Vertical structure of velocity field induced by I and II mode solitary waves in a stratified fluid

OXANA KURKINA, EKATERINA ROUVINSKAYA, ANDREY KURKIN,
LIDIYA TALALUSHKINA, AYRAT GINIYATULLIN
Laboratory of Modeling of Natural and Anthropogenic Disasters
Nizhny Novgorod State Technical University n.a. R.E. Alekseev
603950 Nizhny Novgorod, Minin St., 24
RUSSIAN FEDERATION
aakurkin@gmail.com   http://www.nntu.ru

Abstract: The structure of the velocity field induced by internal solitary waves of the first and second modes is determined. The contribution from second-order terms in asymptotic expansion into the horizontal velocity is estimated for the models of almost two- and three-layer fluid density stratification for solitons of positive and negative polarity. The influence of the nonlinear correction manifests itself firstly in the shape of the lines of zero horizontal velocity: they are curved and the shape depends on the soliton amplitude and polarity while for the leading-order wave field they are horizontal. Also the wave field accounting for the nonlinear correction for mode I waves has smaller maximal absolute values of negative velocities (near-surface for the soliton of elevation, and near-bottom for the soliton of depression) and larger maximums of positive velocities. Thus for the solitary internal waves of positive polarity weakly nonlinear theory overestimates the near-bottom velocities and underestimates the near-surface current. For solitary waves of negative polarity, which are the most typical for hydrological conditions of low and middle latitudes, the situation is the opposite. II mode soliton's velocity field in almost two-layer fluid reaches its maximal absolute values in a middle layer instead of near-bottom and near-surface maximums for I mode solitons.

Key-Words: Internal waves, Gardner equation, near-bottom velocity, near-surface velocity

1 Introduction

Research on nonlinear internal waves' dynamics have quite a long history, originating in the first half of the twentieth century. Such waves have an influence on the hydrological regime of natural water bodies due to horizontal and vertical exchange, redistribution of heat, mixing of water, forming of the bottom topography etc. It is proved that such waves can create considerable loads and bending moments on the underwater parts of the hydraulic engineering constructions, as well as contribute to the sediment resuspension. In the context of the impact on the environment internal solitary and breather-like waves are of greatest interest as the most intensive formations. Solitary waves are observed almost everywhere on the ocean shelves and they are clearly visible on satellite images [1].

The shape and properties of these waves are studied well enough in the framework of various theoretical models, in particular within the weakly nonlinear theory of long waves, represented by the Korteweg-de Vries equation and its extended versions, such as the Gardner equation, the modified Korteweg-de Vries equation, "2 + 4" Korteweg-de Vries equation [2], and others. In the majority of classical studies devoted to internal solitary waves emphasis placed on the dynamics of such waves during the propagation over a rough bottom (see, e.g., paper [3], dedicated to the transformation of solitons over a sloping bottom and the paper [4] devoted to the dynamics of breather-like waves in the shelf zone of the Baltic sea), the interaction of such waves, or estimation of such local characteristics, as near-bottom and near-surface velocity and pressure variations at the bottom and on the pillars induced by the propagating waves. However, it is necessary to represent the structure of the velocity field in the entire water column for a more complete understanding of what is happening in the ocean during the propagation of internal solitary waves. Some peculiarities of the internal solitary waves' spatial structure are investigated in the framework of the Korteweg-de Vries equation [5] and the modified Korteweg-de Vries equation for the ocean with two pycnoclines [6].

2 Theoretical model

The weakly nonlinear theory of long IWs in a vertical section of stratified basin assumes, that the internal wave field (in particular, the vertical isopycnal displacement $\zeta(z, x, t)$) can be expressed as a series (up to the 2nd order in amplitude) [7]:

$$\zeta(z,x,t) = \eta(x,t)\Phi(z) + \eta^2(x,t)F(z), \qquad (1)$$

where x is horizontal axis, z is vertical axis directed upwards, t is time, $\eta(x, t)$ describes the transformation of a wave along the axis of propagation and its evolution in time. Function $\Phi(z)$ (the vertical mode) describes vertical structure of long internal wave, and $F(z)$ is the first nonlinear correction to $\Phi(z)$. $\Phi(z)$ is a solution of an eigenvalue problem, which can be written in the form (in Boussinesq approximation usually valid for natural sea stratifications):

$$\frac{d^2\Phi}{dz^2} + \frac{N^2(z)}{c^2}\Phi = 0, \quad \Phi(0) = \Phi(H) = 0. \qquad (2)$$

Here eigenvalue c is the phase speed of long linear internal wave, H is the total water depth, $N(z)$ is the Brunt-Väisälä frequency (BVF) determined by the expression:

$$N^2(z) = -\frac{g}{\rho(z)}\frac{d\rho(z)}{dz}, \qquad (3)$$

g is gravity acceleration and $\rho(z)$ is undisturbed density profile. It is well known, that problem (2) has an infinite number of eigenvalues $c_1 > c_2 > c_3 > \ldots$ and corresponding eigenfunctions $\Phi_1, \Phi_2, \Phi_3, \ldots$. We consider only the first (lowest) mode, when the function Φ has a single zero at $z=0$ and $z=H$. This mode is usually the most energetic in the internal wave spectrum. It is convenient to normalize the solution so that the maximum of $\Phi(z)$ is $\Phi_{\max} = \Phi(z_{\max}) = 1$.

In this case the leading order solution $\eta(x, t)$ coincides with the isopycnal surface displacement at $z_{\max}$:

$$\varsigma(x, z_{\max}, t) = \eta(x,t). \qquad (4)$$

Function $F(z)$ can be found as a solution of the inhomogeneous boundary problem:

$$\begin{cases} \dfrac{d^2F}{dz^2} + \dfrac{N^2}{c^2}F = -\dfrac{\alpha}{c}\dfrac{d^2\Phi}{dz^2} + \dfrac{3}{2}\dfrac{d}{dy}\left[\left(\dfrac{d\Phi}{dz}\right)^2\right], \\ F(0) = F(H) = 0, \end{cases} \qquad (5)$$

where the auxiliary normalization condition $F(z_{\max})=0$ is used to determine the solution uniquely.

In this model, the function $\eta(x, t)$ satisfies the nonlinear evolution equation (extended Korteweg-de Vries (KdV) or Gardner equation):

$$\frac{\partial\eta}{\partial t} + \left(c + \alpha\eta + \alpha_1\eta^2\right)\frac{\partial\eta}{\partial x} + \beta\frac{\partial^3\eta}{\partial x^3} = 0. \qquad (6)$$

This equation contains cubic nonlinearity, the presence of which provides better predictions of wave form, especially in the coastal zone. The coefficients of this equation are determined through the $\Phi(z)$ and $F(z)$:

$$\beta = \frac{c\int_0^H \Phi^2 dz}{2\int_0^H \left(\frac{d\Phi}{dz}\right)^2 dz}, \quad \alpha = \frac{3c\int_0^H \left(\frac{d\Phi}{dz}\right)^3 dz}{2\int_0^H \left(\frac{d\Phi}{dz}\right)^2 dz}, \qquad (7)$$

$$\alpha_1 = \frac{1}{2\int_0^H \left(\frac{d\Phi}{dz}\right)^2 dz}\int_0^H dz\left\{9c\frac{dF}{dz}\left(\frac{d\Phi}{dz}\right)^2 - 6c\left(\frac{d\Phi}{dz}\right)^4 + \right.$$

$$\left. + 5\alpha\left(\frac{d\Phi}{dz}\right)^3 - 4\alpha\frac{dF}{dz}\frac{d\Phi}{dz} - \frac{\alpha^2}{c}\left(\frac{d\Phi}{dz}\right)^2\right\}.$$

Let us consider the single-soliton solution of Eq. (6):

$$\eta(x,t) = \frac{A}{1 + B\,\mathrm{ch}(\gamma(x - Vt))}, \qquad (8)$$

where the soliton velocity $V = c + \beta\gamma^2$ is expressed through the inverse width of soliton, γ, and the soliton amplitude, a, or the extremum of the function (8), is

$$A = a(1+B),\ \gamma^2 = \frac{A\alpha}{6\beta},\ B^2 = 1 + \frac{6\alpha_1\beta\gamma^2}{\alpha^2}. \qquad (9)$$

When the cubic nonlinear coefficient α_1 is negative, soliton solutions of single polarity, with $\alpha\eta > 0$, exist with amplitudes between zero and a limiting value

$$a_{\lim} = \frac{\alpha}{|\alpha_1|}. \qquad (10)$$

With the use of Eq. (1) the components of velocities of particles (u, w) in the vertical section (x, z) can be expressed as follows:

$$\begin{cases} u(x,z,t) = u_l + u_n, \\ u_l = c\eta(x,t)\dfrac{d\Phi}{dz},\ u_n = \left(\dfrac{\alpha}{2}\dfrac{d\Phi}{dz} + c\dfrac{dF}{dz}\right)\eta^2, \end{cases} \qquad (11)$$

$$\begin{cases} w(x,z,t) = w_l + w_n, \\ w_l = -c\dfrac{\partial\eta}{\partial x}\Phi(z),\ w_n = -\left(\alpha\Phi(z) + 2cF(z)\right)\eta\dfrac{\partial\eta}{\partial x}. \end{cases} \qquad (12)$$

The horizontal velocity component u gives the greatest contribution into the local current speed. This is typical for long waves and this characteristic of internal wave field must be considered in the analysis of near-bottom processes connected with sediment transport. The first terms in Eqs. (11) and (12) correspond to the leading order of the asymptotic expansion. The remaining additives result from the first nonlinear correction in the asymptotic series. Thus, for the forecast of the local current speed one has to determine the isopycnal displacement $\eta(x, t)$ at the level of z_{max} (see (5)), the vertical IW mode $\Phi(z)$ and its nonlinear correction $F(z)$. The amplitude of $\eta(x, t)$ is not known a priori, it depends upon a large number of background conditions of internal wave generation, and can be found by means of the detailed simulation.

3 Results. Quasi-two-layer density stratification

Structure of the linear part and the contribution of the correction terms to the horizontal velocity can be estimated for the model stratification. A typical density stratification profile is taken as a two-layer fluid with a smooth density change on the pycnocline

$$\rho(z) = \frac{\Delta\rho}{2}\,\mathrm{th}\,\frac{z - z_p}{w_p}, \qquad (13)$$

where $\Delta\rho$ is the value of the density “jump” (assumed here equal to 10 kg/m^3), z_p is the position of the “jump” center along the vertical (here it corresponds to 35 m from the surface, with a total depth of 100 m), w_p – typical width of the “jump” along the vertical coordinate (here it is equal 8 m). This profile of the fluid density and the vertical profile of the Brunt - Väisälä frequency are shown in Fig. 1

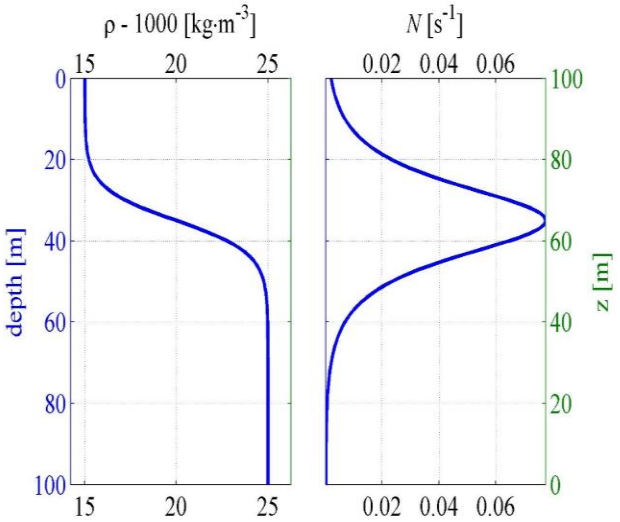

Fig. 1. Left panel: model vertical profile of sea water density; right panel: Brunt - Väisälä frequency

The function $\Phi(z)$ and the nonlinear correction $F(z)$ for the first (lowest) mode are shown in Fig. 2 for such density stratification. The coefficients of the Gardner equation (6) under such background conditions are given in Table. 1.

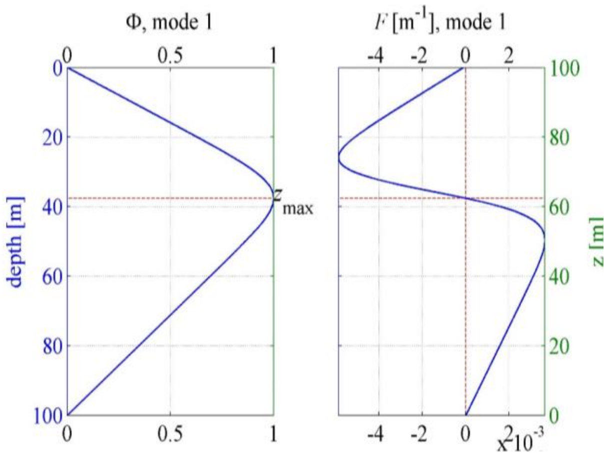

Fig. 2. Left panel: typical form of the first linear mode function for internal waves; right panel: nonlinear correction to it.

Table 1. The coefficients of the Gardner equation for internal waves of the first and second modes

Mode number	1	1
c [m/s]	1.35	0.38
β [m^3/s]	582	27
α [s^{-1}]	-0.0262	0.0622
α_1 [m^{-1}·s^{-1}]	-0.00157	0.00131

It can be noted that for the first mode both nonlinear coefficients α and α_1 are negative and in such a fluid only solitons of negative polarity bounded by the limiting amplitude $a_{\lim}$ = –16.7 m can exist.

The contours of equal density (isopycnic lines) and the contours of the horizontal velocity in the linear approximation during the propagation of internal solitary wave of the first mode with an amplitude of $-H/10$ are shown in Fig. 3.

The spatial structure of the nonlinear correction to the horizontal velocity and its total field are shown in Fig. 4. These figures demonstrate a quasi-two-layer structure of horizontal velocity with a thin transition layer of width $2w_p$. Particle velocity is considered positive when direction of its motion coincides with the direction of the soliton’ propagation (here it is the direction of the positive values of the horizontal axis). For the chosen density stratification, the particle velocities for a negative-polarity soliton are positive in the upper layer and negative in the lower layer.

The velocities in the upper and lower layers reach the maximum absolute values at the surface and at the bottom of the fluid, respectively. The influence of the nonlinear correction is manifested primarily in the form of an isoline of zero horizontal velocity (shown in bold in Fig. 4): it is a horizontal line in the linear approximation, but when the nonlinear correction is taken into account, it is curved. Its shape depends on the amplitude of the soliton. Accounting for the nonlinear correction also leads to a slight increase in the absolute values of the particle velocity near the bottom and to an insignificant decrease in near-surface velocities. Thus, the linear part of the horizontal velocity in (11) gives an upper estimate for the flow velocity at the surface and the lower – for the bottom.

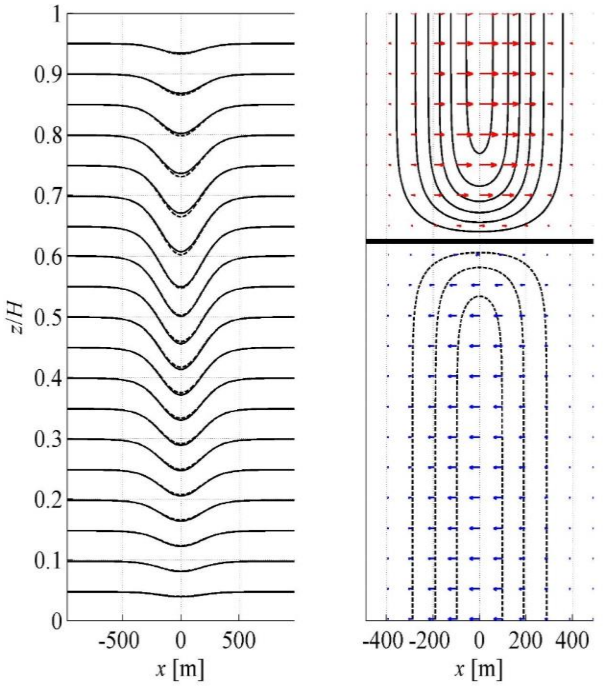

Fig. 3. Left panel: wave field represented as a vertical displacement of isopycnic lines at different horizons, when the perturbation at the maximum of the first linear mode is the soliton of the Gardner equation with an amplitude equal to -H/10 (solid lines are the linear component, the dashed lines – vertical displacements with consideration of nonlinear correction. Right panel: the horizontal velocity field corresponding to a linear wave field on a left panel (bold solid line is the isoline corresponding to the zero velocity value, thin solid lines are the isolines of positive values, the dashed lines are the contours of negative values; the interval between the contours corresponds to 0.05c).

The vertical structure of the second mode and the its nonlinear correction $F(z)$ are shown in Fig. 5. The function $\Phi(z)$ vanishes once inside the interval (0, H) in this case. As one can see from Table 1, both nonlinear coefficients are positive for the second mode. In this case, there are two families of soliton solutions: the first one with positive polarity, for which there is no upper or lower amplitude limitation, and the second family with negative polarity, for which the absolute value of amplitude should be greater than the modulus of the amplitude of the algebraic soliton ($a_{alg} = -2\alpha/\alpha_1$).

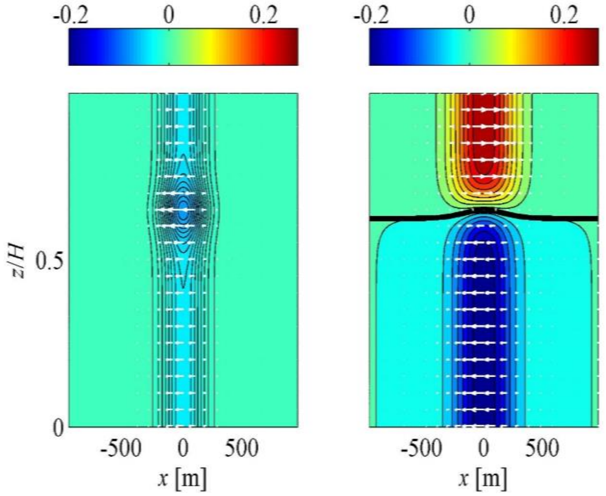

Fig. 4. Left panel: nonlinear correction to the horizontal velocity; right panel: field of horizontal velocity with consideration of nonlinear correction for the internal solitary wave of the first mode. (Solid line - isoline, corresponding to zero velocity value, the interval between the contours corresponds to 0.05c). Both characteristics are normalized to the phase velocity of the first mode.

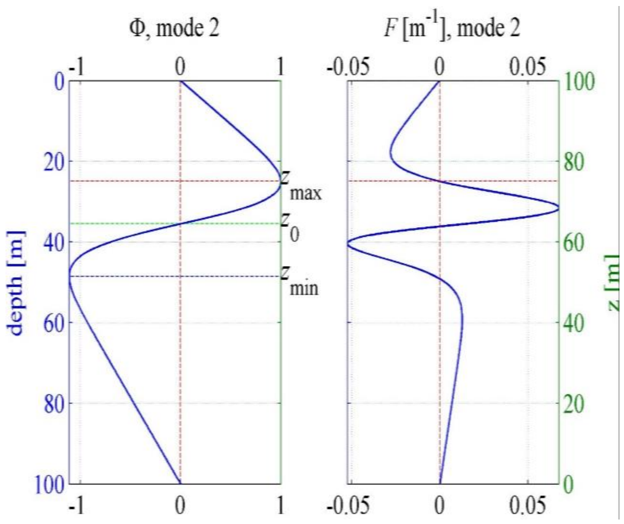

Fig. 5. Structure of the second linear mode function for internal waves (left); nonlinear correction (right).

The displaced isopycnic lines and the contours of the horizontal velocity in the linear approximation during the propagation of internal solitary wave of the second mode with an amplitude of H/20 (having a positive polarity at the maximum of the linear mode) are shown in Fig. 6. Such solitary waves of the second mode are usually called “convex” [8].

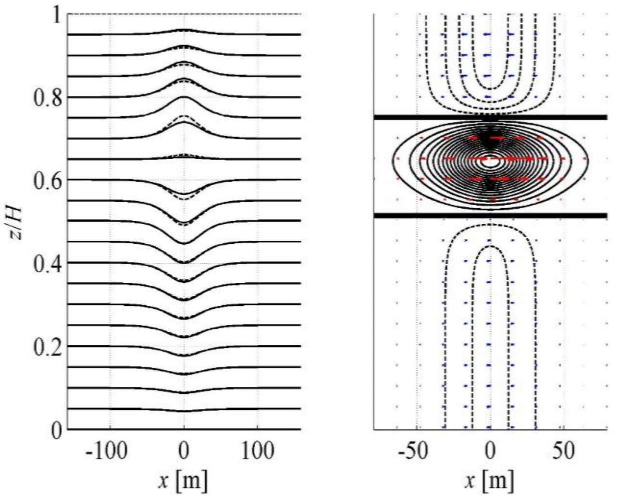

Fig. 6. Left panel: wave field represented as a vertical displacement of isopycnic lines at different horizons, when the perturbation at the maximum of the second linear mode is the soliton of the Gardner equation with an amplitude equal to $H/20$ (solid lines are the linear component, the dashed lines – vertical displacements with consideration of nonlinear correction. Right panel: the horizontal velocity field corresponding to a linear wave field on a left panel (bold solid line is the isoline corresponding to the zero velocity value, thin solid lines are the isolines of positive values, the dashed lines are the contours of negative values; the interval between the contours corresponds to $0.05c$).

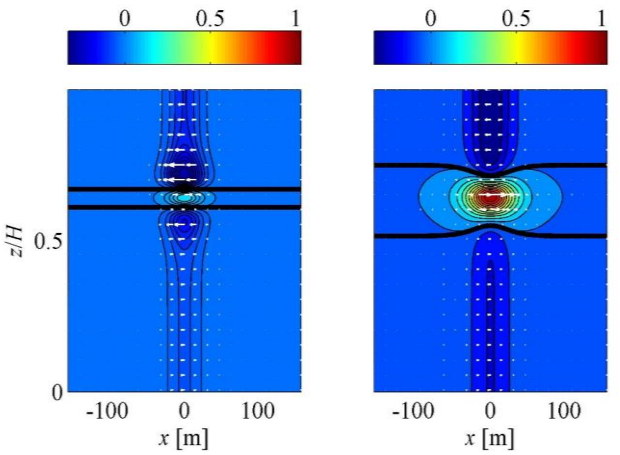

Fig. 7. Left panel: nonlinear correction to horizontal velocity. Right panel: field of horizontal velocity with consideration of nonlinear correction for the internal solitary wave of the second mode. Both characteristics are normalized to the phase velocity of the second mode.

The nonlinear correction to the horizontal velocity and its total value for an internal solitary wave of second mode are shown in Fig. 7. Maximal absolute values of the horizontal velocity are positive located inside the fluid. The range of positive and negative velocity values is asymmetric with respect to zero (the maximal absolute values of negative velocities are much less than maximal positive velocities' values). Near-bottom and near-surface velocities are small in relation to velocities in the "middle" layer of the fluid and they are negative. There are two lines of zero velocity in the water column. Taking into account the nonlinear correction for the vertical structure of the mode leads to their bending.

4 Results. Quasi-three-layer density stratification

It is obvious that the spatial structure of the velocity field of a solitary wave will differ substantially for both modes if we will use, for example, a three-layer model for which the wave regimes in the framework of the Gardner equation were studied in detail in [9].

The profile of the density for three-layer fluid and the vertical profile of the Brunt - Väisälä frequency for such fluid are shown in Fig. 8.

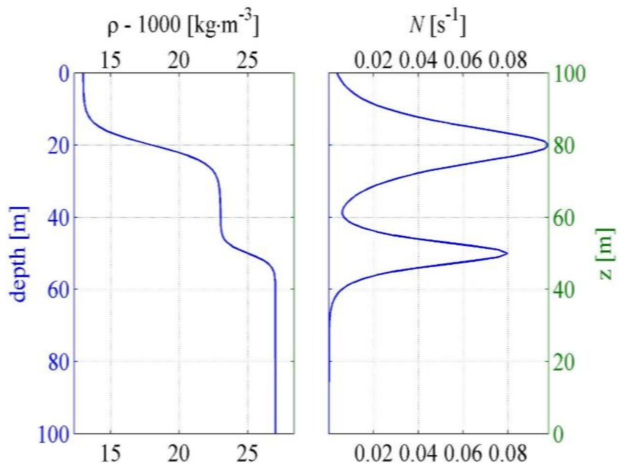

Fig. 8. Left panel: model vertical profile of three-layer fluid; right panel: Brunt - Väisälä frequency

The coefficients of the Gardner equation (6) under such background conditions are given in Table. 2.

Table 2. The coefficients of the Gardner equation for internal waves of the first and second modes in the three-layer fluid

Mode number	1	1
c [m/s]	1.33	0.69
β [m^3/s]	561	88
α [s^{-1}]	–0.0471	0.0538
α_1 [m^{-1}·s^{-1}]	0.0007	–0.0074

The function $\Phi(z)$ and the nonlinear correction $F(z)$ for the first mode are shown in Fig.9 for such density stratification.

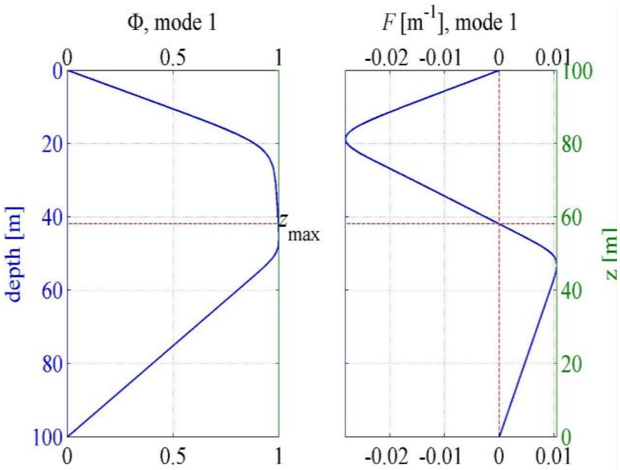

Fig. 9. Left panel: typical form of the first linear mode function for internal waves in a three-layer fluid; right panel: nonlinear correction to it.

The contours of equal density and the contours of the horizontal velocity in the linear approximation during the propagation of internal solitary wave of the first mode with an amplitude of $-H/10$ are shown in Fig. 10.

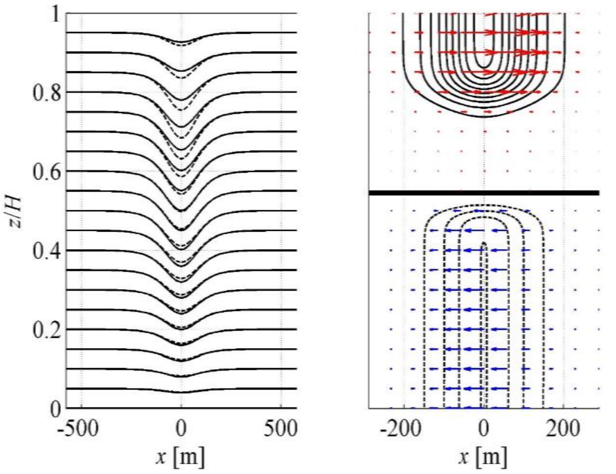

Fig. 10. Left panel: wave field represented as a vertical displacement of isopycnic lines at different horizons, when the perturbation at the maximum of the first linear mode is the soliton of the Gardner equation with an amplitude equal to $-H/10$ (solid lines are the linear component, the dashed lines – vertical displacements with consideration of nonlinear correction. Right panel: the horizontal velocity field corresponding to a linear wave field on a left panel (bold solid line is the isoline corresponding to the zero velocity value, thin solid lines are the isolines of positive values, the dashed lines are the contours of negative values; the interval between the contours corresponds to $0.05c$).

Structure of the isopycnal displacement field and the field of horizontal velocity for the soliton of the first mode in the selected three-layer fluid are qualitatively similar to that for the soliton of the first mode in a two-layer fluid. The spatial structure of the nonlinear correction to the horizontal velocity and its total value for such a case are shown in Fig. 11. Here we can also see qualitative agreement with the results presented in Fig. 4.

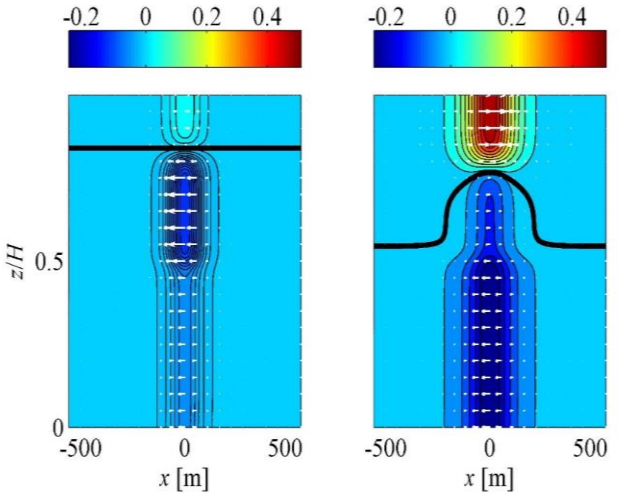

Fig. 11. Left panel: nonlinear correction to the horizontal velocity; right panel: field of horizontal velocity with consideration of nonlinear correction for the internal solitary wave of the first mode. (Solid line - isoline, corresponding to zero velocity value, the interval between the contours corresponds to $0.05c$).

The vertical structure of the second mode and the first nonlinear correction to it are shown in Fig. 12 for three-layer fluid.

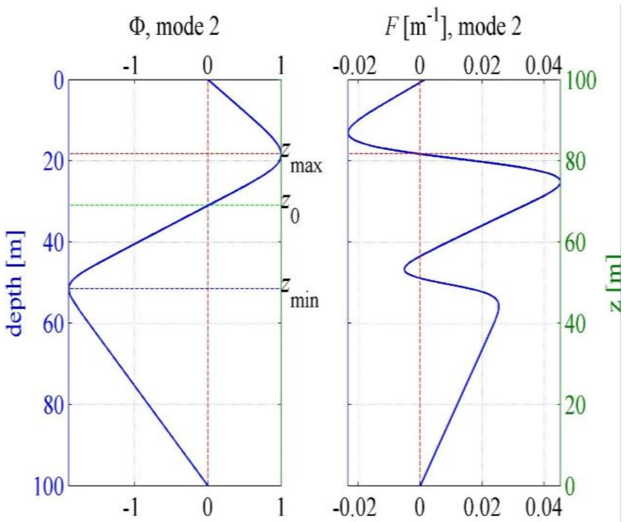

Fig. 12. Left panel: typical form of the second linear mode function for internal waves in a three-layer fluid; right panel: nonlinear correction to it.

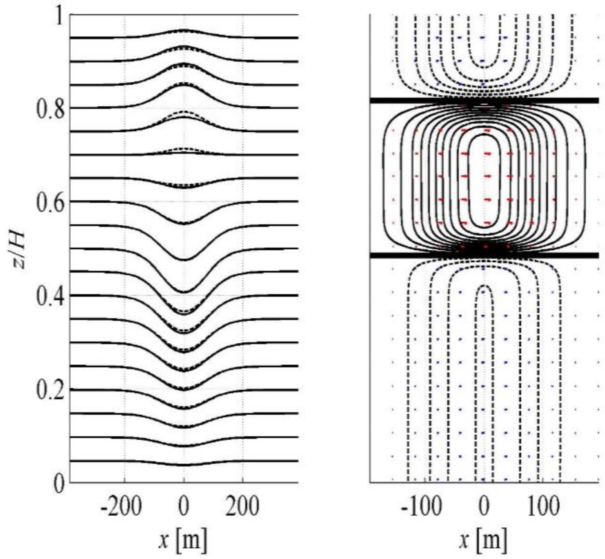

Fig. 13. Left panel: wave field represented as a vertical displacement of isopycnic lines at different horizons, when the perturbation at the maximum of the second linear mode is the soliton of the Gardner equation with an amplitude equal to $H/20$ (solid lines are the linear component, the dashed lines – vertical displacements with consideration of nonlinear correction. Right panel: the horizontal velocity field corresponding to a linear wave field on a left panel (bold solid line is the isoline corresponding to the zero velocity value, thin solid lines are the isolines of positive values, the dashed lines are the contours of negative values; the interval between the contours corresponds to $0.05c$).

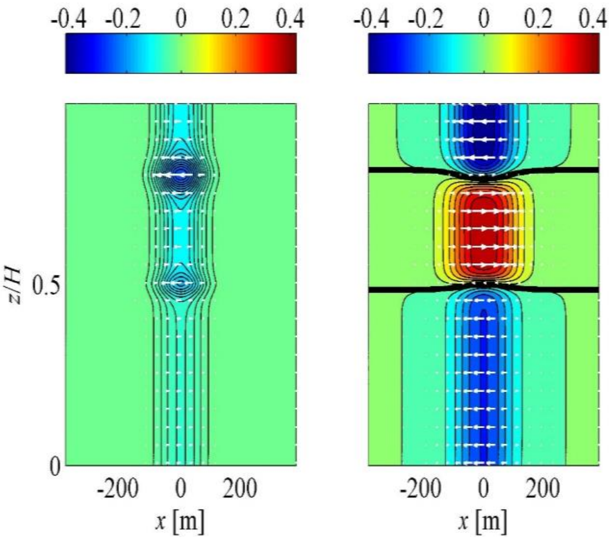

Fig. 14. Left panel: nonlinear correction to horizontal velocity in the three-layer fluid. Right panel: field of horizontal velocity with consideration of nonlinear correction for the internal solitary wave of the second mode. Both characteristics are normalized to the phase velocity of the second mode.

The displaced isopycnic lines and the contours of the horizontal velocity in the linear approximation during the propagation of internal solitary wave of the second mode with an amplitude of $H/20$ that has a positive polarity at the maximum of the linear mode (this wave is also of convex type) are shown in Fig. 13. The spatial structure of the horizontal velocity for an internal solitary wave of the second mode in three-layer fluid is shown in Fig. 14 in detail. From this figures, we can see that spatial structure of horizontal velocity field and field of isopycnal displacements induced by internal solitary wave of the second mode are qualitatively similar in two- and three-layer fluid. Maximal absolute values of the horizontal velocity are obtained in the mid-layer and near the surface for this example. Spatial structure of nonlinear correction to horizontal velocity in a three-layer fluid differs markedly from such a nonlinear correction in a two-layer fluid.

5 Conclusion

The structure of the velocity of flows induced by internal solitary waves of the first and second modes for the model profiles of the fluid density is investigated in the framework of weakly nonlinear theory of ideal stratified fluid. The characteristics of the medium strongly vary along the vertical. The thickness of the pycnocline(s) is rather small, and the horizontal velocity of the current in the soliton varies in a jumplike manner in the pycnocline(s), providing the conditions for the development of a Kelvin–Helmholtz instability and turbulization of the current in case when the viscosity is taken into account [10]. Turbulent processes in the pycnocline and the near-bottom layer will influence particle transport. Meanwhile, integral characteristics of the current, which are necessary for calculations of dynamical loads on underwater elements of marine engineering, are influenced insignificantly, when the viscosity is taken into account, and spatial evolution of the soliton and its attenuation occur at very large distances.

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