addition, we discuss the existence and uniqueness of positive solutions for an elastic beam equation with three parameters and one perturbed loading force.

Key–Words: Nonlinear operator equation; positive solution; elastic beam equation; perturbed loading force

1 Introduction and Preliminaries

It is well known that nonlinear operator equations defined on a cone in Banach spaces play an important role in theory of nonlinear differential and integral equations and has been extensively studied over the past several decades (see [1]-[14]).

In this paper, we consider the operator equation on a Banach space E

$$x = \lambda Ax + Bx, \tag{1}$$

where A is an increasing convexity operator, B is a increasing concavity operator and λ is a parameter.

Many nonlinear problems with a parameter, such as initial value problems, boundary value problems, and impulsive problems, can be transformed into Eq.(1), which shows the importance to study the operator equation (1) both in theory and applications. There are many recent discussions to positive solutions of operator equations. For example [1], [2], [3]-[9] and [10]-[12] investigated operator equations $Ax = \lambda x$ $(\lambda > 0)$, $Ax = x$ and $Ax + Bx = x$, respectively, which are special forms of the Eq.(1). To our knowledge, little has been done on the Eq.(1) in literature, especially on the solution's dependence on the parameter λ, thus it is worthwhile doing this work.

By properties of cones, we study the existence and uniqueness of the positive solutions for the operator equation (1). Moreover we find the value λ^* such that the operator equation has a unique positive solution for $\lambda \in [0, \lambda^*]$, on the other hand, we discuss the elastic beam equation with three parameters and a perturbed loading force, and obtain the concrete interval I such that the problem has a unique positive solution for the parameter $\lambda \in I$. It may be the first time that the simply supported beam equation with three parameters and one perturbed loading force is studied.

Let E be a real Banach space which is partially ordered by a cone P of E, i.e., $x \leq y$ if $y - x \in P$. By θ we denote the zero element of E.

Recall that a nonempty closed convex set $P \subset E$ is a cone if it satisfies

$$\forall x \in P, r \geq 0 \Rightarrow \ rx \in P;$$
$$x \in P, -x \in P \Rightarrow \ x = \theta.$$

Recall that a cone P is said to be normal if there exists a positive number N, called the normal constant of P, such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$.

For a given $e > \theta$, that is, $e \geq \theta$ and $e \neq \theta$, let

$$P_e = \{x \in E \mid \text{there exist } \tau_1(x) > 0 \text{ and } \tau_2(x) > 0 \text{ such that } \tau_1(x)e \leq x \leq \tau_2(x)e\}. \tag{2}$$

Then it is easy to see that

(a) $P_e \subset P$;

(b) for any given $x, y \in P_e$, there exist $0 < \tau_1^* \leq 1 \leq \tau_2^* < \infty$ such that $\tau_1^* y \leq x \leq \tau_2^* y$.

Let $D \subseteq E$ and P be a cone of E. An operator $T : D \to E$ is said to be increasing if for $x_1, x_2 \in D$, with $x_1 \leq x_2$ we have $Tx_1 \leq Tx_2$.

An element $x^* \in D$ is called a fixed point of T if $Tx^* = x^*$.

All the concepts discussed above can be found in [13].

Lemma 1. *Suppose that P is a normal cone of E and $T : P \to P$ be an increasing operator. Assume that*

(L1) there exist $y_0, z_0 \in P_e$ with $y_0 \le z_0$ such that $y_0 \le Ty_0, Tz_0 \le z_0$;

(L2) for any $t \in (0, 1)$, there exists $\eta(t) > 0$ such that

$$T(tx) \ge t(1 + \eta(t))Tx, \ x \in [y_0, z_0].$$

Then the following statements hold

(a) T has a unique fixed point $x^* \in [y_0, z_0]$;

(b) T has not any fixed point in $P_e \setminus [y_0, z_0]$;

(c) for any $u_0 \in P_e$, the sequence $\{u_n, n \ge 1\}$ generated by $u_n = Tu_{n-1}$ has limit x^*, i.e., $\lim_{n\to\infty} \|u_n - x^*\| = 0$.

Proof. Set $y_n = Ty_{n-1}$ and $z_n = Tz_{n-1}$ for $n = 1, 2, \cdots$. The condition (L1) and the fact that T is increasing yield to

$$\begin{aligned} y_0 \le y_1 \le \cdots \le y_n \le \cdots \\ \le z_n \le \cdots \le z_1 \le z_0. \end{aligned} \tag{3}$$

Let

$$\mu_n = \sup\{\tau > 0 | y_n \ge \tau z_n\}, n = 1, 2, \cdots . \tag{4}$$

In view of the property (b) of P_e we get

$$0 < \mu_n \le 1, y_n \ge \mu_n z_n, n = 1, 2, \cdots . \tag{5}$$

From (3) and (5) we infer that

$$0 < \mu_0 \le \mu_1 \le \cdots \le \mu_n \le \cdots \le 1,$$

which means that $\lim_{n\to\infty} \mu_n = \mu \le 1$. We assert that $\mu = 1$. If it is not true, i.e., $0 < \mu_n \le \mu < 1$ for $n \ge 1$, then by (L2) and (3) we deduce that

$$\begin{aligned} y_{n+1} &= Ty_n \ge T(\mu_n z_n) \ge T(\frac{\mu_n}{\mu}\mu z_n) \\ &\ge \frac{\mu_n}{\mu}T(\mu z_n) \ge \mu_n\Big(1 + \eta(\mu)\Big)z_{n+1}. \end{aligned}$$

By (4), we have

$$\mu_{n+1} \ge \mu_n(1 + \eta(\mu)), \ n = 0, 1, 2, \cdots ,$$

and

$$\mu_{n+1} \ge \mu_0(1 + \eta(\mu))^{n+1}, \ n = 0, 1, 2, \cdots .$$

This gives rise to the contradiction $1 > \mu \ge +\infty$.

Note that P is normal. By (3) and (5) we have

$$\|z_n - y_n\| \le N(1 - \mu_n)\|z_0\| \to 0 \text{ as } n \to \infty,$$

which implies that both $\{y_n\}$ and $\{z_n\}$ are Cauchy sequences, and there exist $y^*, z^* \in P$ such that $y_n \to y^*$, $z_n \to z^*$ and $y_n \le y^* \le z^* \le z_n$. Thus $y^* = z^* \in [y_0, z_0]$. Since T is increasing, then

$$y_n \le y_{n+1} \le Ty^* = Tz^* \le z_{n+1} \le z_n.$$

Therefore, we have

$$y^* \le Ty^* = Tz^* \le z^*,$$

which implies that y^* is a fixed point of T in $[y_0, z_0]$.

Now, let y_* is a fixed point of T in $[y_0, z_0]$ and

$$\tilde{\mu} = \sup\{\tau > 0 | y^* \ge \tau y_*\}.$$

Then $0 < \tilde{\mu} \le 1$ and $y^* \ge \tilde{\mu} y_*$. If $\tilde{\mu} \ne 1$, we have

$$y^* = Ty^* \ge T(\tilde{\mu} y_*) \ge \tilde{\mu}(1 + \eta(\tilde{\mu}))y_*,$$

which implies that $\tilde{\mu} \ge \tilde{\mu}(1 + \eta(\tilde{\mu}))$. This is a contradiction. Hence, $\tilde{\mu} = 1$. This means that $y^* \ge y_*$. Similar argument show that $y^* \le y_*$. Consequently, we have $y^* = y_*$.

Next to prove (b). Assume that $\overline{y}$ is a fixed point of T in $P_e \setminus [y_0, z_0]$. Let

$$\overline{\mu} = \sup\left\{\tau > 0 | \tau y^* \le \overline{y} \le \frac{1}{\tau}y^*\right\}. \tag{6}$$

Then $0 < \overline{\mu} \le 1$. We assert that $\overline{\mu} = 1$. If $0 < \overline{\mu} < 1$, by (L2) we have

$$\overline{y} = T\overline{y} \ge T(\overline{\mu} y^*) \ge \overline{\mu}(1 + \eta(\overline{\mu}))y^*$$

and

$$\overline{y} \le T(\frac{1}{\overline{\mu}}y^*) \le \frac{1}{\overline{\mu}(1 + \eta(\overline{\mu}))}y^*.$$

Thus, from (6) we have $\overline{\mu} \ge \overline{\mu}(1 + \eta(\overline{\mu}))$, which is a contradiction. Thus, (6) implies that $\overline{y} = y^*$, which means a contradiction

$$[y_0, z_0] \not\ni \overline{y} \in [y_0, z_0].$$

This end the proof of the conclusion (b).

Note that $y^* \in [x_0, y_0]$ is unique fixed point of T in P_e and

$$T(ty^*) \ge t(1 + \eta(t))Ty^*, \ \ t \in (0, 1).$$

the conclusion (c) can be proved by similar way to the proof of Theorem 3.4 of [13], here is omitted. The proof is complete. □

Lemma 2. *([4, 7]) Let $T : P_e \to P_e$ be an increasing operator. Suppose that*

(L3) there exists $\alpha \in (0, 1)$ such that

$$T(tx) \ge t^\alpha Tx, \ x \in P_e, \ t \in (0, 1).$$

Then T has a unique fixed point x^* in P_e. Moreover, for any $u_0 \in P_e$, letting $u_n = Tu_{n-1}, n = 1, 2, \cdots$, one has $\lim_{n\to\infty} \|u_n - x^*\| = 0$.

2 Positive solutions for operator equation

Throughout this section, we assume that E is a real Banach space, P is a normal cone in E with the normal constant N and P_e is defined by (2), $e > \theta$.

In this section, we investigate the existence and uniqueness of positive solutions of the operator equation (1), where A is a convexity operator and B is a constant operator or an $\alpha-$concave operator.

Firstly, we discuss the case of Eq.(1) with $B \equiv x_0 (x_0 \in E)$ which can be widely applied to various problems for differential equations. We have the following result.

Theorem 3. *Let $x_0 \in P_e$. Suppose that the operator $A : P \to P$ is increasing and satisfies conditions:*

(H1) $Ae > \theta$ and there exists $l > 0$ such that $Ae \le le$;

(H2) there exists a real number $\beta > 1$ such that $A(tx) = t^\beta Ax,\ t \in (0, 1),\ x \in P_e$.

Then the following statements are true:

(a) there exists $\lambda^* > 0$ such that the equation $x = x_0 + \lambda Ax$ has a unique solution $x_\lambda \in [x_0, \frac{\beta}{\beta-1}x_0]$ in P_e for $\lambda \in [0, \lambda^*]$. Moreover, for any $u_0 \in P_e$, set $C_\lambda = x_0 + \lambda A$ and $u_n = C_\lambda u_{n-1}, n = 1, 2, \cdots$, then $\lim_{n\to\infty} \|u_n - x_\lambda\| = 0$;

(b) $x_0 \le x_\lambda \le \frac{\beta}{\beta-1}x_0$ for $\lambda \in [0, \lambda^*]$;

(c) x_λ is increasing in λ for $\lambda \in [0, \lambda^*]$;

(d) $\lim_{\lambda\to 0} \|x_\lambda - x_0\| = 0$;

$\|x_\lambda - x_0\| \le \frac{N}{\beta-1}\|x_0\|, \lambda \in [0, \lambda^*]$.

Proof. We prove all statements by five steps.

Step 1. Define a mapping $\rho : P_e \to [0, +\infty)$ by

$$\rho(x) = \inf\{\tau > 0 | Ax \le \tau x_0\},\ \ x \in P_e. \tag{7}$$

By the property (b) of P_e we get $0 < \rho(x) < +\infty$. In addition, for any $x_1, x_2 \in P_e$, $x_1 \le x_2$, we have

$$Ax_1 \le Ax_2 \le \rho(x_2)x_0,$$

which implies that

$$\rho(x_1) \le \rho(x_2),$$

i.e., $\rho(x)$ is increasing in $x \in P_e$.

Let $C_\lambda = x_0 + \lambda A$. It is obvious from (H1) that $C_\lambda(P_e) \subset P_e$ for $\lambda \ge 0$. Set

$$\begin{array}{l}\Gamma = \{\lambda \ge 0 | \text{there exists } S > 1 \text{ such that} \\ C_\lambda(Sx_0) \le Sx_0 \text{ and } \frac{1}{\beta-1} \ge \lambda S^\beta \rho(x_0)\}\end{array} \tag{8}$$

and

$$\lambda^* = \sup \Gamma. \tag{9}$$

Take $S_0 = \frac{\beta}{\beta-1} > 1$, and set

$$\lambda(S) = \frac{S-1}{S^\beta \rho(x_0)}. \tag{10}$$

Then

$$\begin{array}{rl}\lambda(S_0) = \frac{S_0-1}{S_0^\beta \rho(x_0)} = \frac{(\beta-1)^{\beta-1}}{\beta^\beta \rho(x_0)} \\ \ge \frac{S-1}{S^\beta \rho(x_0)} = \lambda(S) > 0, \forall S > 1.\end{array} \tag{11}$$

Moreover, for any $\lambda \in [0, \lambda(S_0)]$ we have

$$C_\lambda(S_0x_0) \le \Big(1 + \lambda(S_0)S_0^\beta\rho(x_0)\Big)x_0 \le S_0x_0, \tag{12}$$

and

$$\begin{array}{rcl}\lambda S_0^\beta \rho(x_0) & \le & \lambda(S_0)S_0^\beta\rho(x_0) \\ & = & S_0 - 1 = \dfrac{1}{\beta-1}.\end{array}$$

Therefore, $[0, \lambda(S_0)] \subset \Gamma$.

Step 2. Now, we show that

$$\lambda^* = \lambda(S_0). \tag{13}$$

Suppose to the contrary that

$$\lambda^* > \lambda(S_0). \tag{14}$$

By the definition of λ^*, there exists a increasing sequence $\{\lambda_n\}_{n=1}^\infty \subset \Gamma$ such that

$$\lim_{n\to\infty} \lambda_n = \lambda^*.$$

That is, there exists a nonincreasing sequence $\{S_n\}_{n=1}^\infty$ with $S_n > 1$ such that

$$C_{\lambda_n}(S_nx_0) \le S_nx_0.$$

This means that

$$S_nx_0 \ge x_0 + \lambda_nS_n^\beta Ax_0. \tag{15}$$

Set $\lim_{n\to\infty} S_n = S_*$, then (15) implies that

$$S_*x_0 \ge x_0 + \lambda^* S_*^\beta Ax_0.$$

So,

$$Ax_0 \le \frac{S_* - 1}{\lambda^* S_*^\beta} x_0$$

and $S_* > 1$. Thus, from (7) we have

$$\rho(x_0) \le \frac{S_* - 1}{\lambda^*(S_*)^\beta}.$$

Combining (11) and (14) gives

$$\lambda^* \le \frac{S_* - 1}{S_*^\beta \rho(x_0)} \le \lambda(S_0) < \lambda^*,$$

which is a contradiction. Hence,

$$\Gamma = [0, \lambda(S_0)] = [0, \lambda^*].$$

Step 3. Conclusion (a) holds.

Let $z_0 = S_0 x_0$, that is, $z_0 = \frac{\beta}{\beta-1} x_0$. Note that (12) and $x_0 \le C_\lambda x_0$ we obtain that C_λ satisfies the condition (L1) in Lemma 1 for $\lambda \in [0, \lambda^*]$. Note that

$$\frac{1-t}{t(1-t^{\beta-1})} > \frac{1}{\beta - 1}, \; t \in (0,1).$$

(8) implies that

$$\frac{1-t}{t(1-t^{\beta-1})} > \lambda S_0^\beta \rho(x_0), \lambda \in [0, \lambda^*], \; t \in (0,1),$$

For any $t \in (0,1)$, let

$$\eta(t) = \Big(\frac{1-t}{t} - \lambda(1 - t^{\beta-1}) S_0^\beta \rho(x_0)\Big) q(\lambda),$$

where

$$q(\lambda) = \sup\{\tau > 0 | x_0 \ge \tau C_\lambda z_0\}, \;\; \lambda \in [0, \lambda^*].$$

Then, $\eta(t) > 0$. Hence, for any $\lambda \in [0, \lambda^*]$ and $t \in (0,1)$, from (H2) we get that

$$\begin{aligned}
& C_\lambda(tx) \\
&= x_0 + \lambda t^\beta Ax \\
&\ge tC_\lambda x + (1-t)x_0 - \lambda t(1 - t^{\beta-1}) A z_0 \\
&= tC_\lambda x + t\Big(\frac{1-t}{t} - \lambda(1-t^{\beta-1}) S_0^\beta \rho(x_0)\Big) x_0 \\
&\ge t(1+\eta(t)) C_\lambda x, \forall x \in [x_0, z_0].
\end{aligned}$$

Thus, C_λ satisfy the condition (L2) in Lemma 1 for $\lambda \in [0, \lambda^*]$. Consequently, the conclusion (a) follows from Lemma 1.

Step 4. Conclusions (b) and (c) hold.

From the above proof of the conclusion (a), it is easy to see that the conclusion (b) holds.

Next we prove (c). Let $\lambda_1, \lambda_2 \in [0, \lambda^*]$ with $\lambda_1 \le \lambda_2$. Noting that

$$C_{\lambda_1} x_{\lambda_2} = x_0 + \lambda_1 A x_{\lambda_2} \le x_0 + \lambda_2 A x_{\lambda_2} = x_{\lambda_2},$$

we have $x_0 \le C_{\lambda_1} x_0 \le C_{\lambda_1} x_{\lambda_2} \le x_{\lambda_2}$. Similar to the above proof, we know that C_{λ_1} have a unique fixed point $x^* \in [x_0, x_{\lambda_2}]$ in P_e, which implies that $x_{\lambda_1} = x^* \le x_{\lambda_2}$.

Step 5. Finally, we prove (d).

For any $\lambda \in [0, \lambda^*]$, in virtue of the conclusion (a), there exists a unique $x_\lambda \in [x_0, z_0]$ in P_e such that $x_\lambda = x_0 + \lambda A x_\lambda$. Thus, from (7) we have

$$\theta \le x_\lambda - x_0 = \lambda A x_\lambda \le \lambda A z_0 \le \lambda \rho(z_0) x_0,$$

which implies that $\lim_{\lambda \to 0} \|x_\lambda - x_0\| = 0$.

By the conclusion (b), we have

$$\theta \le x_\lambda - x_0 \le \frac{1}{\beta - 1} x_0.$$

This means that $\|x_\lambda - x_0\| \le \frac{N}{\beta-1}\|x_0\|$. The proof is complete. □

Remark 4. *From (10) and (13), we can give the expression of λ^* in Theorem 3 that is, $\lambda^* = \frac{(\beta-1)^{\beta-1}}{\beta^\beta \rho(x_0)}$.*

Corollary 5. *Let operator $A : P \to P$ is increasing and satisfies (H1) and (H2). Suppose that $h \in P_e, 0 < M \le (\beta - 1)\big(\kappa\beta^\beta\big)^{-\frac{1}{\beta-1}}$, where $\kappa = \inf\{\tau > 0 | Ah \le \tau h\}$. Then the operator equation $x = Mh + Ax$ has a unique solution $x^* \in P_e$. Moreover, for any $u_0 \in P_e$, set $C = Mh + A$ and $u_n = Cu_{n-1}, n = 1, 2, \cdots$, then $\lim_{n\to\infty} \|u_n - x^*\| = 0$.*

Proof. In the proof of Theorem 3, let $x_0 = Mh$. So, $\rho(x_0) = M^{\beta-1}\kappa$. Just note that $\lambda^* = \lambda(S_0) \ge 1$ when $0 < M \le (\beta - 1)\big(\kappa_0 \beta^\beta\big)^{-\frac{1}{\beta-1}}$. Taking $x_0 = Mh$ and $\lambda = 1$ in Theorem 3 finishes the proof. □

If B α concavity operator, we can obtain:

Theorem 6. *Let $A, B : P \to P$ be increasing operators. Suppose that the operator A satisfies (H1) and (H2), and the operator B satisfies the following conditions:*

(H3) $B(P_e) \subset P_e$;

(H4) there exists a real number $\alpha \in (0,1)$ such that $B(tx) = t^\alpha Bx, t \in (0,1), x \in P_e$.

Then (a) there exists $\lambda^* > 0$ such that the operator equation $x = \lambda Ax + Bx$ has a unique fixed point x_λ in P_e for $\lambda \in [0, \lambda^*]$. Moreover, for any $u_0 \in P_e$, set $C_\lambda = \lambda A + B$ and $u_n = C_\lambda u_{n-1}, n = 1, 2, \cdots$, then $\lim_{n\to\infty} \|u_n - x_\lambda\| = 0$;

(b) there exist $x_0, z_0 \in P_e$ with $x_0 \le z_0$ such that $x_\lambda \in [x_0, z_0]$;

(c) x_λ is increasing in λ for $\lambda \in [0, \lambda^*]$.

Proof. By virtue of Lemma 2, B has a unique fixed point x_0 in P_e. For any $x \in P_e$ and $\lambda \ge 0$ from (H1) and (H3) we have

$$\lambda l(x) e + Bx \ge C_\lambda x = \lambda Ax + Bx \ge Bx.$$

That is

$$C_\lambda(P_e) \subset P_e, \; \lambda \ge 0. \tag{16}$$

Since x_0 is the unique fixed point of the operator B in P_e, then

$$x_0 \leq C_\lambda x_0, \;\; \lambda \geq 0. \tag{17}$$

The proof of Part (a).

Set

$$\Omega = \left\{ \lambda \geq 0 \,\middle|\, \begin{array}{l} \exists R > 1, s.t\, C_\lambda(Rx_0) \leq Rx_0 \\ \text{and } \frac{1-\alpha}{\beta-1} \geq \lambda R^\beta \rho(x_0) \end{array} \right\}. \tag{18}$$

where $\rho(x)$ is defined by (7). Set

$$\lambda^* = \sup \Omega. \tag{19}$$

Now, we show that $\lambda^* > 0$ and $\Omega = [0, \lambda^*]$.

Let $y = Rx_0$ for $R > 1$, then we have $x_0 \leq y$. Set

$$\lambda_1(R) = \frac{R - R^\alpha}{R^\beta \rho(x_0)}. \tag{20}$$

Then, for any $0 \leq \lambda \leq \lambda_1(R)$, we have

$$\begin{aligned} C_\lambda y &= \lambda R^\beta A x_0 + R^\alpha B x_0 \\ &\leq \lambda_1(R) R^\beta \rho(x_0) x_0 + R^\alpha x_0 \\ &= Rx_0 = y. \end{aligned} \tag{21}$$

Set

$$\lambda_2(R) = \frac{1-\alpha}{\beta-1} \cdot \frac{1}{R^\beta \rho(x_0)}.$$

Then

$$\begin{aligned} \tfrac{1-\alpha}{\beta-1} &= \lambda_2(R) R^\beta \rho(x_0) \\ &\geq \lambda R^\beta \rho(x_0), \; \forall \lambda \in [0, \lambda_2(R)]. \end{aligned} \tag{22}$$

Let

$$\lambda(R) = \min\{\lambda_1(R), \; \lambda_2(R)\}.$$

Taking $F(R) = R - R^\alpha - \frac{1-\alpha}{\beta-1}$ for any $R > 1$, it is easy to check that $\lim\limits_{R \to 1^+} F(R) < 0$ and $F\Big((\frac{\beta-\alpha}{\beta-1})^{\frac{1}{1-\alpha}} \Big) > 0$, which implies that there exists $R_0 \in (1, (\frac{\beta-\alpha}{\beta-1})^{\frac{1}{1-\alpha}})$ such that $F(R_0) = 0$. Note that $F(R)$ is increasing, we obtain that

$$\begin{aligned} \lambda(R) &= \min\{\lambda_1(R), \lambda_2(R)\} \\ &= \begin{cases} \lambda_1(R), & 1 < R < R_0, \\ \lambda_1(R_0) = \lambda_2(R_0), & R = R_0, \\ \lambda_2(S), & R > R_0. \end{cases} \end{aligned} \tag{23}$$

Since $\lambda_1(R)$ is increasing in intervals $(1, R_0)$ and $\lambda_2(R)$ are decreasing in $(R_0, +\infty)$, then $\lambda(R_0) = \max\limits_{R>1} \lambda(R) > 0$. Thus, from (21) and (22) we have

$$C_\lambda(R_0 x_0) \leq R_0 x_0, \;\; \frac{1-\alpha}{\beta-1} \geq \lambda R^\beta \rho(x_0) \tag{24}$$

for any $\lambda \in [0, \lambda(R_0)]$. Therefore, $[0, \lambda(R_0)] \subset \Omega$.

Now, we show that

$$\lambda^* = \lambda(R_0). \tag{25}$$

Suppose to the contrary that $\lambda^* > \lambda(R_0)$. By (19), there exists a increasing sequence $\{\lambda_n\}_{n=1}^\infty \subset \Omega$ with $\lambda_n \geq \lambda(R_0)$ such that $\lim\limits_{n\to\infty} \lambda_n = \lambda^*$. This means that there exists a nonincreasing sequence $\{R_n\}_{n=1}^\infty \subset (1, R_0]$ such that $C_{\lambda_n}(R_n x_0) \leq R_n x_0$. Moreover, we have

$$\begin{aligned} R_n x_0 &\geq \lambda_n A(R_n x_0) + B(R_n x_0) \\ &= \lambda_n R_n^\beta A x_0 + R_n^\alpha x_0. \end{aligned} \tag{26}$$

Set $\lim\limits_{n\to\infty} R_n = R_*$, then (26) implies that

$$R_* x_0 \geq \lambda^* R_*^\beta A x_0 + R_*^\alpha x_0.$$

So,

$$A x_0 \leq \frac{R_* - R_*^\alpha}{\lambda^* R_*^\beta} x_0$$

and $1 < R_* < R_0$. This means that

$$\rho(x_0) \leq \frac{R_* - R_*^\alpha}{\lambda^* R_*^\beta}.$$

Combining (23) and (25) gives

$$\begin{aligned} \lambda^* &\leq \frac{R_* - R_*^\alpha}{R_*^\beta \rho(x_0)} = \lambda_1(R_*) \\ &= \lambda(R_*) \leq \lambda(R_0) < \lambda^*, \end{aligned}$$

which is a contradiction. Hence, $\Omega = [0, \lambda^*]$.

Now, let $z_0 = R_0 x_0$, then, for any fixed $\lambda \in \Omega$, by (17) and (24), we know that $x_0 \leq C_\lambda x_0 \leq C_\lambda(z_0) \leq z_0$ and

$$\frac{1-\alpha}{\beta-1} \geq \lambda R_0^\beta \rho(x_0).$$

Noting that

$$\frac{t^{\alpha-1} - 1}{1 - t^{\beta-1}} > \frac{1-\alpha}{\beta-1}, \;\; t \in (0,1),$$

we have

$$t^{\alpha-1} - 1 - \lambda(1 - t^{\beta-1}) R_0^\beta \rho(x_0) > 0, \; t \in (0,1).$$

Thus, from (H2) and (H4) we get that

$$\begin{aligned} & C_\lambda(tx) \\ &= tC_\lambda x - \lambda t(1 - t^{\beta-1})Ax + t(t^{\alpha-1} - 1)Bx \\ &\geq tC_\lambda x - \lambda t(1 - t^{\beta-1} A z_0 + t(t^{\alpha-1} - 1) B x_0 \\ &\geq tC_\lambda x + t\Big(t^{\alpha-1} - 1 - (1 - t^{\beta-1})\lambda R_0^\beta \rho(x_0)\Big) x_0 \\ &\geq t(1 + \eta(t)) C_\lambda x, \;\; t \in (0,1), x \in [x_0, z_0], \end{aligned}$$

where

$$\eta(t) = \Big(t^{\alpha-1} - 1 - \lambda(1-t^{\beta-1})R_0^{\beta}\rho(x_0)\Big)q(\lambda),$$
$$q(\lambda) = \sup\{\tau > 0 | x_0 \geq \tau C_\lambda z_0\},$$
$$\lambda \in [0, \lambda^*].$$

The application of Lemma 1 concludes the proof of part (a).

From above proof, it is easy to see $x_0 \leq x_\lambda \leq z_0$, where x_0 is the unique fixed point of B in P_e and $z_0 = R_0 x_0$, R_0 is the unique solution of $F(R) = R - R^\alpha - \frac{1-\alpha}{\beta-1}$ in $(1,\infty)$. This ends the proof of part (b).

Next we prove part (C). Let $\lambda_1, \lambda_2 \in [0, \lambda^*]$ with $\lambda_1 \leq \lambda_2$. Noting that $C_{\lambda_1} x_{\lambda_2} = \lambda_1 A x_{\lambda_2} + B x_{\lambda_2} \leq \lambda_2 A x_{\lambda_2} + B x_{\lambda_2} = x_{\lambda_2}$, we have $x_0 \leq C_{\lambda_1} x_0 \leq C_{\lambda_1} x_{\lambda_2} \leq x_{\lambda_2}$. From the above proof, we know that C_{λ_1} have a unique fixed point $x^* \in [x_0, x_{\lambda_2}]$ in P_e, which implies that $x_{\lambda_1} = x^* \leq x_{\lambda_2}$. The proof is complete. □

Remark 7. *From (23) and (25), we can obtain the expression of λ^* in Theorem 6, that is,*

$$\lambda^* = \lambda_2(R_0) = \frac{1-\alpha}{\beta-1} \cdot \frac{1}{R_0^{\beta}\rho(x_0)},$$

where R_0 is the unique solution of $F(R) = R - R^\alpha - \frac{1-\alpha}{\beta-1}$ in $(1,\infty)$ and x_0 is the unique fixed point of B in P_e.

Theorem 8. *Let $A, B : P \to P$ be increasing operators. Suppose that the operator A satisfies (H1) and (H2), and the operator B satisfies (H3) and*

(H5) there exists a real number $\alpha \in (0,1)$ such that $B(tx) \geq t^\alpha Ax$, $t \in (0,1)$, $x \in P_e$.

Then (a) there exists an interval I with $[0, \lambda(R_0)] \subset I \subset [0, +\infty)$ such that $x = \lambda Ax + Bx$ has a unique solution x_λ in P_e for $\lambda \in I$. Moreover, for any $u_0 \in P_e$, set $C_\lambda = \lambda A + B$ and $u_n = C_\lambda u_{n-1}, n = 1, 2, \cdots$, then $\lim\limits_{n\to\infty} \|u_n - x_\lambda\| = 0$;

(b) x_λ is increasing in λ for $\lambda \in I$;

(c) there exist $x_0, z_0 \in P_e$ with $x_0 \leq z_0$ such that $x_\lambda \in [x_0, z_0]$ for $\lambda \in [0, \lambda(R_0))$.

Where $\lambda(R)$ is defined (23) and R_0 is the unique solution of $F(R) = R - R^\alpha - \frac{1-\alpha}{\beta-1}$ in $(1,\infty)$.

Proof. By virtue of Lemma 2, B has a unique fixed point x_0 in P_e. It is easy to see that

$$x_0 \leq C_\lambda x_0, \;\; C_\lambda(P_e) \subset P_e, \; \lambda \geq 0.$$

Similar to the proof of Theorem 6 we obtain

$$[0, \lambda(R_0)] \subset \Omega,$$

where Ω is defined (18). Moreover, $x = \lambda Ax + Bx$ has a unique solution x_λ in P_e for $\lambda \in [0, \lambda(R_0)]$.

On the other hand, for any $\bar{\lambda} \in \Omega$ it is evident that $[0, \bar{\lambda}] \subset \Omega$ and $x = \lambda Ax + Bx$ has a unique solution x_λ in P_e for $\lambda \in [0, \bar{\lambda}]$. Thus, the conclusion (a) can be proved.

The proof of the conclusions (b) and (c) is the same as the proof of Theorem 6. The proof is complete. □

3 Positive solutions for beam equation

In this section, we apply the results of Section 2 to study the existence and uniqueness of positive solutions for the following perturbed elastic beam equations with three parameters

$$\begin{cases} u^{(4)}(t) + \eta u''(t) - \zeta u(t) \\ = \lambda f(t, u(t)) + \varphi(t), 0 < t < 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{27}$$

and

$$\begin{cases} u^{(4)}(t) + \eta u''(t) - \zeta u(t) \\ = \lambda f(t, u(t)) + g(u(t)), 0 < t < 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{28}$$

where ζ, η and λ are parameters.

It is well-known that the deformation of the equilibrium state an elastic beam, its two ends of which are simply supported, can be described by a boundary value problem for a fourth-order ordinary differential equation [15]. The existence and multiplicity of positive solutions for the elastic beam equations without perturbations have been studied extensively, see for example [16]-[27] and references therein. However, there are few papers concerned with the uniqueness of positive solutions for the problem (27) and the problem (28)with three parameters and one perturbed loading force in literatures. In this section, we consider the problems for (27) and (28), and give an example to illustrate the result.

In what follows, set $E = C[0,1]$, the Banach space of continuous functions on $[0,1]$ with the norm $\|x\| = \max\limits_{t\in[0,1]} |x(t)|$. $P = \{x \in C[0,1] | x(t) \geq 0, t \in [0,1]\}$. It is clear that P is a normal cone of which the normality constant is 1.

The following hypotheses are needed in this section.

(H6) $f \in C[[0,1]\times[0,\infty), [0,+\infty)]$ is increasing in $u \in [0,+\infty)$ for fixed $t \in [0,1]$ and $f(t,1) \not\equiv 0$ for $t \in [0,1]$.

(H7) $\varphi : [0,1] \to [0,+\infty)$ is an integrable function, and

$$m = \inf_{t\in[0,1]} \varphi(t) > 0, \;\; M = \sup_{t\in[0,1]} \varphi(t) < +\infty.$$

(H8) there exists a constant $\beta > 1$ such that

$$f(t, ru) = r^{\beta} f(t, u), \ \forall t \in [0, 1], \\ \forall r \in (0, 1), \ \forall u \in [0, +\infty).$$

(H9) $g \in C[[0, \infty), (0, +\infty)]$ is increasing.

(H10) there exists a constant $\alpha \in (0, 1)$ such that

$$g(ru) \geq r^{\alpha} g(u), \ \forall r \in (0, 1), u \in [0, +\infty).$$

(H11) $\zeta, \eta \in R$ and $\eta < 2\pi^2, \zeta \geq -\frac{\eta^2}{4}, \zeta/\pi^4 + \eta/\pi^2 < 1$.

Let γ_1 and γ_2 be the roots of the polynomial $\gamma^2 + \eta\gamma - \zeta$, i.e.,

$$\gamma_1, \gamma_2 = \frac{1}{2}\left\{-\eta \pm \sqrt{\eta^2 + 4\zeta}\right\}.$$

In view of (H11) it is easy to see that $\gamma_1 \geq \gamma_2 > -\pi^2$. Let $G_i(t, s)$ $(i = 1, 2)$ be the Green's functions corresponding to the boundary value problems

$$-u''(t) + \gamma_i u(t) = 0, \ u(0) = u(1) = 0. \tag{29}$$

Moreover,

$$G_i(t, s) = \begin{cases} \frac{\sinh \nu_i t \cdot \sinh \nu_i (1-s)}{\nu_i \sinh \nu_i}, & 0 \leq t \leq s \leq 1 \\ \frac{\sinh \nu_i s \cdot \sinh \nu_i (1-t)}{\nu_i \sinh \nu_i}, & 0 \leq s \leq t \leq 1 \end{cases}$$

for $\gamma_i > 0$;

$$G_i(t, s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1 \\ s(1-t), & 0 \leq s \leq t \leq 1 \end{cases}$$

for $\gamma_i = 0$;

$$G_i(t, s) = \begin{cases} \frac{\sin \nu_i t \cdot \sin \nu_i (1-s)}{\nu_i \sin \nu_i}, & 0 \leq t \leq s \leq 1 \\ \frac{\sin \nu_i s \cdot \sin \nu_i (1-t)}{\nu_i \sin \nu_i}, & 0 \leq s \leq t \leq 1 \end{cases}$$

for $-\pi^2 < \gamma_i < 0$, where $\nu_i = \sqrt{|\gamma_i|}$, and

$$\inf_{0<t,s<1} \frac{G_i(t, s)}{G_i(t, t) G_i(s, s)} = \delta_i > 0, \tag{30}$$

where $\delta_i = \frac{\nu_i}{\sinh \nu_i}$ if $\gamma_i > 0$; $\delta_i = 1$ if $\gamma_i = 0$; $\delta_i = \nu_i \sin \nu_i$ if $-\pi^2 < \gamma_i < 0$. For $\sigma_1, \sigma_2 \in (0, 1)$ with $\sigma_1 \leq \sigma_2$, let

$$\epsilon_i = \min_{\sigma_1 \leq s \leq \sigma_2} G_i(s, s).$$

Then $\epsilon_i > 0$ $(i = 1, 2)$.

For more information on the Green's function of (29), we refer to [19, 22].

Let

$$e(t) = \int_0^1 \int_0^1 G_1(t, \tau) G_2(\tau, s) ds d\tau, \ \ t \in [0, 1],$$

then (H11) implies that $e \in P$. Moreover,

$$e(t) \geq \int_{\sigma_1}^{\sigma_2} \int_{\sigma_1}^{\sigma_2} G_1(t, \tau) G_2(\tau, s) ds d\tau \\ \geq (\sigma_2 - \sigma_1) \delta_1 \delta_2 \epsilon_1^2 \epsilon_2^2 > 0, t \in (0, 1). \tag{31}$$

Hence, $e > \theta$. Define P_e as (2).

Theorem 9. *Assume that (H6)-(H8) and (H11) hold. Then*

(a) there exists $\lambda^* > 0$ such that (27) has a unique positive solution $u_\lambda(t) \in P_e$ for $\lambda \in [0, \lambda^*]$. Moreover, for any $u_0 \in P_e$, set

$$u_n(t) = \lambda \int_0^1 \int_0^1 G_1(t, \tau) G_2(\tau, s) f(s, u_{n-1}(s)) ds d\tau \\ + x_0(t), \quad t \in [0, 1], n = 1, 2, \cdots,$$

then $\lim_{n \to \infty} \|u_n - u_\lambda\| = 0$;

(b) $x_0(t) \leq u_\lambda(t) \leq \frac{\beta}{\beta - 1} x_0(t) (t \in [0, 1])$ for $\lambda \in [0, \lambda^*]$;

(c) u_λ is increasing in λ for $\lambda \in [0, \lambda^*]$;

(d) $\lim_{\lambda \to 0} \|u_\lambda - x_0\| = 0$;

$$\|u_\lambda - x_0\| \leq \frac{1}{\beta - 1} \|x_0\|, \lambda \in [0, \lambda^*],$$

where

$$x_0(t) = \int_0^1 \int_0^1 G_1(t, \tau) G_2(\tau, s) \varphi(s) ds d\tau. \tag{32}$$

Proof. It is easy to see that the problem (27) has an integral formulation given by

$$u(t) = x_0(t) + \lambda \int_0^1 \int_0^1 G_1(t, \tau) G_2(\tau, s) \\ \cdot f(s, u(s)) ds d\tau, \ t \in [0, 1].$$

where $x_0(t)$ is defined by (32). Define operator $A : P \to E$ by

$$Au(t) = \int_0^1 \int_0^1 G_1(t, \tau) G_2(\tau, s) f(s, u(s)) ds d\tau$$

for $t \in [0, 1]$. It is easy to prove that u_λ is a solution of the problem (27) if and only if u_λ is a fixed point of the operator $x_0 + \lambda A$.

In virtue of (H6) and (H7), we know that $x_0 \in P$ and $A : P \to P$ is an increasing operator. Further, from (H7) we have

$$x_0(t) \geq m \int_0^1 \int_0^1 G_1(t, \tau) G_2(\tau, s) ds d\tau = m e(t), \\ x_0(t) \leq M \int_0^1 \int_0^1 G_1(t, \tau) G_2(\tau, s) ds d\tau = M e(t).$$

That is, $x_0 \in P_e$. From (H6), there exist $\sigma_1, \sigma_2 \in (0, 1)$ with $\sigma_1 < \sigma_2$ such that $\inf_{t \in [\sigma_1, \sigma_2]} f(t, 1) > 0$.

Moreover, from (H8) and (31), we obtain

$$\begin{aligned} Ae(t) &\geq \int_{\sigma_1}^{\sigma_2}\int_{\sigma_1}^{\sigma_2} G_1(t,\tau)G_2(\tau,s)f(s,e(s))dsd\tau \\ &\geq \Big((\sigma_2-\sigma_1)\delta_1\delta_2\epsilon_1^2\epsilon_2^2\Big)^{\beta}\inf_{t\in[\sigma_1,\sigma_2]} f(t,1) \\ &\quad\cdot\int_{\sigma_1}^{\sigma_2}\int_{c}^{d} G_1(t,\tau)G_2(\tau,s)dsd\tau \\ &> 0,\ t\in(0,1), \end{aligned}$$

$$\begin{aligned} Ae(t) &\leq \int_0^1\int_0^1 G_1(t,\tau)G_2(\tau,s)f(s,\|e\|)dsd\tau \\ &\leq \Big(\|e\|^{\beta}\sup_{t\in[0,1]} f(t,1)\Big)e(t). \end{aligned}$$

That is, A satisfies (H1).

For any $r\in(0,1)$ and $u\in P_e$, by (H8) we obtain

$$\begin{aligned} &A(ru)(t) \\ &=\int_0^1\int_0^1 G_1(t,\tau)G_2(\tau,s)f(s,ru(s))dsd\tau \\ &= r^{\beta}\int_0^1\int_0^1 G_1(t,\tau)G_2(\tau,s)f(s,u(s))dsd\tau \\ &= r^{\beta}Au(t),\ \ u\in P_e. \end{aligned}$$

That is, A satisfies (H2). Thus, the results of Theorem 9 follows from Theorem 3. The proof is complete. □

Remark 10. *There exist many functions which satisfy (H6) and (H8). For example, $f(t,x)=\psi(t)x^{\beta}$, where $\beta>1,\psi\in C[0,1]$ and $\psi(t)\geq 0$ and $\psi(t)\not\equiv 0$ for $t\in[0,1]$.*

We give a simple example to illustrate Theorem 9 and give an estimate for parameter λ. Consider equation (27) with $f(t,x)=x^2,\varphi(t)=1$ and $\eta=\zeta=0$. Then,

$$\begin{aligned} G_1(t,s) &= G_2(t,s) \\ &= \begin{cases} t(1-s), & 0\leq t\leq s\leq 1, \\ s(1-t), & 0\leq s\leq t\leq 1, \end{cases} \end{aligned}$$

$$\begin{aligned} e(t) &= x_0(t)=\int_0^1\int_0^1 G_1(t,\tau)G_2(\tau,s)dsd\tau \\ &= \frac{t(1-t^3)}{24}. \end{aligned}$$

It is not hard to verify that

$$\|x_0\|=\max_{t\in[0,1]} x_0(t)=\frac{1}{32\sqrt[3]{4}}.$$

It is easy to check that (H6)-(H8) and (H11) hold, where $\beta=2$. Hence, Theorem 9 implies that there exists $\lambda^*>0$ such that (27) has a unique positive solution $u_\lambda(t)\in P_e$ for $\lambda\in[0,\lambda^*]$. Furthermore, such a solution $u_\lambda(t)$ satisfies the following properties:

(a) for any $u_0(t)\in P_e$, set

$$\begin{aligned} u_n(t) &= \lambda\int_0^1\int_0^1 G_1(t,\tau)G_2(\tau,s)u_{n-1}^2(s)dsd\tau \\ &\quad+\frac{t(1-t^3)}{24},\ \ n=1,2,\cdots, \end{aligned}$$

then $\lim_{n\to\infty}\|u_n(\lambda)-u_\lambda\|=0$;

(b) $\frac{t(1-t^3)}{24}\leq u_\lambda(t)\leq\frac{t(1-t^3)}{12}(t\in[0,1])$ for $\lambda\in[0,\lambda^*]$;

(c) $u_\lambda(t)$ is increasing in λ for $\lambda\in[0,\lambda^*]$.

(d) $\lim_{\lambda\to 0}\|u_\lambda-x_0\|=0$;

$\|u_\lambda-x_0\|\leq\frac{1}{32\sqrt[3]{4}},\lambda\in[0,\lambda^*]$.

Now, we give an estimate for λ^*.

Since $x_0(t)=e(t)$ and $\|x_0\|=\frac{1}{32\sqrt[3]{4}}$. Therefore,

$$\begin{aligned} Ax_0(t) &= \int_0^1\int_0^1 G_1(t,\tau)G_2(\tau,s)x_0^2(s)dsd\tau \\ &\leq \|x_0\|^2x_0(t)=\frac{1}{32^2\times 2\sqrt[3]{2}}x_0(t). \end{aligned}$$

This means that $\rho(x_0)\leq\frac{1}{32^2\times 2\sqrt[3]{2}}$. Hence, from Remark 4 we obtain that $\lambda^*\geq 512\sqrt[3]{2}$.

Theorem 11. *Assume that (H6),(H8)-(H11) hold. Then*

(a) there exists an interval I with $[0,\lambda(R_0)]\subset I\subset[0,\infty)$ such that (28) has a unique positive solution $u_\lambda(t)\in P_e$ for $\lambda\in I$. Moreover, for any $u_0\in P_e$, set

$$\begin{aligned} u_n(t) &= \lambda\textstyle\int_0^1\int_0^1 G_1(t,\tau)G_2(\tau,s)f(s,u_{n-1}(s))dsd\tau \\ &\quad+\textstyle\int_0^1\int_0^1 G_1(t,\tau)G_2(\tau,s)g(u_{n-1}(s))dsd\tau, \\ &\quad t\in[0,1],n=1,2,\cdots, \end{aligned}$$

then $\lim_{n\to\infty}\|u_n-u_\lambda\|=0$;

(b) u_λ is increasing in λ for $\lambda\in I$;

(c) there exist $x_0,z_0\in P_e$ with $x_0\leq z_0$ such that $u_\lambda\in[x_0,z_0]$ for $\lambda\in[0,\lambda(R_0)]$.

Where $\lambda(R_0)$ is defined by Theorem 8.

Proof. It is easy to see that the problem (28) has an integral formulation given by

$$\begin{aligned} u(t) &= \lambda\textstyle\int_0^1\int_0^1 G_1(t,\tau)G_2(\tau,s)f(s,u(s))dsd\tau \\ &\quad+\textstyle\int_0^1\int_0^1 G_1(t,\tau)G_2(\tau,s)g(u(s))dsd\tau. \end{aligned}$$

Define operator $A, B : P \to E$ by

$$Au(t) = \int_0^1 \int_0^1 G_1(t,\tau)G_2(\tau,s)f(s,u(s))dsd\tau,$$
$$Bu(t) = \int_0^1 \int_0^1 G_1(t,\tau)G_2(\tau,s)g(u(s))dsd\tau$$

for $t \in [0,1]$. It is easy to prove that u_λ is a solution of the problem (28) if and only if u_λ is a fixed point of the operator $\lambda A + B$.

In virtue of (H6),(H8) and (H9), we know that $A, B : P \to P$ is an increasing operator. By the proof of Theorem 9 we obtain that A satisfies (H1) and (H2). From (H9) we have

$$\begin{aligned} Be(t) &\geq \int_0^1 \int_0^1 G_1(t,\tau)G_2(\tau,s)g(e(s))dsd\tau \\ &\geq g(0)\int_0^1 \int_0^1 G_1(t,\tau)G_2(\tau,s)dsd\tau \\ &> g(0)e(t), \ \ t \in (0,1). \end{aligned}$$

On the other hand,

$$\begin{aligned} Be(t) &\leq \int_0^1 \int_0^1 G_1(t,\tau)G_2(\tau,s)g(\|e\|)dsd\tau \\ &\leq g(\|e\|)e(t), \end{aligned}$$

That is, B satisfies (H3).

For any $r \in (0,1)$ and $u \in P_e$, by (H10) we have

$$\begin{aligned} &B(ru)(t) \\ &= \int_0^1 \int_0^1 G_1(t,\tau)G_2(\tau,s)g(ru(s))dsd\tau \\ &\geq r^\alpha \int_0^1 \int_0^1 G_1(t,\tau)G_2(\tau,s)g(u(s))dsd\tau \\ &= r^\alpha Bu(t), \ \ u \in P_e. \end{aligned}$$

That is, B satisfies (H5). Thus, Theorem 11 follows from Theorem 8. The proof is complete. □

Remark 12. *We can give a simple example to illustrate Theorem 11.*

Consider equation (28) with $f(t,x) = x^2, g(x) = 1 + x^{\frac{1}{2}}$ and $\eta = \zeta = 0$. From Remark 10, $f(t,x)$ satisfies (H6) and (H8) with $\beta = 2$. $G_1(t,s), G_2(t,s)$ and $e(t)$ are the same as Remake 10. It is easy to check that $g(x)$ satisfies (H9) and (H10) with $\alpha = \frac{1}{2}$. Moreover, integral equation

$$u(t) = \int_0^1 \int_0^1 G_1(t,\tau)G_2(\tau,s)(1 + u^{\frac{1}{2}}(s))dsd\tau.$$

has a unique solution $x_0(t)$ in P_e. Note that $\frac{1-\alpha}{\beta-1} = \frac{1}{2}$ and $R_0 = \frac{2+\sqrt{3}}{2}$ is the unique solution of algebraic equation $R - R^{\frac{1}{2}} = \frac{1}{2}$ in $(1, +\infty)$, Then

$$\lambda(R_0) = \frac{1-\alpha}{\beta-1} \cdot \frac{1}{R_0^\beta \rho(x_0)} = \frac{2(7-4\sqrt{3})}{\rho(x_0)}.$$

Hence, Theorem 11 implies that there exists interval $[0, \frac{2(7-4\sqrt{3})}{\rho(x_0)}] \subset I \subset [0,\infty)$ such that (28) has a unique positive solution $u_\lambda(t) \in P_e$ for $\lambda \in I$. Furthermore, such a solution $u_\lambda(t)$ satisfies the following properties:

(a) for any $u_0(t) \in P_e$ and $\lambda \in I$, set

$$\begin{aligned} u_n(t) = \lambda \textstyle\int_0^1 \int_0^1 G_1(t,\tau)G_2(\tau,s)u_{n-1}^2(s)dsd\tau \\ + \textstyle\int_0^1 \int_0^1 G_1(t,\tau)G_2(\tau,s)(1 + u_{n-1}^{\frac{1}{2}}(s))dsd\tau, \end{aligned}$$

then $\lim_{n\to\infty} \|u_n(\lambda) - u_\lambda\| = 0$;

(b) $u_\lambda(t)$ is increasing in λ for $\lambda \in I$;

(c) $x_0(t) \leq u_\lambda(t) \leq \frac{2+\sqrt{3}}{2}x_0(t) (t \in [0,1])$ for $\lambda \in [0, \frac{2(7-4\sqrt{3})}{\rho(x_0)}]$.

Remark 13. *The problem discussed by [19], [22]-[25] is the special case of the problem (28) where $g(u) \equiv 0$.*

Acknowledgements: The authors are grateful to the referees for their comments and suggestions.

This research was partially supported by the NNSF of China (10961020), the Key Project of Chinese Ministry of Education (208154), the Chunhui Project of Chinese Ministry of Education (Z2009-1-81007), the Science Foundation of Shanxi Province of China (2006011013) and the University Natural Science Research Develop Foundation of Shanxi Province of China(20111021).

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