Study on the Periodic Solution and Invariant Tori for Iced Cable

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Abstract: - In this paper, the behavior of iced cable with two degrees of freedom is investigated. With Melnikov function of the system, the sufficient condition for the existence of periodic solutions about the system is obtained. The invariant tori of the system is investigated by using transformations and average equation. The conclusion not only enriches the behavior of nonlinear dynamics about iced cable, but also provides the reference to the study of controlling the icing disaster, which is caused by large amplitude low frequency vibration of iced cable.

Key-Words: - Iced Cable; periodic solution; invariant torus; Melinkov function; transformation

1 Introduction

Because of simply mechanical analysis, convenient design, reliable usage on the cable-suspended structure, it is widely applied to long-span building structure. However, the vibration of cable-suspended structure will be more pronounced as its span increases. Cable-suspend may cover ice under a certain weather condition and its cross section will be noncircular. At the same time, it may cause large amplitude low frequency vibration and bring out great loss to the people's life and asset. Hence, it is important to study the dynamical behavior of iced cable with two degrees of freedom.

In the past 20 years, a number of important results of the existence and numbers about periodic solution have been achieved. In 1991, E. Perdios, C. G. Zagouras and O. Ragos [1] found vertically critical, planar periodic solutions around the triangular equilibrium points of the Restricted Three-Body Problem to exist for values of the mass parameter in the interval [0.03, 0.5], at the same time computed four series of such solutions. In 1996, Y. Agnon and M. Glozman [2] studied the interaction of several two-dimensional standing wave modes using a Hamiltonian formulation, and found four new time-periodic solutions. At the same time, the periodic solutions were useful for understanding the dynamical system of the standing wave modes. In 2000, Chen and Mei [4] obtained the character of the characteristic roots of the Fréchet derivative C for higher order autonomous Birkhoff systems. Furthermore, they obtained the existence theorem of periodic solutions by using Liapunov center theorem, and presented an example to illustrate the results. In 2006, Zhou and Xu [6] analyzed a heteronomy strong nonlinear dynamics system by using the good properties Chebyshev polynomials. At the same time, the method used in the paper did not need to be based on the assumption of small parameters and could be used to analyze strong nonlinear problems. At last, they compared the analytical results of Duffing equation with those obtained via a Runge-Kutta integration algorithm and the standard Harmonic Balance Method and obtained that the suggested approach was extremely accurate and effective. In 2009, M. Bayat and B. Mehri [14] gave a necessary condition for the existence of periodic solutions of certain three dimensional autonomous systems. Their claims were proved and supported by certain examples for the third order autonomous systems. In 2009, Liu and Han [10] considered a four-dimensional system of autonomous ordinary differential equation depending on a small parameter and with the results obtained they discussed a nonlinearly coupled Van der Pol-Duffing oscillator system.

In this paper, the behavior of iced cable with two degrees of freedom is investigated. By computing Melnikov function on symbolic computation software of Maple, the sufficient condition for the existence of periodic solutions about the system is obtained. Meanwhile the invariant tori of the system is investigated by using transformations and average equation.

2 System of Iced Cable

We consider the dynamical equations of iced cable with two degrees of freedom for C_D sufficiently small. With the multi-scale transformation and the reduction of normal forms, the equations are given as follows:

$$\dot{x}_1 = -\frac{1}{4}\sigma_1 x_2 - \mu_2 x_1 + \frac{1}{2}\beta_3 x_2(x_3^2 + x_4^2) + \frac{3}{4}\beta_4 x_2(x_1^2 + x_2^2),$$

$$\dot{x}_2 = \frac{1}{4}\sigma_1 x_1 - \mu_2 x_2 - \frac{1}{2}\beta_3 x_1(x_3^2 + x_4^2) - \frac{3}{4}\beta_4 x_1(x_1^2 + x_2^2),$$

$$\dot{x}_3 = (\frac{1}{2}u_m p_3 - \frac{1}{2}\sigma_2)x_4 - \mu_3 x_3 + \frac{3}{2}\beta_1 x_4(x_3^2 + x_4^2) + \beta_2 x_4(x_1^2 + x_2^2),$$

$$\dot{x}_4 = (\frac{1}{2}\sigma_2 p_3 + \frac{1}{2}u_m)x_3 - \mu_3 x_4 - \frac{3}{2}\beta_1 x_3(x_3^2 + x_4^2) - \beta_2 x_3(x_1^2 + x_2^2). \quad (1)$$

2.1 Nonsingular Linear Transformation

Denote the nonsingular linear transformation by TF, that is,

$$\tau = \tilde{n}t,\ u_1 = \frac{1}{\tilde{m}}(\frac{1}{4}\sigma_1 x_1 + \mu_2 x_2),\ u_2 = x_2,$$

$$v_1 = \frac{1}{\tilde{n}}[\frac{1}{2}(\sigma_2 + u_m p_3)x_3 + \mu_3 x_4],\ v_2 = x_4,$$

where $u_m p_3 \neq \sigma_2$, $\tilde{m} = \frac{1}{2}\sqrt{4\mu_2^2 + \sigma_1^2} > 0$, $\tilde{n} = \frac{1}{2}\sqrt{4\mu_3^2 + \sigma_2^2 - u_m^2 p_3^2} > 0$. Then the system (1) can be transformed into the system (2) as follows:

$$\frac{du_1}{d\tau} = -a_{0100}u_2 + M_{1a}(u_1, u_2, v_1, v_2),$$

$$\frac{du_2}{d\tau} = b_{1000}u_1 - b_{0100}u_2 + M_{1b}(u_1, u_2, v_1, v_2), \quad (2A)$$

$$\frac{dv_1}{d\tau} = -v_2 + M_{2c}(u_1, u_2, v_1, v_2),$$

$$\frac{dv_2}{d\tau} = v_1 - d_{0001}v_2 + M_{2d}(u_1, u_2, v_1, v_2), \quad (2B)$$

where the expressions of coefficient are shown in Appendix 1 and M_{1i} , M_{2j} , $i = a, b$, $j = c, d$ are in Appendix 2.

The perturbed system of the system (2) is given by

$$\frac{du}{d\tau} = f(u) + \varepsilon P(u, v, \varepsilon), \quad (3A)$$

$$\frac{dv}{d\tau} = g(v) + \varepsilon Q(u, v, \varepsilon), \quad (3B)$$

where $u = (u_1, u_2)^T$, $v = (v_1, v_2)^T$,

$$f(u) = (-a_{0100}u_2 + M_{3a}(u,v), b_{1000}u_1 + M_{3b}(u,v))^T$$
$$g(v) = (-v_2, v_1)^T,$$
$$P(u, v, \varepsilon) = (M_{1a}(u,v), -b_{0100}u_2 + M_{1b}(u,v))^T$$
$$Q(u, v, \varepsilon) = (M_{4c}(u,v), -d_{0001}v_2 + M_{4d}(u,v))^T$$

and the expressions of M_{3i} , M_{4j} , $i = a, b$, $j = c, d$ are shown in Appendix 2.

At the same time, for the perturbed system (3), we have

(i) If $\sigma_1 \neq 0$ and $\sigma_1 \neq 4\sqrt{3}\mu_2$, then $u = 0$ is a 1-order weak focus of planar autonomous system

$$\frac{du}{d\tau} = f(u). \quad (4A)$$

(ii) The planar autonomous system

$$\frac{dv}{d\tau} = g(v) \quad (4B)$$

is a Hamiltonian system and there exists an open interval $J \subset R$ such that the system (4B) has a family of periodic orbits

$$\left\{L_h : (u_1, u_2) : u_1^2 + u_2^2 = 2h, h \in J\right\}. \quad (5)$$

2.2 Existence of Periodic Solutions

For simplicity, let us introduce ∂_i :

$$\partial_i^{m,1}(a) = (0, \cdots, a, \cdots, 0)_{m\times 1}^T,$$

$\partial_i^{m,1}(a)$ denotes an $m \times 1$ block matrix, where the i-th element is a and the others are equal to 0.

Let $\partial_{i,j}^{m,n}(a) = \partial_i^m\left[(\partial_j^n(a))^T\right]$,

$$\vec{X} = (x_1, x_2)^T,\ \vec{Y} = (y_1, y_2)^T,$$

then $\vec{X} \Lambda \vec{Y} = \det(\vec{X}, \vec{Y})$,

$$\vec{X}^{\perp} = (\partial_{1,2}^{2,2}(-1) + \partial_{2,1}^{2,2}(1))\vec{X}.$$

When $0 \le \theta \le 2\pi$, $h \in J \subset R$, the transformation $u = u$, $v = G(\theta, h) = (\sqrt{2h}\cos\theta, \sqrt{2h}\sin\theta)^T$ turns the system (3) into a system with the form

$$\frac{du}{d\theta} = \frac{f(u, G(\theta, h)) + \varepsilon P(u, G(\theta, h), \varepsilon)}{1 + \varepsilon G_h(\theta, h)\Lambda Q(u, G(\theta, h), \varepsilon)}, \tag{6A}$$

$$\frac{dh}{d\theta} = -\varepsilon \frac{g(G(\theta, h))\Lambda Q(u, G(\theta, h), \varepsilon)}{1 + \varepsilon G_h(\theta, h)\Lambda Q(u, G(\theta, h), \varepsilon)}. \tag{6B}$$

Suppose that $(u(u_0, \theta, r, \varepsilon), h(u_0, \theta, r, \varepsilon))$, $|u_0| << 1$, $r \in J$ is a solution of the system (6), that is, $(u(u_0, 0, r, \varepsilon), h(u_0, 0, r, \varepsilon)) = (u_0, r)$. At the same time, it has an expansion as follows:

$$u(u_0, \theta, r, \varepsilon) = u_1(\theta, r)u_0 + u_2(\theta, r)\varepsilon + O(|u_0, \varepsilon|^2), \tag{7A}$$

$$h(u_0, \theta, r, \varepsilon) = r + \varepsilon(h_1(\theta, r) + h_2(\theta, r)u_0 + h_3(\theta, r)\varepsilon + O(|u_0, \varepsilon|^2). \tag{7B}$$

Further, we have

$$h_i(0, r) = 0, \quad (i = 1, \cdots, 3),$$

$$u_1(0, r) = I_2, \quad u_2(0, r) = 0, \tag{8}$$

where I_2 is the identity matrix of order 2.

With the system (6) and the equation (7), we have

$$\frac{du_1(\theta, r)}{d\theta} = Bu_1(\theta, r),$$

$$\frac{du_2(\theta, r)}{d\theta} = Bu_2(\theta, r),$$

$$\frac{dh_1(\theta, r)}{d\theta} = -g(G(\theta, r)) \wedge Q(0, G(\theta, r), 0)$$
$$= -2d_{0001} r \sin^2\theta + \sqrt{2r}\cos\theta \cdot M_{5c}(u, \mathrm{G}(\theta, r)) + \sqrt{2r}\sin\theta \cdot M_{5d}(u, \mathrm{G}(\theta, r)),$$

$$\frac{dh_2(\theta, r)}{d\theta} = -g(G(\theta, r))\Lambda(Q_u(0, G(\theta, r), 0)u_1(\theta, r)) = 0$$

$$\frac{dh_3(\theta, r)}{d\theta} = -\frac{1}{2\pi}H(\theta, r) = -\frac{r}{2\pi}, \tag{9}$$

where $B = \partial_{1,2}^{2,2}(-a_{0100}) + \partial_{2,1}^{2,2}(b_{1000})$ and the expressions of M_{5i}, $i = a, b$ are shown in Appendix 2. Hence,

$$u_1(2\pi, r) = \exp(2\pi B), \quad u_2(2\pi, r) = \exp(2\pi B),$$

$$h_1(2\pi, r) = \int_0^{2\pi} (-2b_{0100} r \sin^2\theta + \sqrt{2r}\cos\theta \cdot M_{5c}(u, \mathrm{G}(\theta, r)) + \sqrt{2r}\sin\theta \cdot M_{5d}(u, \mathrm{G}(\theta, r)))d\theta,$$

$$h_2(2\pi, r) = 0, \quad h_3(2\pi, r) = \int_0^{2\pi} -\frac{r}{2\pi}d\theta = -r. \tag{10}$$

Let

$$M_1 = 3c_{0030} + c_{0012} + d_{0021} + 3d_{0003},$$

$$M_1^1 = \frac{2d_{0001}}{3c_{0030} + c_{0012} + d_{0021} + 3d_{0003}}.$$

Lemma 1 For small $\varepsilon \ne 0$, there generates a periodic orbit of the system (3) in the neighborhood of

$$\overline{L}_r \equiv \{(u, v) : u = 0, H(v) = r\}, \quad r \in J$$

if and only if the following bifurcation equations have a solution in (u_0, r) with $r \in J$, $|u_0| << 1$:

$$(\exp(2\pi B) - I_2)u_0 + \exp(2\pi B)\varepsilon + O(|u_0, \varepsilon|^2) = 0, \tag{11A}$$

$$\pi r(M_1 M_1^1 + rM_1) - r\varepsilon + O(|u_0, \varepsilon|^2) = 0. \tag{11B}$$

Hence, we denote $h_1(2\pi, r)$ to be the Melnikov function of (3B). Let

$$M(r) = \int_0^{2\pi} g(G(\theta, r))\Lambda Q(0, G(\theta, r), 0)\, d\theta.$$

Theorem 1 (*Sufficient Condition for the Existence of Periodic Solution*) For $0 < |\varepsilon| << 1$, then

(1) If $M(r) \ne 0$ for any $r \in J$, that is, $d_{0001} \ne 0$ and $M_1 \ne 0$, there does not exist periodic orbit of the system (3) with period near 2π in the neighborhood of $\overline{L}_{h_0}$.

(2) If $a_{0100} \ne k$, there exists $h_0 = M_1^1$, such that

$M(h_0)=0$, $M'(h_0)=2\pi d_{0001}\neq 0$, then there exists an unique periodic orbit of the system (3) with period near 2π in the neighborhood of $\overline{L}_{h_0}$.

Proof: (1) By the theory of successor function, for any $r\in J$, if $M(r)\neq 0$, then there does not exist periodic orbit of the system (3) with period near 2π in the neighborhood of $\overline{L}_{h_0}$.

(2) By $M(h_0)=0$, $M'(h_0)\neq 0$, we have

$$h_1(2\pi, h_0)=0,\ h_1'(2\pi, h_0)\neq 0. \tag{12}$$

Substituting (10) and (12) into (11), we have

$$(\exp(2\pi B)-I_2)u_0+\exp(2\pi B)\varepsilon$$
$$+O(|r-h_0||u_0|+|r-h_0|\varepsilon+|u_0,\varepsilon|^2)=0, \tag{13A}$$
$$2\pi(-2d_{0001}+h_0M_1)(r-h_0)-h_0\varepsilon$$
$$+O(|r-h_0|^2+|r-h_0|\varepsilon+|u_0,\varepsilon|^2)=0. \tag{13B}$$

If $a_{0100}\neq k$, then the matrix $u_1(2\pi, h_0)-I_2$ is invertible. Therefore, by the equations (13), we have

$$u_0=\varepsilon\left[-(\exp(2\pi B)-I_2)^{-1}\exp(2\pi B)\right.$$
$$\left.+O(|r-h_0|+|\varepsilon|)\right],$$
$$r=h_0-\frac{h_3(2\pi, h_0)\varepsilon}{h_1'(2\pi, h_0)}=h_0+\frac{2\varepsilon}{\pi M_1}. \tag{14}$$

Let $r=h_0+c_1\varepsilon+O(\varepsilon^2)$, $u_0=c_2\varepsilon+O(\varepsilon^2)$, where

$$c_1=\frac{2}{\pi M_1},\ c_2=-(\exp(2\pi B)-I_2)^{-1}\exp(2\pi B),$$

then there exists an unique periodic orbit of the system (6) with period 2π as follows:

$$h(\theta,\varepsilon)=h_0+\overline{u}_0(\theta,\varepsilon),$$
$$u(\theta,\varepsilon)=(m_1(\theta), m_2(\theta))^T+O(\varepsilon^2)$$
$$\equiv(u_{01}(\theta,\varepsilon), u_{02}(\theta,\varepsilon)), \tag{15}$$

where

$$\overline{u}_0(\theta,\varepsilon)=(c_1+h_1(\theta, h_0))\varepsilon+O(\varepsilon^2),$$
$$(m_1(\theta), m_2(\theta))^T=\exp(2\pi B)c_2+\exp(2\pi B).$$

Hence, there exists an unique periodic orbit L_ε of the system (3) with period near 2π

$$L_\varepsilon: u=(m_1(\theta), m_2(\theta))^T\varepsilon+O(\varepsilon^2),$$
$$v=G(\theta, h_0+\overline{u}_0(\theta,\varepsilon)).$$

3 Stability of Periodic Solution

In this part, the invariant tori of the system is obtained through using the blow-up transformation and the average method.

3.1 Transformations

With condition (i) in the part 2.1, the system (3) can be written into the system (16)

$$\frac{du_1}{d\tau}=-a_{0100}u_2+M_{3a}(u_1,u_2,v_1,v_2)$$
$$+\varepsilon M_{4a}(u_1,u_2,v_1,v_2),$$
$$\frac{du_2}{d\tau}=b_{1000}u_1-\varepsilon b_{0100}u_2+M_{3b}(u_1,u_2,v_1,v_2)$$
$$+\varepsilon M_{4b}(u_1,u_2,v_1,v_2),$$
$$\frac{dv}{d\tau}=g(v)+\varepsilon Q(u,v,\varepsilon), \tag{16}$$

where

$$g(v)=(-v_2, v_1)^T,$$
$$Q(u,v,\varepsilon)=(M_{1a}(u_1,u_2,v_1,v_2),$$
$$-d_{0001}v_2+M_{1b}(u_1,u_2,v_1,v_2))^T,$$

and the expressions of M_{3i}, $i=a,b$ and M_{4j}, $j=c,d$ are shown in Appendix 2.

From Theorem 1, we know that if $h_0=M_1^1$ and $a_{0100}\neq k$, there exists an unique periodic orbit of the system (3) with period near 2π in the neighborhood of $\overline{L}_{h_0}$. Hence,

Let $h=h_0+\overline{v}$, where $|\overline{v}|<<1$, the system (6) can be transformed into the system (17)

$$\frac{du_1}{d\theta}=-a_{0100}u_2+a_{10}(\theta)\varepsilon+a_{20}(\theta)\overline{v}\varepsilon+a_{30}(\theta)u_1\varepsilon$$
$$+a_{40}(\theta)u_2\varepsilon+M_{6a}(u_1,u_2)+R_2(u,\theta,\overline{v},\varepsilon),$$
$$\frac{du_2}{d\theta}=b_{1000}u_1+a_{01}(\theta)\varepsilon+a_{02}(\theta)\overline{v}\varepsilon+a_{03}(\theta)u_1\varepsilon$$
$$+a_{04}(\theta)u_2\varepsilon+M_{6b}(u_1,u_2)+R_3(u,\theta,\overline{v},\varepsilon),$$

$$\frac{d\bar{v}}{d\theta} = \varepsilon(b_{10}(\theta) + b_{20}(\theta)\bar{v} + b_{30}(\theta)u_1 + b_{40}(\theta)u_2 + R_1(u, \theta, \bar{v}, \varepsilon)), \quad (17)$$

where the expressions of $M_{6i}(u_1, u_2)$, $i = a, b$, $a_{\bar{i}0}(\theta)$, $b_{j0}(\theta)$, $b_{0k}(\theta)$, $\bar{i}, j, k = 1, \cdots, 4$ and $R_l(\theta, \bar{u}, v, \varepsilon)$, $l = 1, \cdots, 3$ are shown in Appendix 2.

Let

$(u_1, u_2)^T = (u_{01}(\theta, \varepsilon), u_{02}(\theta, \varepsilon))^T + (z_1, z_2)^T$, $\bar{v} = \bar{u}_0(\theta, \varepsilon) + p$, the system (17) can be turned into the system (18)

$$\frac{dz_1}{d\theta} = a_{30}(\theta)z_1\varepsilon + a_{40}(\theta)z_2\varepsilon - a_{0100}z_2 + M_{6a}(u_{01} + z_1, u_{02} + z_2),$$

$$\frac{dz_2}{d\theta} = a_{03}(\theta)z_1\varepsilon + a_{04}(\theta)z_2\varepsilon + b_{1000}z_1 + M_{6b}(u_{01} + z_1, u_{02} + z_2),$$

$$\frac{dp}{d\theta} = \varepsilon(b_{20}(\theta)p + b_{30}(\theta)z_1 + b_{40}(\theta)z_2 + \bar{R}_1(z_1, z_2, \theta, p, \varepsilon)). \quad (18)$$

Denote the blow-up transformation by

$$p = \mu\bar{p}, z_1 = \mu\bar{z}_1, \quad z_2 = \mu\bar{z}_2,$$

where $\mu = |\varepsilon|^{1/2}$, then the system (18) can be transformed into the system (19)

$$\frac{d\bar{z}_1}{d\theta} = -a_{0100}z_2 + \mu^2 H_2(\bar{z}_1, \bar{z}_2, \theta, \bar{p}) + \mu^3\bar{R}_{20}(\bar{z}_1, \bar{z}_2, \theta, \bar{p}, \varepsilon),$$

$$\frac{d\bar{z}_2}{d\theta} = b_{1000}z_1 + \mu^2 H_3(\bar{z}_1, \bar{z}_2, \theta, \bar{p}) + \mu^3\bar{R}_{30}(\bar{z}_1, \bar{z}_2, \theta, \bar{p}, \varepsilon),$$

$$\frac{d\bar{p}}{d\theta} = \mu^2 H_1(\bar{z}_1, \bar{z}_2, \theta, \bar{p}) + \mu^3\bar{R}_{10}(\bar{z}_1, \bar{z}_2, \theta, \bar{p}, \varepsilon), \quad (19)$$

where

$$H_1(\bar{z}_1, \bar{z}_2, \theta, \bar{p}) = (b_{20}(\theta)\bar{p} + b_{30}(\theta)\bar{z}_1 + b_{40}(\theta)\bar{z}_2)\operatorname{sgn}(\varepsilon),$$

$$H_2(\bar{z}_1, \bar{z}_2, \theta, \bar{p}) = (a_{30}(\theta)\bar{z}_1 + a_{40}(\theta)\bar{z}_2)\operatorname{sgn}(\varepsilon) + M_{6a}(\bar{z}_1, \bar{z}_2),$$

$$H_3(\bar{z}_1, \bar{z}_2, \theta, \bar{p}) = (a_{03}(\theta)\bar{z}_1 + a_{04}(\theta)\bar{z}_2)\operatorname{sgn}(\varepsilon) + M_{6b}(\bar{z}_1, \bar{z}_2),$$

$$\bar{R}_{i0}(0, 0, \theta, 0, \varepsilon) = 0, \quad i = 1, 2, 3.$$

3.2 Average Equation and Invariant Tori

Let $\bar{z}_1 = \rho\cos\varphi$, $\bar{z}_2 = -\rho\sin\varphi$, $\bar{p} = \rho w$, the system (19) can be transformed into the system (20)

$$\frac{d\varphi}{d\theta} = -a_{0100} - \frac{\mu^2}{\rho}(H_2\sin\varphi + H_3\cos\varphi) - \frac{\mu^3}{\rho}(\bar{R}_{20}\sin\varphi + \bar{R}_{30}\cos\varphi),$$

$$\frac{d\rho}{d\theta} = \mu^2(H_2\cos\varphi - H_3\sin\varphi) + \mu^3(\bar{R}_{20}\cos\varphi - \bar{R}_{30}\sin\varphi),$$

$$\frac{dw}{d\theta} = \frac{\mu^2}{\rho}\left[H_1 - w(H_2\cos\varphi - H_3\sin\varphi)\right] + \frac{\mu^3}{\rho}\left[\bar{R}_{10} - \bar{R}_{20}\cos\varphi - \bar{R}_{30}\sin\varphi\right]. \quad (20)$$

When a_{0100} is an irrational, let

$$\bar{H}_1 = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{(H_2\sin\varphi + H_3\cos\varphi)}{\rho}d\theta d\varphi = \frac{\rho}{4\pi}\int_0^{2\pi}a_{03}(\theta) - a_{40}(\theta)d\theta\operatorname{sgn}(\varepsilon) + \frac{3\rho^3}{8}(b_{3000} - a_{0300}) + \frac{\rho^3}{8}(b_{1200} - a_{2100}),$$

$$\bar{H}_2 = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}(H_2\cos\varphi - H_3\sin\varphi)d\theta d\varphi = \frac{\rho}{4\pi}\int_0^{2\pi}a_{30}(\theta) + a_{04}(\theta)d\theta\operatorname{sgn}(\varepsilon) + \frac{3\rho^3}{8}(a_{3000} + b_{0300}) + \frac{\rho^3}{8}(b_{2100} + a_{1200}),$$

$$\overline{H}_3 = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{1}{\rho}\big[H_1 - w(H_2\cos\varphi - H_3\sin\varphi)\big]d\theta d\varphi$$

$$= \frac{w}{2\pi}\int_0^{2\pi} b_{20}(\theta)\,d\theta\,\mathrm{sgn}(\varepsilon) - \frac{w}{4\pi}\int_0^{2\pi} a_{30}(\theta)+a_{04}(\theta)\,d\theta\,\mathrm{sgn}(\varepsilon) - \frac{\rho^2 w}{8}(3a_{3000}+a_{1200}+b_{2100}+3b_{0300}).$$

With the average method, the average equation of the system (20) is given by

$$\frac{d\varphi}{d\theta} = -a_{0100} - \mu^2\overline{H}_1(\rho, w) + \mu^3 R_1(\rho, w, \varepsilon),$$

$$\frac{d\rho}{d\theta} = \mu^2\overline{H}_2(\rho, w) + \mu^3 R_2(\rho, w, \varepsilon),$$

$$\frac{dw}{d\theta} = \mu^2\overline{H}_3(\rho, w) + \mu^3 R_3(\rho, w, \varepsilon), \qquad (21)$$

where $R_i(\rho, w, \varepsilon)$ are continuous functions and $R_i(0, w, \varepsilon) = 0$, $i = 1, 2, 3$.

Let

$$M(\rho, w) = \begin{pmatrix} \dfrac{\partial\overline{H}_2(\rho, w)}{\partial\rho} & \dfrac{\partial\overline{H}_2(\rho, w)}{\partial w} \\ \dfrac{\partial\overline{H}_3(\rho, w)}{\partial\rho} & \dfrac{\partial\overline{H}_3(\rho, w)}{\partial w} \end{pmatrix}.$$

With the average equation and the theory of manifold, the equations (22) are obtained as follows:

$$\overline{H}_2(\rho, w) = 0,$$

$$\overline{H}_3(\rho, w) = 0. \qquad (22)$$

With Equations (22), we can obtain the following theorems:

Theorem 2 (I) Equations (22) have a zero solution, that is, $(\rho, w) = (0, 0)$ and the matrix $M(0, 0)$ has a pair of eigenvalues $\frac{a_1}{4\pi}\mathrm{sgn}(\varepsilon)$ and $\frac{2a_2 - a_1}{4\pi}\mathrm{sgn}(\varepsilon)$.

(II) Under the condition of $(3a_{3000} + 3b_{0300} + a_{1200} + b_{2100})a_1\varepsilon < 0$, equations (22) have a nonzero solution, that is, $(\rho, w) = (\rho_0, 0)$ and the matrix $M(\rho_0, 0)$ has a pair of eigenvalues $-\frac{a_1}{2\pi}\mathrm{sgn}(\varepsilon)$ and $\frac{a_2}{2\pi}\mathrm{sgn}(\varepsilon)$.

Where

$$a_1 = \int_0^{2\pi} a_{30}(\theta) + a_{04}(\theta)d\theta = 2\pi(a_{1020}h_0 + a_{1002}h_0 + b_{0120}h_0 + b_{0102}h_0 - b_{0001}),$$

$$a_2 = \int_0^{2\pi} b_{20}(\theta)d\theta = \pi\big[(6c_{0030} + 6d_{0003} + 2c_{0012} + 2d_{0021})h_0 - 2d_{0001}\big],$$

$$\rho_0 = \left[\frac{\left|\frac{1}{2}\int_0^{2\pi} b_{30}(\theta) + b_{04}(\theta)d\theta\right|}{\frac{3\pi}{4}(a_{3000} + b_{0300}) + \frac{\pi}{4}(a_{1200} + b_{2100})}\right]^{1/2}.$$

Theorem 3 There exist continuous functions $\psi_i(\theta, \varphi, \varepsilon)$, $i = 1, 2, 3$ under the conditions of $a_1(2a_2 - a_1) \neq 0$ and $a_1 a_2 \neq 0$, which make the system (20) has invariant torus $S_{1\varepsilon}$ and $S_{2\varepsilon}$:

$$S_{1\varepsilon} = \{(\theta, \varphi, \rho, w): \rho = 0, w = \psi_1(\theta, \varphi, \varepsilon), \theta \in R, \varphi \in R\},$$

$$S_{2\varepsilon} = \{(\theta, \varphi, \rho, w): \rho = \rho_0 + \psi_2(\theta, \varphi, \varepsilon), w = \psi_3(\theta, \varphi, \varepsilon), \theta \in R, \varphi \in R\}$$

with $\psi_i(0, \varphi, 0) = 0$, $i = 1, 2, 3$. Meanwhile, if $2a_2\varepsilon < a_1\varepsilon < 0$, $S_{1\varepsilon}$ is exponentially asymptotically stable. If $a_1\varepsilon > 0$, $a_2\varepsilon < 0$, $S_{2\varepsilon}$ is exponentially asymptotically stable.

It is easy to see that the invariant torus $S_{1\varepsilon}$ of the system (20) corresponds to the solution $(p, z_1, z_2) = (0, 0, 0)$ of the system (18) and to the periodic orbit L_ε of the system (16) as a trivial invariant torus. The invariant torus $S_{2\varepsilon}$ of the system (20) is a nontrivial invariant torus of the system (16) with the form

$$\overline{S}_{2\varepsilon} = \{(\theta, \varphi, u, v): u = G(\theta, h_0 + \overline{v}_0(\theta, \varepsilon)) + |\varepsilon|^{1/2}(\rho_0 + \psi_2(\theta, \varphi, \varepsilon))\psi_3(\theta, \varphi, \varepsilon),$$

$$v = (v_{01}(\theta, \varepsilon) + v_{02}(\theta, \varepsilon)) + |\varepsilon|^{1/2}(\cos\varphi - \sin\varphi)(\rho_0 + \psi_2(\theta, \varphi, \varepsilon)),$$

$\theta \in R, \varphi \in R \}.$

4 Conclusion

In this paper, the behavior of iced cable with two degrees of freedom is investigated. By using periodic transformation and Melnikov function, the existence of periodic solutions about the system is obtained, which is shown in Theorem1. Based on Theorem1, with a series of changes of coordinate, the existence of invariant tori for the perturbed system is obtained, which is shown in Theorem 2 and 3. The conclusion not only enriches the behavior of nonlinear dynamics about iced cable, but also provides the reference to the study of controlling the icing disaster, which is caused by large amplitude low frequency vibration of iced cable.

5 Acknowledgment

The authors gratefully acknowledge the support of the National Natural Science Foundation of China (NNSFC) through Grant Nos. 11072007 and 10722020, the Natural Science Foundation of Beijing (NSFB) through Grant No. 1082002.

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Appendix 1

$$a_{3000}=-\frac{48\beta_4\mu_2^3}{\sigma_1^3}-\frac{3\beta_4\mu_2}{\sigma_1}$$

$$a_{2100}=\frac{3\beta_4(64\mu_2^2\sigma_1^2+768\mu_2^4+\sigma_1^4)}{4\sqrt{16\mu_2^2+\sigma_1^2}\sigma_1^3}$$

$$a_{1200}=-\frac{9\beta_4\mu_2(16\mu_2^2+\sigma_1^2)}{\sigma_1^3}$$

$$a_{0300}=\frac{3\beta_4(32\mu_2^2\sigma_1^2+\sigma_1^4+256\mu_2^4)}{4\sqrt{16\mu_2^2+\sigma_1^2}\sigma_1^3}$$

$$a_{1020}=-\frac{2\mu_2\beta_3(4\mu_3^2-u_m^2p_3^2+\sigma_2^2)}{\sigma_1(\sigma_2+u_mp_3)^2}$$

$$a_{1011}=\frac{16\mu_2\beta_3\sqrt{\mu_3^2-\frac{1}{4}u_m^2p_3^2+\frac{1}{4}\sigma_2^2}\mu_3}{\sigma_1(\sigma_2+u_mp_3)^2}$$

$$a_{1002}=-\frac{2\mu_2\beta_3(\sigma_2^2+2\sigma_2u_mp_3+u_m^2p_3^2+4\mu_3^2)}{\sigma_1(\sigma_2+u_mp_3)^2}$$

$$a_{0102}=\frac{\beta_3(\sigma_1^2\sigma_2^2+2\sigma_1^2\sigma_2u_mp_3+\sigma_1^2u_m^2p_3^2+4\sigma_1^2\mu_3^2)}{2\sqrt{16\mu_2^2+\sigma_1^2}\sigma_1(\sigma_2+u_mp_3)^2}+\frac{\beta_3(64\mu_2^2\mu_3^2 16\mu_2^2\sigma_2^2+32\mu_2^2\sigma_2u_mp_3+16\mu_2^2u_m^2p_3^2)}{2\sqrt{16\mu_2^2+\sigma_1^2}\sigma_1(\sigma_2+u_mp_3)^2}$$

$$a_{0120}=\frac{\beta_3(-\sigma_1^2u_m^2p_3^2+\sigma_1^2\sigma_2^2+4\sigma_1^2\mu_3^2+64\mu_2^2\mu_3^2)}{2\sqrt{16\mu_2^2+\sigma_1^2}\sigma_1(\sigma_2+u_mp_3)^2}+\frac{\beta_3(16\mu_2^2\sigma_2^2-16\mu_2^2u_m^2p_3^2)}{2\sqrt{16\mu_2^2+\sigma_1^2}\sigma_1(\sigma_2+u_mp_3)^2}$$

$$a_{0111}=-\frac{2\beta_3\sqrt{4\mu_3^2-u_m^2p_3^2+\sigma_2^2}\mu_3\sqrt{16\mu_2^2+\sigma_1^2}}{\sigma_1(\sigma_2+u_mp_3)^2}$$

$$b_{0300}=\frac{3\beta_4\mu_2(16\mu_2^2+\sigma_1^2)}{\sigma_1^3}$$

$$b_{2100}=\frac{3\beta_4(64\mu_2^2\sigma_1^2+768\mu_2^4+\sigma_1^4)}{4\sqrt{16\mu_2^2+\sigma_1^2}\sigma_1^3}$$

$$b_{1200}=-\frac{9\beta_4\mu_2(16\mu_2^2+\sigma_1^2)}{\sigma_1^3}$$

$$b_{3000}=-\frac{3\beta_4\mu_2(16\mu_2^2+\sigma_1^2)}{\sigma_1^3}$$

$$b_{1020}=-\frac{2\mu_2\beta_3(4\mu_3^2-u_m^2p_3^2+\sigma_2^2)}{\sigma_1(\sigma_2+u_mp_3)^2}$$

$$b_{1011}=\frac{16\mu_2\beta_3\sqrt{\mu_3^2-\frac{1}{4}u_m^2p_3^2+\frac{1}{4}\sigma_2^2}\mu_3}{\sigma_1(\sigma_2+u_mp_3)^2}$$

$$b_{1002}=-\frac{2\mu_2\beta_3(4\mu_3^2+\sigma_2^2+2\sigma_2u_mp_3+u_m^2p_3^2)}{\sigma_1(\sigma_2+u_mp_3)^2}$$

$$b_{0120}=\frac{\beta_3(-\sigma_1^2u_m^2p_3^2+4\sigma_1^2\mu_3^2+\sigma_1^2\sigma_2^2)}{2\sqrt{16\mu_2^2+\sigma_1^2}\sigma_1(\sigma_2+u_mp_3)^2}+\frac{\beta_3(16\mu_2^2\sigma_2^2+64\mu_2^2\mu_3^2-16\mu_2^2u_m^2p_3^2)}{2\sqrt{16\mu_2^2+\sigma_1^2}\sigma_1(\sigma_2+u_mp_3)^2}$$

$$b_{0111}=\frac{16\mu_2\beta_3\sqrt{\mu_3^2-\frac{1}{4}u_m^2p_3^2+\frac{1}{4}\sigma_2^2}\mu_3}{\sigma_1(\sigma_2+u_mp_3)^2}$$

$$b_{0102}=\frac{\beta_3(16\mu_2^2\sigma_2^2+32\mu_2^2\sigma_2u_mp_3+16\mu_2^2u_m^2p_3^2+4\sigma_1^2\mu_3^2)}{2\sqrt{16\mu_2^2+\sigma_1^2}\sigma_1(\sigma_2+u_mp_3)^2}+\frac{\beta_3(\sigma_1^2\sigma_2^2+2\sigma_1^2\sigma_2u_mp_3+\sigma_1^2u_m^2p_3^2+64\mu_2^2\mu_3^2)}{2\sqrt{16\mu_2^2+\sigma_1^2}\sigma_1(\sigma_2+u_mp_3)^2}$$

$$c_{0030}=-\frac{3\mu_3\beta_1(4\mu_3^2-u_m^2p_3^2+\sigma_2^2)}{(\sigma_2+u_mp_3)^3}$$

$$c_{0012}=-\frac{9\mu_3\beta_1(4\mu_3^2+\sigma_2^2+2\sigma_2u_mp_3+u_m^2p_3^2)}{(\sigma_2+u_mp_3)^3}$$

$$c_{0021}=\frac{3\beta_1(-2u_m^3p_3^3\sigma_2-u_m^4p_3^4+16\mu_3^2\sigma_2^2+8\sigma_2\mu_3^2u_mp_3)}{2\sqrt{4\mu_3^2-u_m^2p_3^2+\sigma_2^2}(\sigma_2+u_mp_3)^3}+\frac{3\beta_1(-8\mu_3^2u_m^2p_3^2+2u_mp_3\sigma_2^3+\sigma_2^4+48\mu_3^4)}{2\sqrt{4\mu_3^2-u_m^2p_3^2+\sigma_2^2}(\sigma_2+u_mp_3)^3}$$

$$c_{0003}=\frac{3\beta_1(8\mu_3^2\sigma_2^2+16\sigma_2\mu_3^2u_mp_3+8\mu_3^2u_m^2p_3^2)}{2\sqrt{4\mu_3^2-u_m^2p_3^2+\sigma_2^2}(\sigma_2+u_mp_3)^3}+\frac{3\beta_1(16\mu_3^4+\sigma_2^4+4u_mp_3\sigma_2^3+6u_m^2p_3^2\sigma_2^2)}{2\sqrt{4\mu_3^2-u_m^2p_3^2+\sigma_2^2}(\sigma_2+u_mp_3)^3}+\frac{3\beta_1(4u_m^3p_3^3\sigma_2+u_m^4p_3^4)}{2\sqrt{4\mu_3^2-u_m^2p_3^2+\sigma_2^2}(\sigma_2+u_mp_3)^3}$$

$$c_{2010}=\frac{\beta_2(16\mu_2^2\sigma_2^2+32\mu_2^2\sigma_2u_mp_3)}{\sqrt{4\mu_3^2-u_m^2p_3^2+\sigma_2^2}(\sigma_2+u_mp_3)\sigma_1^2}$$

$$+\frac{\beta_2(16\mu_2^2u_m^2p_3^2+4\sigma_1^2\mu_3^2\sigma_1^2\sigma_2^2+2\sigma_1^2\sigma_2u_mp_3)}{\sqrt{4\mu_3^2-u_m^2p_3^2+\sigma_2^2}(\sigma_2+u_mp_3)\sigma_1^2}$$
$$+\frac{\beta_2(\sigma_1^2u_m^2p_3^2+64\mu_2^2\mu_3^2)}{\sqrt{4\mu_3^2-u_m^2p_3^2+\sigma_2^2}(\sigma_2+u_mp_3)\sigma_1^2}$$

$$c_{2001}=\frac{\beta_2(16\mu_2^2\sigma_2^2+32\mu_2^2\sigma_2u_mp_3+16\mu_2^2u_m^2p_3^2)}{\sqrt{4\mu_3^2-u_m^2p_3^2+\sigma_2^2}(\sigma_2+u_mp_3)\sigma_1^2}$$
$$+\frac{\beta_2(4\sigma_1^2\mu_3^2+\sigma_1^2\sigma_2^2+2\sigma_1^2\sigma_2u_mp_3)}{\sqrt{4\mu_3^2-u_m^2p_3^2+\sigma_2^2}(\sigma_2+u_mp_3)\sigma_1^2}$$
$$+\frac{\beta_2(\sigma_1^2u_m^2p_3^2+64\mu_2^2\mu_3^2)}{\sqrt{4\mu_3^2-u_m^2p_3^2+\sigma_2^2}(\sigma_2+u_mp_3)\sigma_1^2}$$

$$c_{1110}=\frac{64\mu_3\beta_2\sqrt{\mu_2^2+\frac{1}{16}\sigma_1^2}\,\mu_2}{(\sigma_2+u_mp_3)\sigma_1^2}$$

$$c_{1101}=-\frac{8\beta_2\sqrt{16\mu_2^2+\sigma_1^2}\,\mu_2(4\mu_3^2+\sigma_2^2)}{\sqrt{4\mu_3^2-u_m^2p_3^2+\sigma_2^2}(\sigma_2+u_mp_3)\sigma_1^2}$$
$$-\frac{8\beta_2\sqrt{16\mu_2^2+\sigma_1^2}\,\mu_2(2\sigma_2u_mp_3+u_m^2p_3^2)}{\sqrt{4\mu_3^2-u_m^2p_3^2+\sigma_2^2}(\sigma_2+u_mp_3)\sigma_1^2}$$

$$c_{0210}=-\frac{2\mu_3\beta_2(16\mu_2^2+\sigma_1^2)}{(\sigma_2+u_mp_3)\sigma_1^2}$$

$$c_{0201}=\frac{\beta_2(16\mu_2^2\sigma_2^2+32\mu_2^2\sigma_2u_mp_3+16\mu_2^2u_m^2p_3^2)}{\sqrt{4\mu_3^2-u_m^2p_3^2+\sigma_2^2}(\sigma_2+u_mp_3)\sigma_1^2}$$
$$+\frac{\beta_2(4\sigma_1^2\mu_3^2+\sigma_1^2\sigma_2^2+2\sigma_1^2\sigma_2u_mp_3)}{\sqrt{4\mu_3^2-u_m^2p_3^2+\sigma_2^2}(\sigma_2+u_mp_3)\sigma_1^2}$$
$$+\frac{\beta_2(\sigma_1^2u_m^2p_3^2+64\mu_2^2\mu_3^2)}{\sqrt{4\mu_3^2-u_m^2p_3^2+\sigma_2^2}(\sigma_2+u_mp_3)\sigma_1^2}$$

$$d_{0003}=\frac{3\mu_3\beta_1(4\mu_3^2+\sigma_2^2+2\sigma_2u_mp_3+u_m^2p_3^2)}{(\sigma_2+u_mp_3)^3}$$

$$d_{0021}=\frac{9\mu_3\beta_1(4\mu_3^2-u_m^2p_3^2+\sigma_2^2)}{(\sigma_2+u_mp_3)^3}$$

$$d_{0012}=-\frac{3\beta_1\sqrt{4\mu_3^2-u_m^2p_3^2+\sigma_2^2}(12\mu_3^2+\sigma_2^2)}{2(\sigma_2+u_mp_3)^3}$$
$$-\frac{3\beta_1\sqrt{4\mu_3^2-u_m^2p_3^2+\sigma_2^2}(2\sigma_2u_mp_3+u_m^2p_3^2)}{2(\sigma_2+u_mp_3)^3}$$

$$d_{0030}=-\frac{3\beta_1(4\mu_3^2-u_m^2p_3^2+\sigma_2^2)^{\frac{3}{2}}}{2(\sigma_2+u_mp_3)^3}$$

$$d_{2010}=-\frac{\beta_2\sqrt{4\mu_3^2-u_m^2p_3^2+\sigma_2^2}(16\mu_2^2+\sigma_1^2)}{(\sigma_2+u_mp_3)\sigma_1^2}$$

$$d_{2001}=\frac{2\mu_3\beta_2(16\mu_2^2+\sigma_1^2)}{(\sigma_2+u_mp_3)\sigma_1^2}$$

$$d_{1110}=\frac{64\beta_2\sqrt{\mu_3^2-\frac{1}{4}u_m^2p_3^2+\frac{1}{4}\sigma_2^2}\sqrt{\mu_2^2+\frac{1}{16}\sigma_1^2}\,\mu_2}{(\sigma_2+u_mp_3)\sigma_1^2}$$

$$d_{1101}=\frac{\beta_2(32\mu_3v_2u_1\mu_2^2+2\mu_3v_2u_1\sigma_1^2-8\mu_3v_2\sqrt{16\mu_2^2+\sigma_1^2}\,\mu_2u_2)}{(\sigma_2+u_mp_3)\sigma_1^2}$$
$$+\frac{\beta_2(-16\sqrt{4\mu_3^2-u_m^2p_3^2+\sigma_2^2}v_1u_1\mu_2^2-\sqrt{4\mu_3^2-u_m^2p_3^2+\sigma_2^2}v_1u_1\sigma_1^2)}{(\sigma_2+u_mp_3)\sigma_1^2}$$
$$+\frac{\beta_2(+4\sqrt{4\mu_3^2-u_m^2p_3^2+\sigma_2^2}v_1\sqrt{16\mu_2^2+\sigma_1^2}\,\mu_2u_2)}{(\sigma_2+u_mp_3)\sigma_1^2}$$

Appendix 2

$$M_{1i}(u_1,u_2,v_1,v_2)=\sum_{\substack{m+n+p+q=3\\(p+q)^{m+n}=1\\0\le m,n,p,q\le 3}} i_{m,n,p,q}u_1^mu_2^nv_1^pv_2^q$$

$$M_{2j}(u_1,u_2,v_1,v_2)=\sum_{\substack{m+n+p+q=3\\(p+q)^{m+n}=1\\0\le m,n,p,q\le 3}} i_{m,n,p,q}u_1^mu_2^nv_1^pv_2^q$$

$$M_{3i}(u_1,u_2,v_1,v_2)=\sum_{\substack{m+n+p+q=3\\(m+n)^{p+q}=1\\m+n\ne 1\\0\le m,n,p,q\le 3}} i_{m,n,p,q}u_1^mu_2^nv_1^pv_2^q$$

$$M_{4j}(u_1,u_2,v_1,v_2)=\sum_{\substack{m+n+p+q=3\\(p+q)^{m+n}=1\\m+n\ne 0\\0\le m,n,p,q\le 3}} i_{m,n,p,q}u_1^mu_2^nv_1^pv_2^q$$

$$M_{5i}(u_1,u_2,v_1,v_2)=\sum_{\substack{m+n+p+q=3\\(m+n)^{p+q}=1\\m+n\ne 1\\0\le m,n,p,q\le 3}} i_{m,n,p,q}u_1^mu_2^nv_1^pv_2^q$$

$$M_{6j}(v_1,v_2)=\sum_{\substack{p+q=3\\0\le p,q\le 3}} i_{m,n,p,q}v_1^pv_2^q,$$

$$R_1(u,\theta,\bar{v},\varepsilon)=O(\bar{v}^2+|u|^2+|\varepsilon|+|u\bar{v}|)$$

$$R_2(u,\theta,\bar{v},\varepsilon)=O(|u|^4+|u|^3|\bar{v}|+|u|\bar{v}^3+|\bar{v}u|^2+|\varepsilon|(\bar{v}^2+|u|^2+|u\bar{v}|)+\varepsilon^2)$$

$$R_3(u,\theta,\bar{v},\varepsilon)=O(|u|^4+|u|^3|\bar{v}|+|u|\bar{v}^3+|\bar{v}u|^2+|\varepsilon|(\bar{v}^2+|u|^2+|u\bar{v}|)+\varepsilon^2)$$

$$a_{30}=2a_{1020}h\cos^2\theta+2a_{1011}h\cos\theta\sin\theta+2a_{1002}h\sin^2\theta$$

$$\begin{aligned}a_{40}&=-2a_{0100}d_{0003}h\sin\theta\cos^3\theta+2a_{0100}c_{0003}h\sin^2\theta\cos^2\theta+2a_{0100}d_{0003}h\sin\theta\cos\theta\\&-2a_{0100}c_{0003}h\sin^2\theta+2a_{0100}d_{0021}h\sin\theta\cos^3\theta-2a_{0100}c_{0021}h\sin^2\theta\cos^2\theta\\&+2a_{0111}h\sin\theta\cos\theta-a_{0100}d_{0001}\sin\theta\cos\theta+2a_{0102}h-2a_{0100}d_{0012}h\cos^4\theta\\&+2a_{0100}c_{0012}h\sin\theta\cos^3\theta-2a_{0102}h\cos^2\theta+2a_{0100}d_{0030}h\cos^4-2a_{0100}c_{0030}h\sin\theta\cos^3\theta\\&+2a_{0120}h\cos^2\theta+2a_{0100}d_{0012}h\cos^2\theta-2a_{0100}c_{0012}h\sin\theta\cos\theta\end{aligned}$$

$$\begin{aligned}a_{03}&=2b_{1000}d_{0003}h\sin\theta\cos^3\theta-2b_{1000}c_{0003}h\sin^2\theta\cos^2\theta-2b_{1000}d_{0003}h\sin\theta\cos\theta\\&+2b_{1000}c_{0003}h\sin^2\theta-2b_{1000}d_{0021}h\sin\theta\cos^3\theta+2b_{1000}c_{0021}h\sin^2\theta\cos^2\theta\\&+2b_{1011}h\sin\theta\cos\theta+b_{1000}d_{0001}\sin\theta\cos\theta+2b_{1002}h+2b_{1000}d_{0012}h\cos^4\theta\\&-2b_{1000}c_{0012}h\sin\theta\cos^3\theta-2b_{1002}h\cos^2\theta-2b_{1000}d_{0030}h\cos^4+2b_{1000}c_{0030}h\sin\theta\cos^3\theta\\&+2b_{1020}h\cos^2\theta-2b_{1000}d_{0012}h\cos^2\theta+2b_{1000}c_{0012}h\sin\theta\cos\theta,\end{aligned}$$

$$a_{04}=-b_{0001}+2b_{0120}h\cos^2\theta+2b_{0111}h\cos\theta\sin\theta+2b_{0102}h\sin^2\theta,$$

$$\begin{aligned}b_{10}&=2h(-2d_{0012}h\sin\theta\cos^3\theta-2c_{0003}h\sin\theta\cos^3\theta+2c_{0003}h\sin\theta\cos\theta+2d_{0030}h\sin\theta\cos^3\theta\\&+2c_{0021}h\sin\theta\cos^3\theta+2d_{0012}h\sin\theta\cos\theta-2d_{0021}h\cos^4\theta+d_{0001}\cos^2\theta+2c_{0030}h\cos^4\theta\\&+2c_{0012}h\cos^2\theta+2d_{0021}h\cos^2\theta-d_{0001}+2d_{0003}h-4d_{0003}h\cos^2\theta\\&-2c_{0012}h\cos^4\theta+2d_{0003}h\cos^4\theta\end{aligned}$$

$$\begin{aligned}b_{20}&=8d_{0003}h\sin^2\theta+4c_{0012}h\sin^2\theta\cos^2\theta-2c_{0003}h\sin\theta\cos^3\theta+2c_{0003}h\sin\theta\cos\theta\\&-2d_{0012}h\sin\theta\cos^3\theta-8d_{0003}h\sin^2\theta\cos^2\theta-2d_{0001}\sin^2\theta+6c_{0021}h\sin\theta\cos^3\theta\\&+6d_{0030}h\sin\theta\cos^3\theta+4d_{0021}h\sin^2\theta\cos^2\theta+2d_{0012}h\sin\theta\cos\theta+4c_{0012}h\cos^2\theta\\&+6c_{0003}h\sin\theta\cos\theta+4d_{0021}h\cos^2\theta+6d_{0012}h\sin\theta\cos\theta-4c_{0012}h\cos^4\theta\\&+8c_{0030}h\cos^4\theta+2c_{0021}h\sin\theta\cos^3\theta+2d_{0030}h\sin\theta\cos^3\theta-6d_{0012}h\sin\theta\cos^3\theta\\&-6c_{0003}h\sin\theta\cos^3\theta-4d_{0021}h\cos^4\theta\end{aligned}$$

$$\begin{aligned}b_{30}&=-6\sqrt{2}h^{3/2}(-c_{0003}\sin\theta+c_{0003}\sin\theta\cos^2\theta-c_{0021}\sin\theta\cos^2\theta-c_{0012}\cos\theta\\&-c_{0030}\cos^3\theta+c_{0012}\cos^3\theta)\end{aligned}$$

$$\begin{aligned}b_{40}&=-\sqrt{2h}(-6d_{0021}\sin\theta\cos^2\theta-6d_{0003}h\sin\theta+6d_{0003}h\sin\theta\cos^2\theta+d_{0001}\sin\theta\\&-6d_{0012}h\cos\theta-6d_{0030}h\cos^3\theta+6d_{0012}h\cos^3\theta)\end{aligned}$$