# Qualitative Analysis in A Predator-prey Model with Sigmoidal Type Functional Response

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*Abstract:* In this paper, a cyclic predator-prey system with Sigmoidal type functional response is considered. The stability of the positive equilibrium and existence of Hopf bifurcation is studied by analyzing the distribution of the roots of associated characteristic equation. It is shown that the positive equilibrium is locally asymptotically stable when the time delay is small enough, while change of stability of the positive equilibrium will cause a bifurcating periodic solution as the time delay passes through a sequence of critical values. An explicit formula for determining the stability and the direction of the Hopf bifurcation periodic solutions bifurcating from Hopf bifurcations is derived, using the normal form theory and center manifold argument. Finally, numerical simulations supporting the theoretical results are carried out.

*Key–Words:* Predator-prey system, Stability, Hopf bifurcation, Sigmoidal type functional response, Time delay

## 1 Introduction

In recent years, the interest in study of the dynamical properties occurring in the predator-prey system with delay has been growing rapidly. For example, Liu [2] had made discussion about the global asymptotic stability and uniqueness of periodic solutions of a cyclic and predator-prey system of three species with Holling's type II functional response. Liu et al. [3] analyze the permanence, almost periodic phenomena and the global asymptotically stability of the unique positive periodic solution for a three species clockwise chain predator-prey model with Holling IV functional response. Tang et al.[4] investigated the permanence, the global asymptotically stability of the unique positive periodic solution in a three species clockwise chain predator-prey model with Holling IV functional response. Yu [7] studied the existence and uniqueness of uniformly asymptotically stable almost periodic solution for a cyclic predator-prey system with Functional Response. For more investigation about predator-prey, one can see [5-6,8-16]. Recently, by using comparison theory and Lyapunov functional methods, Ma and Jia [1] investigated the global asymptotic stability and uniqueness of periodic solutions of the following cyclic predator-prey system with Sigmoidal type functional response

$$\begin{cases} \dot{x}_1(t) = x_1(t)\Big[r_1(t) - a_1(t)x_1(t) \\ \qquad - \frac{d_1(t)x_1(t)x_2(t)}{c_1(t)+b_1(t)x_1(t)+x_1^2(t)} \\ \qquad + \frac{k_3(t)d_3(t)x_3^2(t)}{c_3(t)+b_3(t)x_3(t)+x_3^2(t)}\Big], \\ \dot{x}_2(t) = x_2(t)\Big[r_2(t) - a_2(t)x_2(t) \\ \qquad - \frac{d_2(t)x_2(t)x_3(t)}{c_2(t)+b_2(t)x_2(t)+x_2^2} \\ \qquad + \frac{k_1(t)d_1(t)x_1^2(t)}{c_1(t)+b_1(t)x_1(t)+x_1^2(t)}\Big], \\ \dot{x}_3(t) = x_3(t)\Big[r_3(t) - a_3(t)x_3(t) \\ \qquad - \frac{d_3(t)x_1(t)x_3(t)}{c_3(t)+b_3(t)x_3(t)+x_3^2(t)} \\ \qquad + \frac{k_2(t)d_2(t)x_2^2(t)}{c_2(t)+b_2(t)x_2(t)+x_2^2(t)}\Big], \end{cases} \tag{1}$$

where $x_2$ is the predator of $x_1$, $x_3$ is the predator of $x_2$ and $x_1$ is the predator of $x_3$, they have dependent density and Sigmoidal functional response. $a_i(t), b_i(t), c_i(t), d_i(t), k_i(t), r_i(t) (i = 1, 2, 3)$ are continuous nonnegative and bounded function within $[0, +\infty)$. Moreover, $a_i(t), c_i(t) (i = 1, 2, 3) > 0$. It is well known that in the implementation of predator-prey systems, time delays are inevitably encountered because of the finite development speed of predators and preys. Motivated by the viewpoint, in the following, we assume that time delay occurs in the Sigmoidal Functional Response, i.e., the Sigmoidal Functional Response takes the form $f(x) = \frac{x^2(t-\tau)}{c(t)+b(t)x(t-\tau)+x^2(t-\tau)}$. Furthermore, the parameters of system (1) keep unchange in time, then we have the following predator-prey system which delays are introduced:

$$\begin{cases} \dot{x}_1(t) = x_1(t)\Big[r_1 - a_1x_1(t) \\ \qquad - \frac{d_1x_1(t)x_2(t)}{c_1+b_1x_1(t)+x_1^2(t)} \\ \qquad + \frac{k_3d_3x_3^2(t-\tau)}{c_3+b_3x_3(t-\tau)+x_3^2(t-\tau)}\Big], \\ \dot{x}_2(t) = x_2(t)\Big[r_2 - a_2x_2(t) \\ \qquad - \frac{d_2x_2(t)x_3(t)}{c_2+b_2x_2(t)+x_2^2(t)} \\ \qquad + \frac{k_1d_1x_1^2(t-\tau)}{c_1+b_1x_1(t-\tau)+x_1^2(t-\tau)}\Big], \\ \dot{x}_3(t) = x_3(t)\Big[r_3 - a_3x_3(t) \\ \qquad - \frac{d_3x_1(t)x_3(t)}{c_3+b_3x_3(t)+x_3^2(t)} \\ \qquad + \frac{k_2d_2x_2^2(t-\tau)}{c_2+b_2x_2(t-\tau)+x_2^2(t-\tau)}\Big]. \end{cases} \tag{2}$$

In particular, the appearance of a cycle bifurcating from an equilibrium of an ordinary or a delayed predator-prey with a single parameter, which is known as a Hopf bifurcation, has attracted much attention (see [8-16]). We all know that time delays that occurred in the predator-prey will affect the stability of a systemn by creating instability, oscillation and chaos phenomena. The purpose of this paper is to discuss the stability and the properties of Hopf bifurcation of model (2). To the best of our knowledge, it is the first to deal with the stability and Hopf bifurcation of system (2).

This paper is organized as follows. In Section 2, linearizing the system at the positive constant steady-state solution and the analyzing the corresponding characteristic equation, the stability of the positive constant steady-state solution and the existence of Hopf bifurcation are studied. In Section 3, the direction of Hopf bifurcation and the stability and periodic of bifurcating periodic solutions are investigated by using the normal form theory and center manifold theorem presented in Hassard et al.[17]. In Section 4, we illustrate the procedure with a particular example, in which numerical simulations support our results. Some main conclusions are drawn in Section 5.

## 2 Stability of the Equilibrium and Local Hopf Bifurcations

Throughout this paper, we assume that system (2) has a unique positive equilibrium $E_*(x_1^*, x_2^*, x_3^*)$.

Linearized system of (2) near $E_*(x_1^*, x_2^*, x_3^*)$ takes the form:

$$\begin{cases} \dot{x}_1(t) = m_1x_1(t) + m_2x_2(t) + m_3x_3(t-\tau), \\ \dot{x}_2(t) = n_1x_2(t) + n_2x_3(t) + n_3x_1(t-\tau), \\ \dot{x}_3(t) = p_1x_1(t) + p_2x_3(t) + p_3x_2(t-\tau), \end{cases} \tag{3}$$

where $m_i, n_i, p_i (i = 1, 2, 3)$ are defined by Appendix A.

The associated characteristic equation of (3) is

$$\lambda^3 + \rho_2\lambda^2 + \rho_1\lambda + \rho_0 + \varepsilon_0 e^{-3\lambda\tau} + (\theta_1\lambda + \theta_0)e^{-\lambda\tau} = 0, \tag{4}$$

where $\rho_0 = -m_1n_1p_2, \rho_1 = m_1n_1 + m_1p_2 + n_1p_2, \rho_2 = -(m_1 + n_1 + p_2), \varepsilon_0 = -m_3n_3p_3, \theta_0 = m_3n_1p_1 + m_2n_3p_2 + m_1n_2p_3, \theta_1 = -(m_3p_1 + m_2n_3 + n_2p_3)$.

In order to investigate the distribution of roots of the transcendental equation (4), the following Lemma that is stated in [15] is useful.

**Lemma 1** [15] *For the transcendental equation*

$$\begin{aligned} &P(\lambda, e^{-\lambda\tau_1}, \cdots, e^{-\lambda\tau_m}) = \\ &\lambda^n + p_1^{(0)}\lambda^{n-1} + \cdots + p_{n-1}^{(0)}\lambda + p_n^{(0)} \\ &+ \left[p_1^{(1)}\lambda^{n-1} + \cdots + p_{n-1}^{(1)}\lambda + p_n^{(1)}\right] e^{-\lambda\tau_1} + \cdots \\ &+ \left[p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda + p_n^{(m)}\right] e^{-\lambda\tau_m} = 0, \end{aligned}$$

*as $(\tau_1, \tau_2, \tau_3, \cdots, \tau_m)$ vary, the sum of orders of the zeros of $P(\lambda, e^{-\lambda\tau_1}, \cdots, e^{-\lambda\tau_m})$ in the open right half plane can change, and only a zero appears on or crosses the imaginary axis.*

Now we make the following assumption:

(H1) $\rho_0 + \varepsilon_0 + \theta_0 > 0, \rho_2(\rho_1 + \theta_1) > \rho_0 + \varepsilon_0 + \theta_0.$

**Lemma 2** *If (H1) holds, then we have the following : (i) When*

$$\tau = \tau_k \stackrel{\text{def}}{=} \frac{1}{2\omega_0}\Big[\arccos\theta^* - \psi + 2k\pi\Big], \tag{5}$$

*where* $\theta^* = \frac{(\rho_0-\rho_2\omega_0^2)^2+(\omega_0^3-\rho_1\omega_0)^2-\varepsilon_0^2}{2\sqrt{\theta_0^2\varepsilon_0^2+\varepsilon_0^2\theta_1^2}}, k = 0, 1, 2, \cdots,$. *Eq.(4) has a simple pair of imaginary roots $\pm i\omega_0$, where $\omega_0$ is the positive root of Eq. (15)and $\psi$ satisfies (16).*
*(ii) For $\tau \in [0, \tau_0)$, all roots of Eq. (4) have strictly negative real parts.*
*(iii) When $\tau = \tau_k$, Eq. (4) has a pair of imaginary roots $\pm i\omega_0$ and all other roots have strictly negative real parts.*

**Proof:** Obviously, by the assumption (H1), $\lambda = 0$ is not the root of Eq. (4). When $\tau = 0$, then Eq. (4) becomes

$$\lambda^3 + \rho_2\lambda^2 + (\rho_1 + \theta_1)\lambda + \rho_0 + \varepsilon_0 + \theta_0 = 0. \quad (6)$$

It is easy to see from condition (H1) that all roots of Eq. (7) have negative real parts.

Multiplying $e^{\lambda\tau}$ on both sides of the two equations of (4), it is obvious to obtain

$$\begin{aligned}&(\lambda^3 + \rho_2\lambda^2 + \rho_1\lambda + \rho_0)e^{\lambda\tau} + \varepsilon_0 e^{-2\lambda\tau}\\ &+\theta_1\lambda + \theta_0 = 0.\end{aligned} \quad (7)$$

$\pm i\omega_0$ $(\omega_0 > 0)$ is a pair of purely imaginary roots of (4) if and only if $\omega$ satisfies

$$\begin{aligned}&(-i\omega_0^3 - i\rho_2\omega_0^2 + i\rho_1\omega_0 + \rho_0)e^{i\omega_0\tau} + \varepsilon_0 e^{-2i\omega_0\tau}\\ &+i\theta_1\omega_0 + \theta_0 = 0.\end{aligned}$$

Separating the real and imaginary parts, we get

$$\begin{cases}(\rho_0 - \rho_2\omega_0^2)\cos\omega\tau + (\omega_0^3 - \rho_1\omega_0)\sin\omega\tau\\ = \varepsilon_0\cos 2\omega\tau - \theta_0,\\ (\rho_0 - \rho_2\omega_0^2)\sin\omega\tau - (\omega_0^3 - \rho_1\omega_0)\cos\omega\tau\\ = \varepsilon_0\sin 2\omega\tau - \theta_1\omega_0.\end{cases} \quad (8)$$

Taking square on the both sides of the equation in (8) and summing up, we get

$$\begin{aligned}&(\rho_0 - \rho_2\omega_0^2)^2 + (\omega_0^3 - \rho_1\omega_0)^2 =\\ &\varepsilon_0^2 + 2\theta_0\varepsilon_0\cos 2\omega_0\tau + 2\varepsilon_0\theta_1\sin 2\omega_0\tau.\end{aligned} \quad (9)$$

According to $\sin 2\omega\tau = \pm\sqrt{1-\cos^2 2\omega\tau}$, it follows that

$$\begin{aligned}&(\rho_0 - \rho_2\omega_0^2)^2 + (\omega_0^3 - \rho_1\omega_0)^2 = \varepsilon_0^2 + 2\theta_0\varepsilon_0\\ &\times\cos 2\omega_0\tau \pm 2\sqrt{1-\cos^2 2\omega\tau}\,\varepsilon_0\theta_1.\end{aligned} \quad (10)$$

It is easy to see that (10) is equivalent to

$$q_1\cos^2 2\omega\tau + q_2\cos 2\omega\tau + q_3 = 0, \quad (11)$$

where

$$\begin{aligned}q_1 &= 4(\theta_0^2 + \theta_1^2)\varepsilon_0^2,\\ q_2 &= -4\theta_0\varepsilon_0[(\rho_0 - \rho_2\omega_0^2)^2\\ &\quad + (\omega_0^3 - \rho_1\omega_0)^2 - \varepsilon_0^2],\\ q_3 &= [(\rho_0 - \rho_2\omega_0^2)^2 + (\omega_0^3 - \rho_1\omega_0)^2 - \varepsilon_0^2]^2\\ &\quad - 4\varepsilon_0^2\theta_1^2.\end{aligned}$$

It follows from (11) that

$$\cos 2\omega\tau = \frac{-q_2 \pm \sqrt{q_2^2 - 4q_1q_3}}{2q_1} := f_1(\omega), \quad (12)$$

where $f_1(\omega)$ is a function with respect to $\omega$. Substitute (12) into (9), we get

$$\sin 2\omega\tau = \frac{\chi^*}{4\varepsilon_0\theta_1 q_1} := f_2(\omega), \quad (13)$$

where $\chi^* = 2q_1[(\rho_0 - \rho_2\omega_0^2)^2 + (\omega_0^3 - \rho_1\omega_0)^2 - \varepsilon_0^2] - 2\theta_0\varepsilon_0(-q_2 \pm \sqrt{q_2^2 - 4q_1q_3})$, $f_2(\omega)$ is a function with respect to $\omega$. According to $\sin^2 2\omega\tau + \cos^2 2\omega\tau = 1$, it follows from (12) and (13) that

$$f_1{}^2(\omega) + f_2{}^2(\omega) = 1. \quad (14)$$

If $a_i, b_i, c_i, d_i, k_i (i = 1, 2, 3)$ of the system (2) are given, it is easy to use computer to calculate the roots of (14).

We assume that (14) has at least one positive real root. From (9), we derive

$$\begin{aligned}&(\rho_0 - \rho_2\omega_0^2)^2 + (\omega_0^3 - \rho_1\omega_0)^2 =\\ &\varepsilon_0^2 + 2\sqrt{\theta_0^2\varepsilon_0^2 + \varepsilon_0^2\theta_1^2}\cos(2\omega_0\tau + \psi),\end{aligned} \quad (15)$$

where $\psi$ satisfies

$$\tan\psi = \frac{\theta_1\omega_0}{\varepsilon_0\theta_0}. \quad (16)$$

From (15), it is easy to obtain

$$\tau_k = \frac{1}{2\omega_0}\Big[\arccos\theta^* - \psi + 2k\pi\Big], \quad (17)$$

where $\theta^* = \frac{(\rho_0 - \rho_2\omega_0^2)^2 + (\omega_0^3 - \rho_1\omega_0)^2 - \varepsilon_0^2}{2\sqrt{\theta_0^2\varepsilon_0^2 + \varepsilon_0^2\theta_1^2}}, k = 0, 1, 2, \cdots,$.

From (7), we know that Eq. (4) with $\tau = \tau_k (k = 0, 1, 2, \cdots)$ has a pair of imaginary roots $\pm i\omega_0$, which are simple.

According the discussion and applying the Lemma 1 and Cooke and Grossman [18], we obtain the conclusion (ii) and (iii). This completes the proof.

Let $\lambda_(\tau) = \alpha(\tau) + i\omega(\tau)$ be a root of (4) near $\tau = \tau_k$, and $\alpha(\tau_k) = 0$, and $\omega(\tau_k) = \omega_0, (k = 0, 1, 2, \cdots)$. Due to functional differential equation theory, for every $\tau_k, k = 0, 1, 2, \cdots$ there exists $\varepsilon > 0$ such that $\lambda(\tau)$ is continuously differentiable in $\tau$ for $|\tau - \tau_k| < \varepsilon$. Substituting $\lambda(\tau)$ into the left hand of (4) and taking derivative with respect to $\tau$, we have

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = -\frac{(3\lambda^2 + 2\rho_2\lambda + \rho_1)e^{-\lambda\tau} + \theta_1}{\lambda[\lambda^3 + \rho_2\lambda^2 + \rho_1\lambda + \rho_0)e^{\lambda\tau} - \varepsilon_0 e^{-2\lambda\tau}]} - \frac{\tau}{\lambda},$$

which leads to

$$\mathrm{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_k} = \frac{A_1B_1 - A_2B_2}{A_1^2 + A_2^2},$$

where

$$\begin{aligned} A_1 &= [(\rho_0 - \rho_2\omega_0^2 + \varepsilon_0)\sin\omega_0\tau_k \\ &\quad + (\rho_1\omega_0 - \omega_0^3)\cos\omega_0\tau_k]\omega_0, \\ A_2 &= [(\rho_0 - \rho_2\omega_0^2 + \varepsilon_0)\cos\omega_0\tau_k \\ &\quad - (\rho_1\omega_0 - \omega_0^3)\sin\omega_0\tau_k]\omega_0, \\ B_1 &= (\rho_1 - 3\omega_0^2)\cos\omega_0\tau_k - 2\rho_2\omega_0\sin\omega_0\tau_k + \theta_1, \\ B_2 &= (\rho_1 - 3\omega_0^2)\sin\omega_0\tau_k + 2\rho_2\omega_0\cos\omega_0\tau_k + \theta_1. \end{aligned}$$

We assume that

$$\text{(H2)}\ \ A_1B_1 \neq A_2B_2.$$

**Lemma 3** *Let $\tau = \tau_k$, then the following transversality condition*

$$\frac{dRe[\lambda(\tau)]}{d\tau}\Big|_{\tau=\tau_k} \neq 0$$

*is satisfied.*

From Lemma 2-3, we have the following results on the local stability and Hopf bifurcation for system (2).

**Theorem 4** *For system (2), let $\tau_k$ be defined by (17) and assume that (H1) and (H2) hold.*
*(i) If $\tau \in [0, \tau_0)$, then the equilibrium point of system (2) is asymptotically stable and $\tau = \tau_k(k = 0, 1, 2, \cdots)$ are Hopf bifurcation values for system (2).*

# 3 Direction and stability of the Hopf bifurcation

In the previous section, we have obtained some conditions which guarantee that the two-neuron networks with resonant bilinear terms undergoes the Hopf bifurcation at some values of $\tau = \tau_k(k = 0, 1, 2, \cdots)$. In this section, we shall derive the explicit formulae determining the direction, stability, and period of these periodic solutions bifurcating from the positive equilibrium $E_*(x_1^*, x_2^*, x_3^*)$ at these critical value of $\tau$, by using techniques from normal form and center manifold theory [17], Throughout this section, we always assume that system (3) undergoes Hopf bifurcation at the positive equilibrium $E_*(x_1^*, x_2^*, x_3^*)$ for $\tau = \tau_k$, and then $\pm i\omega_0$ are corresponding purely imaginary roots of the characteristic equation at the positive equilibrium $E_*(x_1^*, x_2^*, x_3^*)$.

For convenience, let $\tau = \tau_k + \mu, \mu \in R$. Then $\mu = 0$ is the Hopf bifurcation value of (2). Thus, we shall study Hopf bifurcation of small amplitude periodic solutions of (2) from the equilibrium point for $\mu$ close to 0. We can consider the fixed phase space $C = C([-1, 0], R^3)$.

For $(\phi_1, \phi_2, \phi_3) \in C$, define

$$L_\mu\phi = \tau_k A\phi(0) + \tau_k B\phi(-1), \qquad (18)$$

where

$$A = \begin{pmatrix} m_1 & m_2 & 0 \\ 0 & n_1 & n_2 \\ p_1 & 0 & p_2 \end{pmatrix},\ B = \begin{pmatrix} 0 & 0 & m_3 \\ n_3 & 0 & 0 \\ 0 & p_3 & 0 \end{pmatrix}.$$

We expand the nonlinear part of system (2) and derive the following expression

$$f(\mu, \phi) = \begin{pmatrix} f_1(\mu, \phi) \\ f_2(\mu, \phi) \\ f_3(\mu, \phi) \end{pmatrix}, \qquad (19)$$

where

$$\begin{aligned} f_1(\mu, \phi) &= (\tau_k + \mu)[l_1x_1^2(t) + l_2x_1(t)x_2(t) \\ &\quad + l_3x_3^2(t-\tau) + l_4x_1(t)x_3(t-\tau) \\ &\quad + l_5x_1^3(t) + l_6x_3^3(t-\tau) + l_7x_1^2(t)x_2(t) \\ &\quad + l_8x_1(t)x_3^2(t-\tau) + \text{h.o.t.}], \\ f_2(\mu, \phi) &= (\tau_k + \mu)[s_1x_1^2(t) + s_2x_1(t)x_2(t) \\ &\quad + s_3x_3^2(t-\tau) + s_4x_1(t)x_3(t-\tau) \\ &\quad + s_5x_1^3(t) + s_6x_3^3(t-\tau) + s_7x_1^2(t)x_2(t) \\ &\quad + s_8x_1(t)x_3^2(t-\tau) + \text{h.o.t.}], \\ f_3(\mu, \phi) &= (\tau_k + \mu)[v_1x_1^2(t) + v_2x_1(t)x_2(t) \\ &\quad + v_3x_3^2(t-\tau) + v_4x_1(t)x_3(t-\tau) \\ &\quad + v_5x_1^3(t) + v_6x_3^3(t-\tau) \\ &\quad + v_7x_1^2(t)x_2(t) + v_8x_1(t)x_3^2(t-\tau) \\ &\quad + \text{h.o.t.}], \end{aligned}$$

where $l_i, s_i, v_i(i = 1, 2, 3, 4, 5, 6, 7)$ are defined by Appendix B.

By the representation theorem, there is a matrix function with bounded variation components $\eta(\theta, \mu), \theta \in [-1, 0]$ such that

$$L_\mu\phi = \int_{-1}^{0} d\eta(\theta, \mu)\phi(\theta)\ \text{ for }\ \phi \in C. \qquad (20)$$

In fact, we can choose

$$\begin{aligned} \eta(\theta, \mu) &= (\tau_k + \mu)\begin{pmatrix} m_1 & m_2 & 0 \\ 0 & n_1 & n_2 \\ p_1 & 0 & p_2 \end{pmatrix}\delta(\theta) \\ &\quad -(\tau_k + \mu)\begin{pmatrix} 0 & 0 & m_3 \\ n_3 & 0 & 0 \\ 0 & p_3 & 0 \end{pmatrix}\delta(\theta + 1), \qquad (21) \end{aligned}$$

where $\delta$ is the Dirac delta function.

For $\phi \in C([-1,0], R^3)$, define

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & -1 \le \theta < 0, \\ \int_{-1}^{0} d\eta(s,\mu)\phi(s), & \theta = 0 \end{cases} \tag{22}$$

and

$$R(\mu)\phi = \begin{cases} 0, & -1 \le \theta < 0, \\ f(\mu,\phi), & \theta = 0. \end{cases} \tag{23}$$

Then (2) is equivalent to the abstract differential equation

$$\dot{x_t} = A(\mu)x_t + R(\mu)x_t, \tag{24}$$

where $x = (x_1, x_2, x_3)^T, x_t(\theta) = x(t+\theta), \theta \in [-1, 0]$.

For $\psi \in C([0,1], (R^3)^*)$, define

$$A^*\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0,1], \\ \int_{-1}^{0} d\eta^T(t,0)\psi(-t), & s = 0. \end{cases} \tag{25}$$

For $\phi \in C([-1,0], R^3)$ and $\psi \in C([0,1], (R^3)^*)$, define the bilinear form

$$< \psi, \phi >= \overline{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{\psi}(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi, \tag{26}$$

where $\eta(\theta) = \eta(\theta, 0)$. We have the following result on the relation between the operators $A = A(0)$ and $A^*$.

**Lemma 5** *$A = A(0)$ and $A^*$ are adjoint operators.*

**Proof:** Let $\phi \in C^1([-1,0], R^3)$ and $\psi \in C^1([0,1], (R^3)^*)$. It follows from (26) and the definitions of $A = A(0)$ and $A^*$ that

$$\begin{aligned}
&< \psi(s), A(0)\phi(\theta) >= \bar{\psi}(0)A(0)\phi(0) \\
&- \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{\psi}(\xi-\theta)d\eta(\theta)A(0)\phi(\xi)d\xi \\
&+\bar{\psi}(0)\int_{-1}^{0} d\eta(\theta)\phi(\theta) \\
&- \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{\psi}(\xi-\theta)d\eta(\theta)A(0)\phi(\xi)d\xi \\
&= \bar{\psi}(0)\int_{-1}^{0} d\eta(\theta)\phi(\theta) \\
&- \int_{-1}^{0} [\bar{\psi}(\xi-\theta)d\eta(\theta)\phi(\xi)]_{\xi=0}^{\theta} \\
&+ \int_{-1}^{0}\int_{\xi=0}^{\theta} \frac{d\bar{\psi}(\xi-\theta)}{d\xi} d\eta(\theta)\phi(\xi)d\xi \\
&= \int_{-1}^{0} \bar{\psi}(-\theta)d\eta(\theta)\phi(0) \\
&- \int_{-1}^{0}\int_{\xi=0}^{\theta} \left[-\frac{d\bar{\psi}(\xi-\theta)}{d\xi}\right] d\eta(\theta)\phi(\xi)d\xi \\
&= A * \bar{\psi}(0)\phi(0) \\
&- \int_{-1}^{0}\int_{\xi=0}^{\theta} A^*\bar{\psi}(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi \\
&=< A^*\psi(s), \phi(\theta) > .
\end{aligned}$$

This shows that $A = A(0)$ and $A^*$ are adjoint operators and the proof is complete.

By the discussions in Section 2, we know that $\pm i\omega_0\tau_k$ are eigenvalues of $A(0)$, and they are also eigenvalues of $A^*$ corresponding to $i\omega_0$ and $-i\omega_0\tau_k$, respectively. We have the following result.

**Lemma 6** *The vector*

$$q(\theta) = (1, \alpha, \beta)^T e^{i\omega_0\tau_k\theta}, \ \ \theta \in [-1, 0],$$

*is the eigenvector of $A(0)$ corresponding to the eigenvalue $i\omega_0\tau_k$ and*

$$q^*(s) = D(1, \alpha^*, \beta^*)e^{i\omega_0\tau_k s}, \quad s \in [0, 1],$$

*is the eigenvector of $A^*$ corresponding to the eigenvalue $-i\omega_0\tau_k$, moreover, $< q^*(s), q(\theta) >= 1$, where*

$$\begin{aligned}
\alpha &= \frac{n_2(i\omega_0 - m_1) + m_3n_3e^{-2i\omega_0\tau_k}}{m_2n_2 + (i\omega_0 - n_1)m_3e^{-i\omega_0\tau_k}}, \\
\beta &= \frac{(i\omega_0 - m_1)(i\omega_0 - n_1) - m_2n_3e^{-i\omega_0\tau_k}}{m_2n_2 + (i\omega_0 - n_1)m_3e^{-i\omega_0\tau_k}}, \\
\alpha^* &= \frac{p_1m_2 - (i\omega_0 + m_1)p_3e^{-i\omega_0\tau_k}}{n_3p_3e^{-2i\omega_0\tau_k} - p_1(i\omega_0 + n_1)}, \\
\beta^* &= \frac{(i\omega_0 + m_1)(i\omega_0 + n_1) - m_2n_3e^{-i\omega_0\tau_k}}{n_3p_3e^{-2i\omega_0\tau_k} - p_1(i\omega_0 + n_1)}, \\
D &= 1 + \bar{\alpha}\alpha^* + \bar{\beta}\beta^* + (n_3\alpha^* + \bar{\alpha}\beta^*p_3 + \bar{\beta}m_3) \\
&\quad \times \tau_k e^{i\omega_0\tau_k}
\end{aligned}$$

**Proof:** Let $q(\theta)$ be the eigenvector of $A(0)$ corresponding to the eigenvalue $i\omega_0$ and $q^*(s)$ be the eigenvector of $A^*$ corresponding to the eigenvalue $-i\omega_0\tau_k$, namely, $A(0)q(\theta) = i\omega_0\tau_k q(\theta)$ and $A^*q^{*T}(s) = -i\omega_0\tau_k q^{*T}(s)$. From the definitions of $A(0)$ and $A^*$, we have $A(0)q(\theta) = dq(\theta)/d\theta$ and $A^*q^{*T}(s) = -dq^{*T}(s)/ds$. Thus, $q(\theta) = q(0)e^{i\omega_0\tau_k\theta}$ and $q^*(s) = q^*(0)e^{i\omega_0\tau_k s}$. In addition,

$$\begin{aligned}
&\int_{-1}^{0} d\eta(\theta)q(\theta) = \tau_k Aq(0) + \tau_k Bq(-1) \\
&= A(0)q(0) = i\omega_0\tau_k q(0).
\end{aligned} \tag{27}$$

That is

$$\begin{pmatrix} i\omega_0 - m_1 & -m_2 & -m_3e^{-i\omega_0\tau_k} \\ -n_3e^{-i\omega_0\tau_k} & i\omega_0 - n_1 & -n_2 \\ -p_1 & -p_3e^{-i\omega_0\tau_k} & i\omega_0 - p_2 \end{pmatrix} q(0)$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{28}$$

Therefore, we can easily obtain

$$\alpha = \frac{n_2(i\omega_0 - m_1) + m_3 n_3 e^{-2i\omega_0\tau_k}}{m_2 n_2 + (i\omega_0 - n_1)m_3 e^{-i\omega_0\tau_k}},$$
$$\beta = \frac{(i\omega_0 - m_1)(i\omega_0 - n_1) - m_2 n_3 e^{-i\omega_0\tau_k}}{m_2 n_2 + (i\omega_0 - n_1)m_3 e^{-i\omega_0\tau_k}}.$$

On the other hand,

$$\int_{-1}^{0} q^*(-t)d\eta(t) = \tau_k A^T q^{*T}(0) + \tau_k B^T q^{*T}(-1)$$
$$= A^* q^{*T}(0) = -i\omega_0\tau_k q^{*T}(0). \tag{29}$$

Namely,

$$\begin{pmatrix} -i\omega_0 - m_1 & -n_3 e^{-i\omega_0\tau_k} & -p_1 \\ -m_2 & -i\omega_0 - n_1 & -p_3 e^{-i\omega_0\tau_k} \\ -m_3 e^{-i\omega_0\tau_k} & -n_2 & -i\omega_0 - p_2 \end{pmatrix} q^*(0)$$
$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{30}$$

Therefore, we can easily obtain

$$\alpha^* = \frac{p_1 m_2 - (i\omega_0 + m_1)p_3 e^{-i\omega_0\tau_k}}{n_3 p_3 e^{-2i\omega_0\tau_k} - p_1(i\omega_0 + n_1)},$$
$$\beta^* = \frac{(i\omega_0 + m_1)(i\omega_0 + n_1) - m_2 n_3 e^{-i\omega_0\tau_k}}{n_3 p_3 e^{-2i\omega_0\tau_k} - p_1(i\omega_0 + n_1)}.$$

In the sequel, we shall verify that $< q^*(s), q(\theta) >= 1$. In fact, from (26), we have

$$< q^*(s), q(\theta) >= \bar{D}(1, \bar{\alpha^*}, \bar{\beta^*})(1, \alpha, \beta)^T$$
$$- \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{D}(1, \bar{\alpha^*}, \bar{\beta^*})e^{-i\omega_0(\xi-\theta)}d\eta(\theta)$$
$$(1, \alpha, \beta)^T e^{i\omega_0\tau_k\xi}d\xi$$
$$= \bar{D}\Big[1 + \alpha\bar{\alpha^*} + \beta\bar{\beta}^*$$
$$- \int_{-1}^{0}(1, \bar{\alpha^*}, \bar{\beta^*})\theta e^{i\omega_0\tau_k\theta}d\eta(\theta)(1, \alpha, \beta)^T\Big]$$
$$= \bar{D}\Big\{1 + \alpha\bar{\alpha^*} + \beta\bar{\beta}^* + (1, \bar{\alpha^*}, \bar{\beta^*})$$
$$\Big[Be^{-i\omega_0\tau_k}\Big](1, \alpha, \beta)^T\Big\}$$
$$= \bar{D}\Big[1 + \alpha\bar{\alpha^*} + \beta\bar{\beta}^* + (n_3\bar{\alpha}^* + \alpha\bar{\beta}^* p_3 + m_3\beta)$$
$$\tau_0 e^{-i\omega_0\tau_k}\Big] = 1.$$

Next, we use the same notations as those in Hassard, Kazarinoff and Wan [17], and we first compute the coordinates to describe the center manifold $C_0$ at $\mu = 0$. Let $x_t$ be the solution of Eq. (2) when $\mu = 0$.

Define

$$z(t) =< q^*, x_t >, W(t,\theta) = x_t(\theta) - 2\mathrm{Re}\{z(t)q(\theta)\}. \tag{31}$$

on the center manifold $C_0$, and we have

$$W(t,\theta) = W(z(t), \bar{z}(t), \theta), \tag{32}$$

where

$$W(z(t), \bar{z}(t), \theta) = W(z, \bar{z}) = W_{20}\frac{z^2}{2}$$
$$+ W_{11}z\bar{z} + W_{02}\frac{\bar{z}^2}{2} + \cdots \tag{33}$$

and $z$ and $\bar{z}$ are local coordinates for center manifold $C_0$ in the direction of $q^*$ and $\bar{q}^*$. Noting that $W$ is also real if $x_t$ is real, we consider only real solutions. For solutions $x_t \in C_0$ of (2),

$$\begin{aligned} \dot{z}(t) &= < q^*(s), \dot{x}_t >=< q^*(s), A(0)x_t + R(0)x_t > \\ &= < q^*(s), A(0)x_t > + < q^*(s), R(0)x_t > \\ &= < A^*q^*(s), x_t > + \bar{q^*}(0)R(0)x_t \\ &\quad - \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{q^*}(\xi - \theta)d\eta(\theta)A(0)R(0)x_t(\xi)d\xi \\ &= < i\omega_0\tau_k q^*(s), x_t > + \bar{q^*}(0)f(0, x_t(\theta) \\ &\stackrel{\text{def}}{=} i\omega_0\tau_k z(t) + \bar{q^*}(0)f_0(z(t), \bar{z}(t)). \end{aligned} \tag{34}$$

That is

$$\dot{z}(t) = i\omega_0\tau_k z + g(z, \bar{z}), \tag{35}$$

where

$$g(z, \bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots. \tag{36}$$

Hence, we have

$$g(z, \bar{z}) = \bar{q}^*(0)f_0(z, \bar{z})$$
$$= f(0, x_t) = \bar{D}\tau_k(1, \alpha^*, \beta^*)$$
$$(f_1(0, x_t), f_2(0, x_t), f_3(0, x_t))^T, \tag{37}$$

where

$$\begin{aligned} f_1(0, x_t) &= \tau_k[l_1 x_{1t}^2(0) + l_2 x_{1t}(0)x_2(0) \\ &\quad + l_3 x_{3t}^2(-1) + l_4 x_{1t}(0)x_{3t}(-1) \\ &\quad + l_5 x_{1t}^3(0) + l_6 x_{3t}^3(-1) \\ &\quad + l_7 x_{1t}^2(0)x_{2t}(0) + l_8 x_{1t}(0)x_{3t}^2(-1) \\ &\quad + \text{h.o.t.}], \\ f_2(0, x_t) &= \tau_k[s_1 x_{1t}^2(0) + s_2 x_1(0)x_{2t}(0) \\ &\quad + s_3 x_{3t}^2(-1) + s_4 x_{1t}(0)x_{3t}(-1) \\ &\quad + s_{5t} x_{1t}^3(0) + s_6 x_{3t}^3(-1) \end{aligned}$$

$$
\begin{array}{rcl}
& & + s_7 x_{1t}^2(0)x_{2t}(0) + s_8 x_{1t}(0)x_{3t}^2(-1) \\
& & + \text{h.o.t.}], \\
f_3(0,x_t) & = & \tau_k[v_1 x_{1t}^2(0) + v_2 x_{1t}(0)x_{2t}(0) \\
& & + v_3 x_{3t}^2(-1) + v_4 x_{1t}(0)x_{3t}(-1) \\
& & + v_5 x_{1t}^3(0) + v_6 x_{3t}^3(-1) \\
& & + v_7 x_{1t}^2(0)x_{2t}(0) + v_8 x_{1t}(0)x_{3t}^2(-1) \\
& & + \text{h.o.t.}].
\end{array}
$$

Noticing $x_t(\theta) = (x_{1t}(\theta), x_{2t}(\theta), x_{3t}(\theta))^T = W(t,\theta) + zq(\theta) + \bar{z}\bar{q}(\theta)$ and $q(\theta) = (1,\gamma)^T e^{i\omega_0\theta}$, we have

$$
\begin{array}{rcl}
x_{1t}(0) & = & z + \bar{z} + W_{20}^{(1)}(0)\dfrac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} \\
& & +W_{02}^{(1)}(0)\dfrac{\bar{z}^2}{2} + \cdots, \\
x_{2t}(0) & = & \gamma z + \bar{\gamma}\bar{z} + W_{20}^{(2)}(0)\dfrac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z} \\
& & +W_{02}^{(2)}(0)\dfrac{\bar{z}^2}{2} + \cdots, \\
x_{3t}(-1) & = & \beta e^{-i\omega_0 2\tau_k} z + \bar{\beta} e^{i\omega_0\tau_k}\bar{z} \\
& & +W_{20}^{(1)}(-1)\dfrac{z^2}{2} + W_{11}^{(1)}(-1)z\bar{z} \\
& & +W_{02}^{(1)}(-1)\dfrac{\bar{z}^2}{2} + \cdots.
\end{array}
$$

From (36) and (37), we have

$$
\begin{array}{l}
g(z,\bar{z}) = \bar{q}^*(0)f_0(z,\bar{z}) \\
= \bar{D}\tau_k\left[f_1(0,x_t) + \bar{\alpha}^* f_2(0,x_t) + \bar{\beta}^* f_3(0,x_t)\right] \\
= \bar{D}\tau_k(K_{11} + \bar{\alpha}^* K_{12} + \bar{\beta}^* K_{13})z^2 \\
+2\bar{D}\tau_k(K_{21} + \bar{\alpha}^* K_{22} + \bar{\beta}^* K_{23})z\bar{z} \\
+\bar{D}\tau_k(K_{31} + \bar{\alpha}^* K_{32} + \bar{\beta}^* K_{33})\bar{z}^2 \\
+\bar{D}\tau_k(K_{41} + \bar{\alpha}^* K_{42} + \bar{\beta}^* K_{43})z^2\bar{z} + \text{h.o.t.},
\end{array}
$$

where $K_{ij}(i,j = 1,2,3)$ are defined by Appendix C. Then we obtain

$$
\begin{array}{rcl}
g_{20} & = & 2\bar{D}\tau_k(K_{11} + \bar{\alpha}^* K_{12} + \bar{\beta}^* K_{13}), \\
g_{11} & = & 2\bar{D}\tau_k(K_{21} + \bar{\alpha}^* K_{22} + \bar{\beta}^* K_{23}), \\
g_{02} & = & 2\bar{D}\tau_k(K_{31} + \bar{\alpha}^* K_{32} + \bar{\beta}^* K_{33}), \\
g_{21} & = & 2\bar{D}\tau_k(K_{31} + \bar{\alpha}^* K_{32} + \bar{\beta}^* K_{33})\bar{z}^2 \\
& & +\bar{D}\tau_0(K_{41} + \bar{\alpha}^* K_{42} + \bar{\beta}^* K_{43}).
\end{array}
$$

For unknown $W_{20}^{(1)}(0), W_{20}^{(2)}(0), W_{20}^{(3)}(-1), W_{11}^{(1)}(0), W_{11}^{(3)}(-1), W_{11}^{(2)}(0)$ in $g_{21}$, we still need to compute them. From (24) and (35), we have

$$
W' = \begin{cases} AW - 2\text{Re}\{\bar{q}^*(0)fq(\theta)\}, & -\tau_1^0 \le \theta < 0, \\ AW - 2\text{Re}\{\bar{q}^*(0)fq(\theta)\} + f, & \theta = 0. \end{cases}
$$

$$
\stackrel{\text{def}}{=} AW + H(z,\bar{z},\theta), \tag{38}
$$

where

$$
H(z,\bar{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots. \tag{39}
$$

Comparing the coefficients, we obtain

$$
(A - 2i\omega_0)W_{20} = -H_{20}(\theta), \tag{40}
$$

$$
AW_{11}(\theta) = -H_{11}(\theta). \tag{41}
$$

We know that for $\theta \in [-1,0)$,

$$
\begin{array}{l}
H(z,\bar{z},\theta) = -\bar{q}^*(0)f_0 q(\theta) - q^*(0)\bar{f}_0\bar{q}(\theta) \\
= -g(z,\bar{z})q(\theta) - \bar{g}(z,\bar{z})\bar{q}(\theta).
\end{array} \tag{42}
$$

Comparing the coefficients of (39) with (42) gives that

$$
H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), \tag{43}
$$

$$
H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta). \tag{44}
$$

From (3.23),(3.26) and the definition of $A$ , we get

$$
\dot{W}_{20}(\theta) = 2i\omega_0\tau_k W_{20}(\theta) + g_{20}q(\theta) + \bar{g_{02}}\bar{q}(\theta). \tag{45}
$$

Noting that $q(\theta) = q(0)e^{i\omega_0\tau_k\theta}$, we have

$$
\begin{array}{l}
W_{20}(\theta) = \dfrac{ig_{20}}{\omega_0\tau_k}q(0)e^{i\omega_0\tau_k\theta} + \\
\dfrac{i\bar{g}_{02}}{3\omega_0\tau_k}\bar{q}(0)e^{-i\omega_0\tau_k\theta} + E_1 e^{2i\omega_0\tau_k\theta},
\end{array} \tag{46}
$$

where $E_1 = (E_1^{(1)}, E_1^{(2)}, E_1^{(3)})^T$ is a constant vector. Similarly, from (41), (44) and the definition of $A$, we have

$$
\dot{W}_{11}(\theta) = g_{11}q(\theta) + \bar{g_{11}}\bar{q}(\theta), \tag{47}
$$

$$
\begin{array}{l}
W_{11}(\theta) = -\dfrac{ig_{11}}{\omega_0\tau_k}q(0)e^{i\omega_0\tau_k\theta} \\
+\dfrac{i\bar{g}_{11}}{\omega_0\tau_k}\bar{q}(0)e^{-i\omega_0\tau_k\theta} + E_2.
\end{array} \tag{48}
$$

where $E_2 = (E_2^{(1)}, E_2^{(2)}, E_2^{(3)})^T$ is a constant vector.

In what follows, we shall seek appropriate $E_1$,$E_2$ in (46), (48), respectively. It follows from the definition of $A$ and (43), (44) that

$$
\int_{-1}^0 d\eta(\theta)W_{20}(\theta) = 2i\omega_0\tau_k W_{20}(0) - H_{20}(0) \tag{49}
$$

and

$$
\int_{-1}^0 d\eta(\theta)W_{11}(\theta) = -H_{11}(0), \tag{50}
$$

where $\eta(\theta) = \eta(0,\theta)$.
From (40), we have

$$H_{20}(0) = -g_{20}q(0) - \bar{g_{02}}\bar{q}(0) + 2\tau_k(K_{11}, K_{12}, K_{13})^T, \tag{51}$$

From (41), we have

$$H_{11}(0) = -g_{11}q(0) - \bar{g_{11}}(0)\bar{q}(0) + 2\tau_k(K_{21}, K_{22}, K_{23})^T, \tag{52}$$

Noting that

$$\left(i\omega_0\tau_k I - \int_{-1}^{0} e^{i\omega_0\tau_k\theta} d\eta(\theta)\right) q(0) = 0, \tag{53}$$

$$\left(-i\omega_0\tau_k I - \int_{-1}^{0} e^{-i\omega_0\tau_k\theta} d\eta(\theta)\right) \bar{q}(0) = 0 \tag{54}$$

and substituting (46) and (51) into (49), we have

$$\left(2i\omega_0\tau_k I - \int_{-1}^{0} e^{2i\omega_0\tau_k\theta} d\eta(\theta)\right) E_1 = 2\tau_k(K_{11}, K_{12}, K_{13})^T. \tag{55}$$

That is

$$\left(2i\omega_0\tau_k I - \tau_k A - \tau_k B e^{-2i\omega_0\tau_k}\right) E_1 = 2\tau_k(K_{11}, K_{12}, K_{13})^T, \tag{56}$$

then

$$\begin{pmatrix} 2i\omega_0 - m_1 & -m_2 & -m_3 e^{-2i\omega_0\tau_k} \\ -n_3 e^{-2i\omega_0\tau_k} & 2i\omega_0 - n_1 & -n_2 \\ -p_1 & -p_3 e^{-2i\omega_0\tau_k} & 2i\omega_0 - p_2 \end{pmatrix} \begin{pmatrix} E_1^{(1)} \\ E_1^{(2)} \\ E_1^{(3)} \end{pmatrix} = 2 \begin{pmatrix} K_{11} \\ K_{12} \\ K_{13} \end{pmatrix}. \tag{57}$$

Hence

$$E_1^{(1)} = \frac{\Delta_{11}}{\Delta_1},\ E_1^{(2)} = \frac{\Delta_{12}}{\Delta_1},\ E_1^{(3)} = \frac{\Delta_{13}}{\Delta_1},$$

where

$$\Delta_1 = \det \begin{pmatrix} 2i\omega_0 - m_1 & -m_2 & -m_3 e^{-2i\omega_0\tau_k} \\ -n_3 e^{-2i\omega_0\tau_k} & 2i\omega_0 - n_1 & -n_2 \\ -p_1 & -p_3 e^{-2i\omega_0\tau_k} & 2i\omega_0 - p_2 \end{pmatrix},$$

$$\Delta_{11} = 2\det \begin{pmatrix} K_{11} & -m_2 & -m_3 e^{-2i\omega_0\tau_k} \\ K_{12} & 2i\omega_0 - n_1 & -n_2 \\ K_{13} & -p_3 e^{-2i\omega_0\tau_k} & 2i\omega_0 - p_2 \end{pmatrix},$$

$$\Delta_{12} = 2\det \begin{pmatrix} 2i\omega_0 - m_1 & K_{11} & -m_3 e^{-2i\omega_0\tau_k} \\ -n_3 e^{-2i\omega_0\tau_k} & K_{12} & -n_2 \\ -p_1 & K_{13} & 2i\omega_0 - p_2 \end{pmatrix},$$

$$\Delta_{13} = 2\det \begin{pmatrix} 2i\omega_0 - m_1 & -m_2 & K_{11} \\ -n_3 e^{-2i\omega_0\tau_k} & 2i\omega_0 - n_1 & K_{12} \\ -p_1 & -p_3 e^{-2i\omega_0\tau_k} & K_{13} \end{pmatrix}.$$

Similarly, substituting (47) and (52) into (50), we have

$$\left(\int_{-1}^{0} d\eta(\theta)\right) E_2 = (K_{21}, K_{22}, K_{23})^T. \tag{58}$$

Then,

$$(A+B)E_2 = (-K_{21}, -K_{22}, -K_{23})^T. \tag{59}$$

That is

$$\begin{pmatrix} m_1 & m_2 & m_3 \\ n_3 & n_1 & n_2 \\ p_1 & p_3 & p_2 \end{pmatrix} \begin{pmatrix} E_2^{(1)} \\ E_2^{(2)} \\ E_2^{(3)} \end{pmatrix} = \begin{pmatrix} -K_{21} \\ -K_{22} \\ -K_{23} \end{pmatrix}. \tag{60}$$

Hence

$$E_2^{(1)} = \frac{\Delta_{21}}{\Delta_2},\ E_2^{(2)} = \frac{\Delta_{22}}{\Delta_2},\ E_2^{(3)} = \frac{\Delta_{23}}{\Delta_2},$$

where

$$\begin{aligned} \Delta_2 &= \det \begin{pmatrix} m_1 & m_2 & m_3 \\ n_3 & n_1 & n_2 \\ p_1 & p_3 & p_2 \end{pmatrix}, \\ \Delta_{21} &= \det \begin{pmatrix} -K_{21} & m_2 & m_3 \\ -K_{22} & n_1 & n_2 \\ -K_{23} & p_3 & p_2 \end{pmatrix}, \\ \Delta_{22} &= \det \begin{pmatrix} m_1 & -K_{21} & m_3 \\ n_3 & -K_{22} & n_2 \\ p_1 & -K_{23} & p_2 \end{pmatrix}, \\ \Delta_{23} &= \det \begin{pmatrix} m_1 & m_2 & -K_{21} \\ n_3 & n_1 & -K_{22} \\ p_1 & p_3 & -K_{23} \end{pmatrix}. \end{aligned}$$

From (46),(48), we can calculate $g_{21}$ and derive the following values:

$$\begin{aligned} c_1(0) &= \frac{i}{2\omega_0\tau_k}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{\mathrm{Re}\{c_1(0)\}}{\mathrm{Re}\{\lambda'(\tau_k)\}}, \\ \beta_2 &= 2\mathrm{Re}(c_1(0)), \\ T_2 &= -\frac{\mathrm{Im}\{c_1(0)\} + \mu_2 \mathrm{Im}\{\lambda'(\tau_k)\}}{\omega_0\tau_k}. \end{aligned}$$

These formulaes give a description of the Hopf bifurcation periodic solutions of (2) at $\tau = \tau_k$ on the center manifold. From the discussion above, we have the following result:

**Theorem 7** *The periodic solution is supercritical (subcritical) if $\mu_2 > 0$ ($\mu_2 < 0$); The bifurcating periodic solutions are orbitally asymptotically stable with asymptotical phase (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$); The periods of the bifurcating periodic solutions increase (decrease) if $T_2 > 0$ ($T_2 < 0$).*

## 4 Numerical Examples

We have derived analytical understanding of possible dynamics of a cyclic predator-prey system with Sigmoidal type functional response to some extent. In this section, we now perform some numerical simulations work(using MATLAB dde23) to verify the analytical predictions obtained in the previous section. As an example, we consider the following special case of system (2) with the parameters $r_1 = 0.5, r_2 = 0.4, r_3 = 0.6, a_1 = 0.6, a_2 = 0.7, a_3 = 0.2, b_1 = 0.5, b_2 = 0.6, b_3 = 0.3, c_1 = 0.3, c_2 = 0.4, c_3 = 0.2, d_3 = 0.6, d_2 = 0.5, d_3 = 0.7, k_1 = \frac{14}{3}, k_2 = 4.8, k_3 = \frac{24}{35}$, then system (2) becomes

$$\left\{\begin{aligned} \dot{x}_1(t) &= x_1(t)\left[0.5 - 0.6x_1(t) - \frac{0.6x_1(t)x_2(t)}{0.3+0.5x_1(t)+x_1^2(t)} + \frac{0.48x_3^2(t-\tau)}{0.3+0.5x_3(t-\tau)+x_3^2(t-\tau)}\right], \\ \dot{x}_2(t) &= x_2(t)\left[0.4 - 0.7x_2(t) - \frac{0.5x_2(t)x_3(t)}{0.4+0.6x_2(t)+x_2^2(t)} + \frac{0.28x_1^2(t-\tau)}{0.4+0.6x_1(t-\tau)+x_1^2(t-\tau)}\right], \\ \dot{x}_3(t) &= x_3(t)\left[0.6 - 0.2x_3(t) - \frac{0.7x_1(t)x_3(t)}{0.2+0.3x_3(t)+x_3^2(t)} + \frac{0.24x_2^2(t-\tau)}{0.2+0.5x_2(t-\tau)+x_2^2(t-\tau)}\right], \end{aligned}\right. \tag{61}$$

which has a positive equilibrium $E_*(1.0082, 0.4395, 0.9091)$. By some complicated computation by means of Matlab 7.0, we get $\omega_0 \approx 2.0653, \tau_0 \approx 4.5, \lambda'(\tau_0) \approx 1.2247 - 2.2556i$. Thus we get $c_1(0) \approx -1.6138 - 9.1355i, \mu_2 \approx 1.3177, \beta_2 \approx -3.2276, T_2 \approx 0.5625$. We obtain the conditions stated in Theorem 4 are fulfilled. Furthermore, it follows that $\mu_2 > 0$ and $\beta_2 < 0$. Thus, the positive equilibrium $E_*(1.0082, 0.4395, 0.9091)$ is stable when $\tau < \tau_0$ which is illustrated by the computer simulations (see Figs.1-7). When $\tau$ passes through the critical value $\tau_0$, the positive equilibrium $E_*(1.0082, 0.4395, 0.9091)$ loses its stability and a Hopf bifurcation occurs, i.e., a family of periodic solutions bifurcate from the positive equilibrium $E_*(1.0082, 0.4395, 0.9091)$. Since $\mu_2 > 0$ and $\beta_2 < 0$, the direction of the Hopf bifurcation is $\tau > \tau_0$, and these bifurcating periodic solutions from $E_*(1.0082, 0.4395, 0.9091)$ at $\tau_0$ are stable, which are depicted in Figs.8-14.

Fig.1

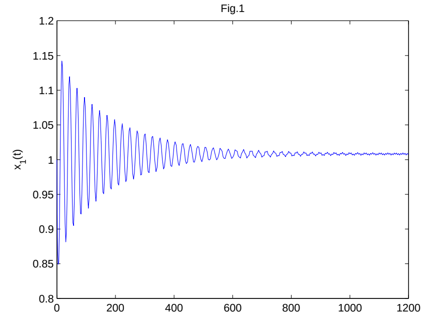



Fig.2

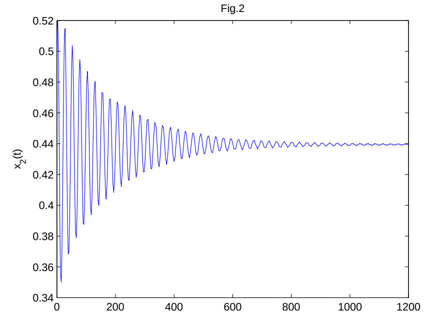



Fig.3

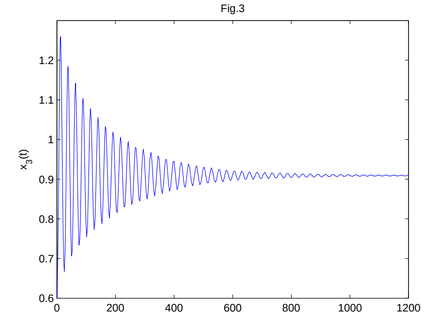



Fig.4

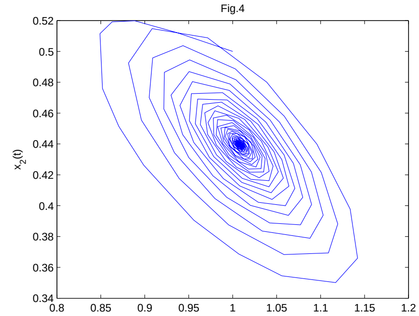
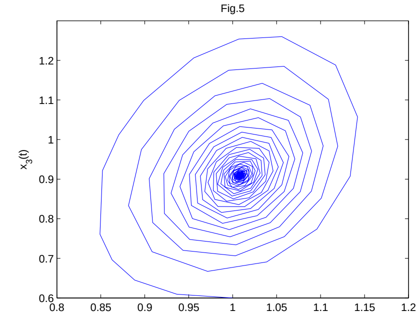
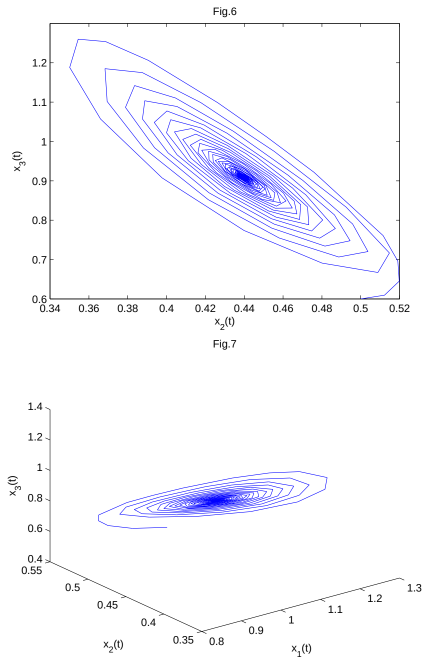
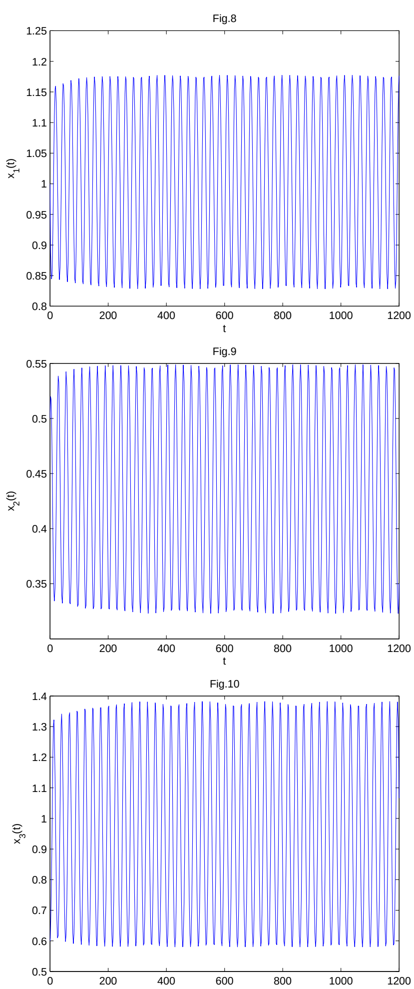


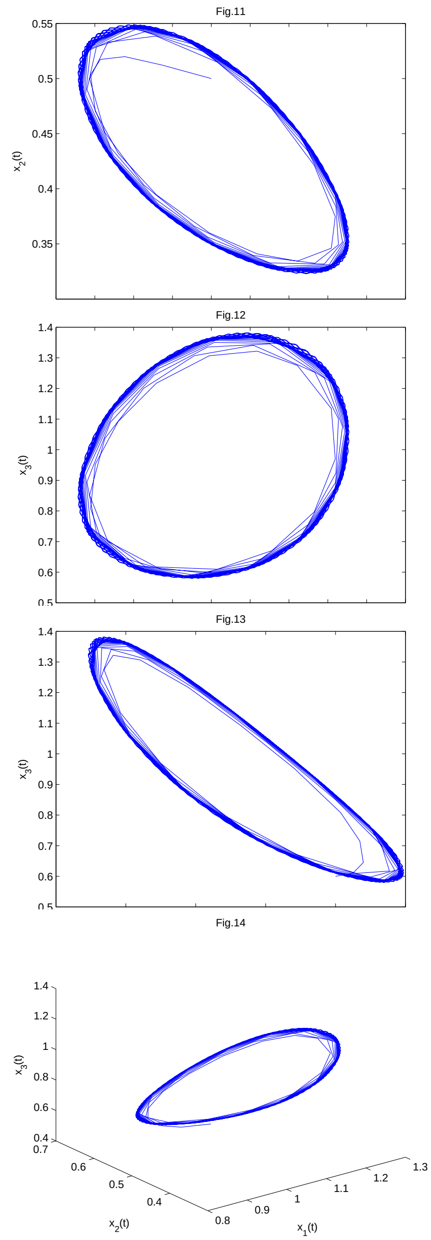

Fig.5






Figs.1-7. Trajectories graphs of system (61) with $\tau = 4.3 < \tau_0 \approx 4.5$. The positive equilibrium $E_*(1.0082, 0.4395, 0.9091)$ is asymptotically stable. The initial value is (1,0.5,0.6).




Figs.8-14. Trajectories graphs of system (61) with $\tau = 5 > \tau_0 \approx 4.5$. Hopf bifurcation occurs from the positive equilibrium $E_*(1.0082, 0.4395, 0.9091)$. The initial value is (1,0.5,0.6).

# 5 Conclusions

In this paper, we have analyzed a cyclic predator-prey system with Sigmoidal type functional response. We studied the effect of time delays on its dynamics. Firstly, we obtained the sufficient conditions to ensure local stability of the equilibrium $E_*(x_1^*, x_2^*, x_3^*)$. Taking the delay as parameter, we investigate the existence of local Hopf bifurcation. Applying normal form theory and center manifold reduction, the stability and direction of the Hopf bifurcation induced by time delay are determined.

**Appendix A** The expressions of $m_i, n_i, p_i (i, j = 1, 2, 3)$ are as follows:

$$\begin{aligned}
m_1 &= x_1^*\Big[\frac{b_1d_1x_1^*x_2^*}{(c_1+2b_1x_1^*+4x_1^{*2})^2} \\
&\quad -\frac{d_1x_2^*}{c_1+b_1x_1^*+x_1^{*2}}-a_1\Big],\\
m_2 &= -\frac{d_1x_1^{*2}}{c_1+b_1x_1^*+x_1^{*2}},\\
m_3 &= x_1^*\Big[\frac{2k_3d_3}{c_3+b_3x_3^*+x_3^{*2}} \\
&\quad -\frac{k_3b_3d_3x_3^*}{(c_3+2b_3x_3^*+4x_3^{*2})^2}\Big],\\
n_1 &= -\frac{d_2x_2^{*2}}{c_2+b_2x_2^*+x_2^{*2}},\\
n_2 &= x_2^*\Big[\frac{b_2d_2x_1^*x_2^*}{(c_2+2b_2x_1^*+4x_2^{*2})^2} \\
&\quad -\frac{d_2x_1^*}{c_2+b_2x_2^*+x_1^{*2}}-a_2\Big],\\
n_3 &= x_2^*\Big[\frac{2k_1d_1}{c_1+b_1x_1^*+x_1^{*2}} \\
&\quad -\frac{k_1b_1d_1x_1^*}{(c_1+2b_1x_1^*+4x_1^{*2})^2}\Big],\\
p_1 &= x_1^*\Big[\frac{b_1d_1x_1^*x_3^*}{(c_1+2b_1x_1^*+4x_1^{*2})^2} \\
&\quad -\frac{d_1x_3^*}{c_1+b_1x_1^*+x_1^{*2}}-a_3\Big],\\
p_2 &= -\frac{d_1x_1^{*2}}{c_1+b_1x_1^*+x_1^{*2}},\\
p_3 &= x_3^*\Big[\frac{2k_2d_2}{c_2+b_2x_2^*+x_2^{*2}} \\
&\quad -\frac{k_2b_2d_2x_2^*}{(c_2+2b_2x_2^*+4x_2^{*2})^2}\Big].
\end{aligned}$$

**Appendix B** The expressions of $l_i, s_i, v_i (i, j = 1, 2, 3)$ are as follows:

$$\begin{aligned}
l_1 &= \frac{2b_1x_1^*(b_1+4x_1^*)}{(c_1+2b_1x_1^*+4x_1^{*2})^3}-\frac{d_1x_2^*}{c_1+b_1x_1^*+x_1^{*2}} \\
&\quad -\frac{d_1x_1^*x_2^*}{c_1+2b_1x_1^*+4x_1^{*2}}-a_1, l_2=-2d_1x_1^*,\\
l_3 &= x_1^*\Big[\frac{k_3d_3}{c_3+b_3x_3^*+x_3^{*2}}+\frac{2b_3k_3d_3x_3^{*2}(b_3+4x_3^*)}{(c_3+2b_3x_3^*+4x_3^{*2})^3} \\
&\quad -\frac{2b_3k_3d_3}{(c_1+2b_1x_1^*+4x_1^{*2})^2}\Big],\\
l_4 &= \frac{k_3d_3}{c_3+b_3x_3^*+x_3^{*2}}-\frac{2b_3k_3d_3x_3^{*2}}{(c_3+2b_3x_1^*+4x_3^{*2})^3},\\
l_5 &= -\frac{8b_1x_1^{*2}(b_1+4x_1^{*2}}{(c_1+2b_1x_1^*+4x_1^{*2})^4},\\
l_6 &= -\frac{8b_3k_3d_3x_1^*x_3^{*2}(b_3+4x_3^{*2}}{(c_1+2b_1x_1^*+4x_1^{*2})^4},\\
l_7 &= -d_1, l_8=\frac{k_3d_3}{c_3+b_3x_3^*+x_3^{*2}},\\
s_1 &= \frac{2b_2x_2^*(b_2+4x_2^*)}{(c_2+2b_21x_2^*+4x_2^{*2})^3}-\frac{d_2x_3^*}{c_2+b_2x_2^*+x_2^{*2}} \\
&\quad -\frac{d_2x_2^*x_3^*}{c_2+2b_2x_2^*+4x_2^{*2}}-a_2, s_2=-2d_2x_2^*,\\
s_3 &= x_2^*\Big[\frac{k_1d_1}{c_1+b_1x_1^*+x_1^{*2}}+\frac{2b_1k_1d_1x_1^{*2}(b_1+4x_1^*)}{(c_1+2b_1x_1^*+4x_1^{*2})^3} \\
&\quad -\frac{2b_1k_1d_1}{(c_1+2b_1x_1^*+4x_1^{*2})^2}\Big],\\
s_4 &= \frac{k_1d_1}{c_1+b_1x_1^*+x_1^{*2}}-\frac{2b_1k_1d_1x_1^{*2}}{(c_1+2b_1x_1^*+4x_1^{*2})^3},\\
s_5 &= -\frac{8b_2x_2^{*2}(b_2+4x_2^{*2}}{(c_2+2b_2x_2^*+4x_2^{*2})^4},\\
s_6 &= -\frac{8b_1k_1d_1x_3^*x_1^{*2}(b_1+4x_3^{*2}}{(c_3+2b_3x_3^*+4x_3^{*2})^4},\\
s_7 &= -d_3, s_8=\frac{k_1d_1}{c_1+b_1x_1^*+x_1^{*2}},\\
v_1 &= \frac{2b_3x_3^*(b_3+4x_3^*)}{(c_3+2b_3x_3^*+4x_3^{*2})^3}-\frac{d_3x_1^*}{c_3+b_3x_3^*+x_3^{*2}} \\
&\quad -\frac{d_3x_3^*x_1^*}{c_3+2b_31x_3^*+4x_3^{*2}}-a_3, v_2=-2d_3x_3^*,\\
v_3 &= x_3^*\Big[\frac{k_2d_2}{c_2+b_2x_2^*+x_2^{*2}}+\frac{2b_2k_2d_2x_2^{*2}(b_2+4x_2^*)}{(c_2+2b_2x_23^*+4x_2^{*2})^3} \\
&\quad -\frac{2b_2k_2d_3}{(c_3+2b_3x_3^*+4x_3^{*2})^2}\Big],\\
v_4 &= \frac{k_2d_2}{c_2+b_2x_2^*+x_2^{*2}}-\frac{2b_2k_2d_2x_2^{*2}}{(c_2+2b_2x_3^*+4x_2^{*2})^3},\\
v_5 &= -\frac{8b_3x_3^{*2}(b_3+4x_3^{*2}}{(c_3+2b_3x_3^*+4x_3^{*2})^4},\\
v_6 &= -\frac{8b_2k_2d_2x_3^*x_2^{*2}(b_2+4x_2^{*2}}{(c_3+2b_3x_3^*+4x_3^{*2})^4},\\
v_7 &= -d_3, v_8=\frac{k_2d_2}{c_2+b_2x_2^*+x_2^{*2}}.
\end{aligned}$$

**Appendix C** The expressions of $K_{ij} (i, j = 1, 2, 3)$ are as follows:

$$\begin{aligned}
K_{11} &= l_1+l_2\alpha+l_3\beta^2e^{-2i\omega_0\tau_k}+l_4\beta e^{-i\omega_0\tau_k},\\
K_{12} &= s_1+s_2\alpha+s_3\beta^2e^{-2i\omega_0\tau_k}+s_4\beta e^{-i\omega_0\tau_k},\\
K_{13} &= v_1+v_2\alpha+v_3\beta^2e^{-2i\omega_0\tau_k}+v_4\beta e^{-i\omega_0\tau_k},\\
K_{21} &= 2(l_1+l_2\mathrm{Re}\{\alpha\}+l_3|\beta|+l_4\mathrm{Re}\{\beta e^{-i\omega_0\tau_k}\}),
\end{aligned}$$

$$
\begin{aligned}
K_{22} &= 2(s_1 + s_2\mathrm{Re}\{\alpha\} + s_3|\beta| + s_4\mathrm{Re}\{\beta e^{-i\omega_0\tau_k}\}),\\
K_{23} &= 2(v_1 + v_2\mathrm{Re}\{\alpha\} + v_3|\beta| + v_4\mathrm{Re}\{\beta e^{-i\omega_0\tau_k}\}),\\
K_{31} &= l_1 + l_2\bar{\alpha} + l_3\bar{\beta}^2 e^{2i\omega_0\tau_k} + l_4\bar{\beta}e^{i\omega_0\tau_k},\\
K_{32} &= s_1 + s_2\bar{\alpha} + s_3\bar{\beta}^2 e^{2i\omega_0\tau_k} + s_4\bar{\beta}e^{i\omega_0\tau_k},\\
K_{33} &= v_1 + v_2\bar{\alpha} + v_3\bar{\beta}^2 e^{2i\omega_0\tau_k} + v_4\bar{\beta}e^{i\omega_0\tau_k},\\
K_{41} &= l_1(W_{20}^{(1)}(0) + 2W_{11}^{(1)}(0)) + l_2(\frac{1}{2}\bar{\alpha}W_{20}^{(2)}(0)\\
&\quad +\frac{1}{2}W_{20}^{(2)}(0) + \alpha W_{11}^{(1)}(0) + W_{11}^{(2)}(0))\\
&\quad +l_3(W_{20}^{(3)}(-1)e^{i\omega_0\tau_k}\mathrm{Re}\{\beta\} + 2\beta W_{11}^{(3)}(-1))\\
&\quad +l_4(\frac{1}{2}\bar{\beta}W_{20}^{(1)}(0)e^{i\omega_0\tau_k} + \frac{1}{2}W_{20}^{(3)}(-1)\\
&\quad +\beta W_{11}^{(1)}(0)e^{-i\omega_0\tau_k} + W_{11}^{(3)}(-1))\\
&\quad +3l_5 + 3l_6\beta^2\bar{\beta}e^{-i\omega_0\tau_k} + l_7(2\alpha + \bar{\alpha})\\
&\quad +l_8(\beta^2 e^{-2i\omega_0\tau_k} + 2|\beta|^2),\\
K_{42} &= s_1(W_{20}^{(1)}(0) + 2W_{11}^{(1)}(0))\\
&\quad +s_2(\frac{1}{2}\bar{\alpha}W_{20}^{(2)}(0) + \frac{1}{2}W_{20}^{(2)}(0)\\
&\quad +\alpha W_{11}^{(1)}(0) + W_{11}^{(2)}(0))\\
&\quad +s_3(W_{20}^{(3)}(-1)e^{i\omega_0\tau_k}\mathrm{Re}\{\beta\} + 2\beta W_{11}^{(3)}(-1))\\
&\quad +s_4(\frac{1}{2}\bar{\beta}W_{20}^{(1)}(0)e^{i\omega_0\tau_0} + \frac{1}{2}W_{20}^{(3)}(-1)\\
&\quad +\beta W_{11}^{(1)}(0)e^{-i\omega_0\tau_k} + W_{11}^{(3)}(-1))\\
&\quad +3s_5 + 3s_6\beta^2\bar{\beta}e^{-i\omega_0\tau_k} + s_7(2\alpha + \bar{\alpha})\\
&\quad +s_8(\beta^2 e^{-2i\omega_0\tau_k} + 2|\beta|^2),\\
K_{43} &= v_1(W_{20}^{(1)}(0) + 2W_{11}^{(1)}(0))\\
&\quad +v_2(\frac{1}{2}\bar{\alpha}W_{20}^{(2)}(0) + \frac{1}{2}W_{20}^{(2)}(0)\\
&\quad +\alpha W_{11}^{(1)}(0) + W_{11}^{(2)}(0))\\
&\quad +v_3(W_{20}^{(3)}(-1)e^{i\omega_0\tau_0}\mathrm{Re}\{\beta\} + 2\beta W_{11}^{(3)}(-1))\\
&\quad +v_4(\frac{1}{2}\bar{\beta}W_{20}^{(1)}(0)e^{i\omega_0\tau_k} + \frac{1}{2}W_{20}^{(3)}(-1)\\
&\quad +\beta W_{11}^{(1)}(0)e^{-i\omega_0\tau_k} + W_{11}^{(3)}(-1)) + 3v_5\\
&\quad +3v_6\beta^2\bar{\beta}e^{-i\omega_0\tau_k} + v_7(2\alpha + \bar{\alpha})\\
&\quad +v_8(\beta^2 e^{-2i\omega_0\tau_k} + 2|\beta|^2).
\end{aligned}
$$

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