A Design of Cross-Coupled Oscillator with approximate 4th order polynomial rooting formula

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Abstract: - It is well known that input impedance of cross-coupled oscillator circuit has a fourth order denominator polynomial in frequency domain. Thus, it is important to design pole frequency by symbolic formula so that it is oscillate as a sinusoidal signal. It should be noted that in order for cross-coupled oscillator to oscillate, all of four pole frequencies should be positions on imaginary axis. The graph of time domain response is plotted by using inverse Laplace's transform. Four unknowns and four brackets of poles can be factored which were derived by partial fraction expansion. Time domain response of cross-coupled oscillator is plotted by using 0.5 micron CMOS level1 transistor model.

Key-Words: - CMOS, Cross-Coupled Oscillator, Input impedance

1 Introduction

The cross-coupled oscillator circuit diagram in MOSFET technology was proposed long time ago but no one indicate when it is first proposed. The cross-coupled oscillator can be designed based on two concepts. The first concept is based on Barkhausen criterion which is based on grouping of symbolic parameters in the denominator polynomial, then the symbolic denominator, then the symbolic denominator polynomial can be separate into real and imaginary part. The symbolic real and imaginary part can be equated to zero so that the oscillation frequency can be derived. The second concept is based on inverse Laplace's transform of the transfer function which can be separated into two methodologies. The first methodology is based on feedback concept. For example, it is the cascade of resonance amplifier one time which can be seen as a cross-coupled oscillator. It is composed of amplifier stage and feedback stage. The second methodology is based on input impedance derivation. This paper proposed how to design and optimize solution of input impedance in time domain by using partial fraction expansion. Pole frequencies of input impedance can be designed by approximate rooting formulas so that pole frequencies have only imaginary frequencies which is very difficult situation because usually pole frequencies are complex numbers.

The circuit diagram idea of the cross-coupled oscillator might come from the paper which is published by A. Abidi [1]. The circuit diagram of relaxation oscillator is different from the first cross-coupled oscillator which is appeared in the textbook of B. Razavi [2] because that circuit diagram did not use BJT transistor, but it used MOSFET transistor. It did not used current source at emitter terminal of the BJT transistor, it did not used capacitive coupling between emitter terminal. It did not used parallel inductor and capacitor at the collector terminal. It did not used parallel inductor and capacitor at the collector terminal. But the collector terminal of Q1 was connected with the base terminal of M1 with the gate terminal of M2. Since 1996, B. Razavi analyzed phase noise of CMOS oscillators of ring oscillator and relaxation oscillator. Input impedance function of CMOS cross-coupled oscillator was published by K. Tripetch since 2016 [3]. The mathematical empirical function were derived as a function of the DC bias point and geometrical parameters which was published since 2004 [4]. A comprehensive analysis of novel cross-coupled oscillator was derived by I. R. Chamas [5]. Common-Centroid layout of CMOS cross-coupled oscillator was proposed since 2009 [6] to make perfect symmetrical layout. Because non-symmetrical layout can make some offset in parasitic capacitance extraction and mismatch of aspect ratio.

CMOS fundamental oscillator was designed by B. Razavi since 2011 [7] at various frequencies. It is the highest oscillation frequency ever reported before 2011. The cross-coupled oscillator can be seen as a two-stage ring oscillator without parallel LC load between supply voltage and drain terminal but the paper reported three stage fully differential ring oscillator which can be seen as a cascade of three delay cells. The Barkhausen stability criterion was applied to a cascade three stage transfer function to derive the oscillation frequency formula which was published by P. Lucchi since 2011 [8]. The PLL synthesizer was designed for Bluetooth transmitter which was published by S. Saad [9]. The VCO of this PLL synthesizer used the cross-coupled oscillator with current source but they did not proposed design equation. The cross-coupled oscillator with current source could not be derived with feedback model because gain stage and feedback stage could not be separated because the sharing between gain stage and feedback stage is the current source. The closed form expressions for the oscillation frequency amplitude of CMOS LC tank oscillator were derived for the first time in weak inversion region of operation which was published since 2013 [10]. They solve stochastic differential equations which including two noise sources. The cross-coupled oscillator was designed by admittance parameters. Bostini [11] published the paper which they used the Barkhausen criterion to derived oscillation frequency.

2 Factorization of the fourth order polynomial

Factorization may not be derived directly without non-linear algebra and without approximation. The document which is published by [12] indicated strange solution of cubic polynomial and quartic polynomial which are the solutions of the third order polynomial rooting and fourth order polynomial rooting. But this paper proposed the approximate fourth order polynomial rooting by multiply fourth brackets of symbolic rooting which can be written as follow.

$$F(s)=(s+a)(s+b)(s+c)(s+d)=\left(s^2+(a+b)s+ab\right)\left(s^2+(c+d)s+cd\right)$$
$$F(s)=\begin{pmatrix} s^4+(a+b+c+d)s^3 \\ +\left((a+b)(c+d)+ab+cd\right)s^2 \\ +\left((a+b)cd+(c+d)ab\right)s+abcd \end{pmatrix}=s^4+gs^3+hs^2+ks+l \tag{1}$$

With approximation by eliminate one term, equation (1) can be written as follow.

$$F(s)=\begin{pmatrix} s^4+(a+b+c+d)s^3 \\ +\left((a+b)(c+d)+ab+cd\right)s^2 \\ +\left((a+b)cd+(c+d)ab\right)s+abcd \end{pmatrix}=s^4+gs^3+hs^2+ks+l \tag{2}$$

The parameter names a,b,c and d can be seen as the pole frequencies if $F(s)=0$. It can be written as a function of all coefficients in the right-hand side of equation (2)

$$x=\frac{h\pm\sqrt{h^2-4l}}{2}$$
$$m=\frac{g}{2}+\frac{k}{2x}$$
$$n=\frac{g}{2}-\frac{k}{2x} \tag{3}$$

$$a=\frac{m\mp\sqrt{m^2+4x}}{2} \tag{4}$$

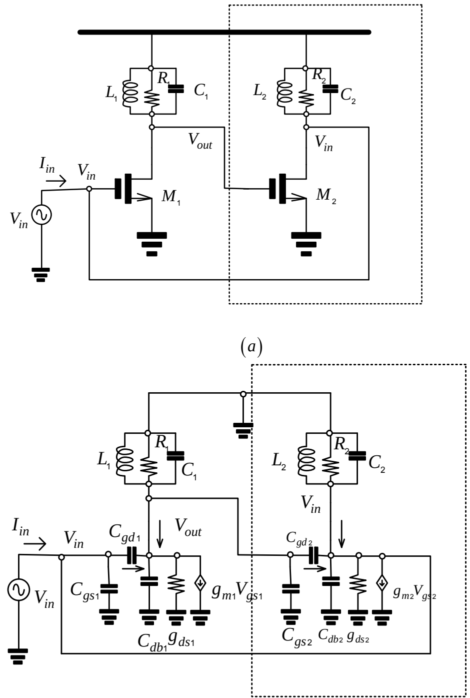
$$b=\frac{m\pm\sqrt{m^2+4x}}{2} \tag{5}$$

$$c=\frac{n\mp\sqrt{n^2-4x}}{2} \tag{6}$$

$$d=\frac{n\pm\sqrt{n^2-4x}}{2} \tag{7}$$

3 Input Impedance of Simple Cross-Coupled Oscillator

The CMOS cross-coupled oscillator circuit diagram is shown in figure 1 (a), its high frequency equivalent circuit of figure 1 (a) is shown in figure 1 (b). Input impedance can be derived [3] to have four pole frequencies. It can be written as follow.



(a)

(b)

Figure1 (a) Cross-Coupled Oscillator Circuit Diagram (b) High Frequency Equivalent circuit diagram of (a)

$$Z_{in} = \frac{V_{in}}{I_{in}} = \frac{sL_1\left(s^2\left(L_2C_{x1}\right) + sL_2\left(g_{ds2} + \frac{1}{R_2}\right) + 1\right)}{s^4a_4 + s^3a_3 + s^2a_2 + sa_1 + 1}$$

$$Z_{in} = \frac{\left(s^3\left[\frac{\left(L_1L_2C_{x1}\right)}{a_4}\right] + s^2\left[\frac{L_1L_2\left(g_{ds2} + \frac{1}{R_2}\right)}{a_4}\right] + s\left(\frac{L_1}{a_4}\right)\right)}{s^4 + s^3\left(\frac{a_3}{a_4}\right) + s^2\left(\frac{a_2}{a_4}\right) + s\left(\frac{a_1}{a_4}\right) + \left(\frac{1}{a_4}\right)}$$

$$Z_{in} = \frac{\left(s^3\left[\frac{\left(L_1L_2C_{x1}\right)}{a_4}\right] + s^2\left[\frac{L_1L_2\left(g_{ds2} + \frac{1}{R_2}\right)}{a_4}\right] + s\left(\frac{L_1}{a_4}\right)\right)}{s^4 + s^3g + s^2h + sk + l} \tag{8}$$

Partial fraction of fourth order polynomial can be written as follows with four unknown and four pole frequencies.

$$Z_{in} = \frac{A}{s+a} + \frac{B}{s+b} + \frac{C}{s+c} + \frac{D}{s+d}$$
$$C_{x1} = C_{db2} + C_{gs1} + C_{gd1} + C_{gd2} + C_2$$
$$C_{x2} = C_{gs2} + C_{gd2} + C_{db1} + C_{gd1} + C_1$$
$$a_4 = L_1C_{x2}L_2C_{x1} - L_1C_2\left(C_{gd1} + C_{gd2}\right)^2$$
$$a_3 = L_1C_{x2}L_2\left(g_{ds2} + \frac{1}{R_2}\right) + L_1L_2C_{x1}\left(\frac{1}{R_1} + g_{ds1}\right) + 2L_1L_2g_{m2}\left(C_{gd1} + C_{gd2}\right)$$
$$a_2 = L_1C_{x2} + L_2C_{x1} + L_1L_2\left(g_{ds1} + \frac{1}{R_1}\right)\left(g_{ds2} + \frac{1}{R_2}\right) - L_2^2g_{m2}^2$$
$$a_1 = L_1\left(\frac{1}{R_1} + g_{ds1}\right) + L_2\left(\frac{1}{R_2} + g_{ds2}\right), a_0 = 1 \tag{9}$$

4 Three conditions for designing imaginary pole frequency from formulas

The first condition is that parameters x in equation (3) should be real number if h can be designed to have value negative and l is positive so that the root on the right hand-side of equation (3) can be imaginary number.

$$h = \frac{a_2}{a_4} < 0, h = L_1C_{x2} + L_2C_{x1} + L_1L_2\left(g_{ds1} + \frac{1}{R_1}\right)\left(g_{ds2} + \frac{1}{R_2}\right) - L_2^2g_{m2}^2 < 0 \tag{10}$$

If the inductors of resonance circuit are equal then $L_1 = L_2 = L_3$

$$L_1C_{x2} + L_2C_{x1} + L_1L_2\left(g_{ds1} + \frac{1}{R_1}\right)\left(g_{ds2} + \frac{1}{R_2}\right) - L_2^2g_{m2}^2 < 0$$
$$L_3\left(C_{x2} + C_{x1}\right) + L_3^2\left[\left(g_{ds1} + \frac{1}{R_1}\right)\left(g_{ds2} + \frac{1}{R_2}\right) - g_{m2}^2\right] < 0$$
$$L_3\left[\left(C_{x2} + C_{x1}\right) + L_3\left[\left(g_{ds1} + \frac{1}{R_1}\right)\left(g_{ds2} + \frac{1}{R_2}\right) - g_{m2}^2\right]\right] < 0$$
$$L_3\left[\left(C_{x2} + C_{x1}\right) + L_3\left[\left(g_{ds1} + \frac{1}{R_1}\right)\left(g_{ds2} + \frac{1}{R_2}\right) - g_{m2}^2\right]\right] < 0$$
$$\left(C_{x2} + C_{x1}\right) + L_3\left[\left(g_{ds1} + \frac{1}{R_1}\right)\left(g_{ds2} + \frac{1}{R_2}\right) - g_{m2}^2\right] < 0 \tag{11}$$

$$L_3 > -\frac{\left(C_{x2} + C_{x1}\right)}{\left(g_{ds1} + \frac{1}{R_1}\right)\left(g_{ds2} + \frac{1}{R_2}\right) - g_{m2}^2}$$
$$Q = \left(g_{ds1} + \frac{1}{R_1}\right)\left(g_{ds2} + \frac{1}{R_2}\right) - g_{m2}^2 < 0 \tag{12}$$

The denominator should be negative so that the inductor can be positive. Isn't it possible to determine the dc operating point so that the condition $Q < 0$ is existed.

The second condition is that $l = 0 = \left(1/a_4\right)$. This condition might be impossible. The third condition is that parameter x in equation (3) should be pure imaginary number. Thus, parameter h should be equal with zero. Thus, transconductance in the circuit can be designed such that it is a function of passive inductor, passive capacitor, parasitic capacitances and passive resistor. It can be written as follows.

$$\begin{aligned}
&h = \frac{a_2}{a_4} = 0 \\
&h = L_1 C_{x2} + L_2 C_{x1} + L_1 L_2 \left(g_{ds1} + \frac{1}{R_1} \right)\left(g_{ds2} + \frac{1}{R_2} \right) - L_2^2 g_{m2}^2 = 0 \\
&L_3 \left(C_{x2} + C_{x1} \right) + L_3^2 \left[\left(g_{ds1} + \frac{1}{R_1} \right)\left(g_{ds2} + \frac{1}{R_2} \right) - g_{m2}^2 \right] = 0 \\
&L_3 \left[\left(C_{x2} + C_{x1} \right) + L_3 \left[\left(g_{ds1} + \frac{1}{R_1} \right)\left(g_{ds2} + \frac{1}{R_2} \right) - g_{m2}^2 \right] \right] = 0
\end{aligned} \tag{13}$$

$$L_3 = \frac{-\left(C_{x2} + C_{x1} \right)}{\left[\left(g_{ds1} + \frac{1}{R_1} \right)\left(g_{ds2} + \frac{1}{R_2} \right) - g_{m2}^2 \right]} \tag{14}$$

But from equation (14), it can be think that the result of inductance value is negative if the denominator is not designed to have negative value. Thus, the condition for equation (14) should be the same as equation (12).

5 Solution of inverse Laplace's transform of input impedance of cross-coupled oscillator

The solution of inverse Laplace's of general transfer function which have numerator polynomial divided by denominator polynomial of any order can be written as follow. It can also be written in factor form and residue form as follow. Residue is the form which describes unknown value which must be determined by comparing with coefficients which have the same order after multiply both sides of the equation with the factored form of pole polynomial which is the roots of denominator polynomial

$$\begin{aligned}
Z_{in}(s) &= \frac{a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + a_{n-3} s^{n-3} + \ldots + a_0}{b_n s^n + b_{n-1} s^{n-1} + b_{n-2} s^{n-2} + b_{n-3} s^{n-3} + \ldots + b_0} \\
Z_{in}(s) &= \frac{\left(s + f_{z1}\right)\left(s + f_{z2}\right) \times \ldots \times \left(s + f_{zn}\right)}{\left(s + f_{p1}\right)\left(s + f_{p2}\right) \times \ldots \times \left(s + f_{pn}\right)}
\end{aligned} \tag{15}$$

$$Z_{in}(s) = \frac{A_1}{s + f_{p1}} + \frac{A_2}{s + f_{p2}} + \frac{A_3}{s + f_{p3}} + \ldots + \frac{A_n}{s + f_{pn}} \tag{16}$$

Each pole frequency in equation (16) can be real number, imaginary number and complex number. From simulation with MATLAB with CMOS transistor level the solution usually contained conjugate complex numbers. Thus, the solution of inverse Laplace's transform of complex pole number can be written as follow.

$$\begin{aligned}
Z_{in}(s) &= \frac{A}{s + f_{p1}} + \frac{B}{s + f_{p2}} + \frac{C}{s + f_{p3}} + \frac{D}{s + f_{p4}} \\
z_{in}(t) &= \frac{v_{in}}{i_{in}} = Ae^{-f_{p1 \times t}} + Be^{-f_{p2 \times t}} + Ce^{-f_{p3 \times t}} + De^{-f_{p4 \times t}}
\end{aligned} \tag{17}$$

The solution of voltage input waveform is a function of time but unknown constant solution can be complex numbers. Real number and imaginary number. It is dependent on design and simulation. The unknown constants are programed after simulation as follow.

$$\begin{aligned}
A &= A_{21} + iA_{22} \\
B &= B_{21} + iB_{22} \\
C &= C_{21} + iC_{22} \\
D &= D_{21} + iD_{22}
\end{aligned} \tag{18}$$

$$\begin{aligned}
v_{in}(t) &= i_{in} \left[Ae^{-f_{p1 \times t}} + Be^{-f_{p2 \times t}} + Ce^{-f_{p3 \times t}} + De^{-f_{p4 \times t}} \right] \\
v_{in}(t) &= i_{in} \begin{bmatrix} \left(A_{21} + iA_{22}\right)e^{-(p_1 + ip_2) \times t} + \left(B_{21} + iB_{22}\right)e^{-(p_1 - ip_2) \times t} \\ + \left(C_{21} + iC_{22}\right)e^{-(p_3 + ip_4) \times t} + \left(D_{21} + iD_{22}\right)e^{-(p_3 - ip_4) \times t} \end{bmatrix} \\
v_{in}(t) &= i_{in} \begin{bmatrix} \left(A_{21} + iA_{22}\right)e^{-(p_1) \times t} e^{-ip_2 t} + \left(B_{21} + iB_{22}\right)e^{-(p_1) \times t} e^{ip_2 t} \\ + \left(C_{21} + iC_{22}\right)e^{-(p_3) \times t} e^{-ip_4 t} + \left(D_{21} + iD_{22}\right)e^{-(p_3) \times t} e^{ip_4 t} \end{bmatrix}
\end{aligned} \tag{19}$$

The solution of time domain waveform is still not finished. It must multiply symbolic complex number with complex trigonometry function which expand from Euler's identity. After that separate real and imaginary signal as follows.

$$v_{in}(t) = i_{in} \begin{bmatrix} \left(A_{21} + iA_{22}\right)e^{-(p_1) \times t} \left[\cos\left(p_2 t\right) - i \sin\left(p_2 t\right) \right] \\ + \left(B_{21} + iB_{22}\right)e^{-(p_1) \times t} \left[\cos(p_2 t) + i \sin(p_2 t) \right] \\ + \left(C_{21} + iC_{22}\right)e^{-(p_3) \times t} \left[\cos(p_4 t) - i \sin(p_4 t) \right] \\ + \left(D_{21} + iD_{22}\right)e^{-(p_3) \times t} \left[\cos(p_4 t) + i \sin(p_4 t) \right] \end{bmatrix} \tag{20}$$

$$v_{in}(t) = i_{in} \begin{bmatrix} \begin{pmatrix} A_{21}\cos(p_2 t) + A_{22}\sin(p_2 t) \\ + i\left(A_{22}\cos(p_2 t) - A_{21}\sin(p_2 t)\right) \end{pmatrix} e^{-(p_1) \times t} + \begin{pmatrix} B_{21}\cos\left(p_2 t\right) - B_{22}\sin(p_2 t) \\ + i\left(B_{22}\cos(p_2 t) + B_{21}\sin(p_2 t)\right) \end{pmatrix} e^{-(p_1) \times t} \\ + \begin{pmatrix} C_{21}\cos\left(p_4 t\right) + C_{22}\sin(p_4 t) \\ + i\left(C_{22}\cos\left(p_4 t\right) - C_{21}\sin(p_4 t)\right) \end{pmatrix} e^{-(p_3) \times t} + \begin{pmatrix} D_{21}\cos(p_4 t) - D_{22}\sin(p_4 t) \\ + i\left(D_{22}\cos(p_4 t) + D_{21}\sin(p_4 t)\right) \end{pmatrix} e^{-(p_3) \times t} \end{bmatrix} \tag{21}$$

6 Simulation of time domain waveform

Time domain waveform can be plotted by assigned drain current value with number. Then, voltage biased must be designed by number. The resistive load can be designed with Ohm's law and supply voltage. Then, aspect ratio was calculated with many constants in level1 0.5 micron transistor model such as oxide thickness, mobility of electron and hole, permittivity of dielectric which all parameters and design equation in level1 were shown in textbook which was published in [13]. Eight parameters which are four poles complex frequencies and four unknown constants are solved with MATLAB after all equations are programed in MATLAB.

The drain current is designed to have 0.1 microampere. The passive capacitors are designed to have 500 nF and passive inductors are designed to have 40 pH.

The period of the waveform can be approximated to be 28 ns. Thus, oscillation frequency is calculated to be 35.71 MHz. Oscillation amplitude is in the range plus and minus 200 mV.

$$
\begin{aligned}
f_{p1,p2} &= p_1 \pm i p_2 = \left(-2.1494\times10^{-1}\right) \pm \left(2.236\times10^{8}\right) i \\
f_{p3,p4} &= p_3 \pm i p_4 = \left(7.7299\times10^{-2}\right) \pm \left(2.236\times10^{8}\right) i \\
A &= \left(9.9999\times10^{5}\right) + \left(2.937\times10^{-2}\right) i \\
B &= \left(1.000\times10^{6}\right) + \left(2.937\times10^{-2}\right) i \\
C &= \left(2.906\times10^{-2}\right) - \left(2.9068\times10^{-2}\right) i \\
D &= \left(2.906\times10^{-2}\right) - \left(2.9684\times10^{-2}\right) i
\end{aligned} \tag{22}
$$

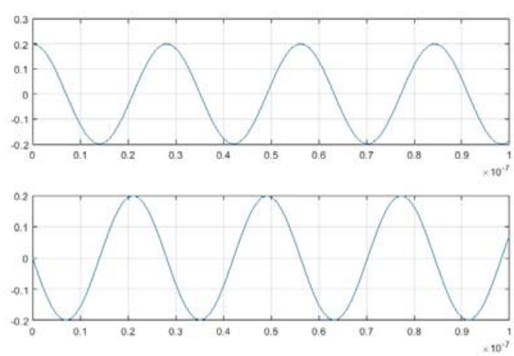


Figure2 Time domain waveform the upper waveform is real signal and lower waveform is imaginary signal

7 Conclusion

From simulation results with MATLAB with design equation in (12) and (14), the inductance which is designed from simulation by procedure in section 6, it indicates that the passive inductor value is too low which is less than picohenry. It might be unrealizable because it is too low. Another issue is that imaginary pole which is designed with approximated fourth order polynomial has no error. But the real pole has some error which were derived By eliminate one term from nine terms. The real part of input impedance should be minimized because of the solution of inverse Laplace's transform is exponential of complex number multiply with time. If the real part of input impedance is more than one, the graph in time domain can be sawtooth wave. If the real part of input impedance is much more than 100. The time domain graph is seen to have a very high slope, its amplitude is risen very fast as a function of time. Its amplitude will reach supply voltage and stay constant as a function of time.

References:

[1] A. Abidi, R. G. Meyer, "Noise in Relaxation Oscillators", IEEE Journal of Solid-State Circuits, Vol. SC-18, No. 6, December 1983, pp. 794-802

[2] B. Razavi, " Design of Active CMOS Integrated Circuits", Copyright 2000

[3] K. Tripetch, " Symbolic Analysis of Communication Circuits in CMOS Technology", Lap Lambert Academic Publishing, Copyright 2016, pp. 13-14

[4] T. Riad, " Behavior modelling for LC tank based oscillators", ISCAS2004, pp. IV-149 to IV-152

[5] I. R. Chamas, S. Raman, " A Comprehensive Analysis of Quadrature Signal Synthesis in Cross-Coupled RF VCOs", IEEE Transactions on Circuit and Systems I: Regular Papers, Vol.54, No.4, April 2007, pp. 689-704

[6] I-Y Lee, S-J Yun, S-M Oh, S-G Lee, "A low-parasitic and common-centroid Cross-Coupled CMOS transistor structure for High-Frequency VCO Design", IEEE Electron Device Letters, Vol.30, No.5, May 2009, pp. 532-534

[7] B. Razavi, " A 300-GHz Fundamental Oscillator in 65-nm CMOS Technology", IEEE Journal of Solid-State Circuit, Vol.46, No.4, April 2011, pp. 894-903

[8] P. Lucchi, M. Borgarino, J. B. Begueret, G. Jacquemod, " Low Phase Noise 130 nm CMOS Ring VCO", 2011 IEEE 9th International New Circuit and Systems Conference, pp. 89-92

[9] S. Saad, M. Mhiri, A. B. Hammadi, K. Besbes, " A CMOS 0.35 micron, 3.3 V PLL Synthesizer for

Bluetooth Transmitter", 2012 International Conference on Design & Technology of Integrated Systems in Nanoscale Era, pp. 1-4

[10] J. Anders and M. Ortmanns, " Frequency noise of CMOS LC tank oscillators operating in weak inversion", 2013 European Conference on Circuit Theory & Design (ECCTD2013)

[11] R. Bostini, G. Ardeshir, H. M-Naimi, " Analysis of Millimeter-Wave LC Oscillators based on Two-Port Network Theory", IEEE Transactions on Circuits and Systems II: Express Briefs, Vol.64, No.3, March 2017, pp. 239-243

[12] M. Abramowitz (Editor), I. A. Stegun (Editor), " Handbook of Mathematical Functions: with Formulas, Graphs, and Mathematical Tables, Dover Publications, copyright 1965, 1046 pages

[13] A. S Sedra, K. C Smith, " Microelectronic Circuits", Oxford University Press, Copyright 2004, pp. 355