$$h\left(Val_r^{\beta}\left(\&_{B\in Y}B\right)\right)<h\left(\frac{1}{2}\right)$$

resp.

$$h\left(\max\left\{Val_r^{\beta}\left(\&_{B\in Y}B\right),Val_r^{\beta}\left(\&_{C\in Z}C\right)\right\}\right) <h\left(\frac{1}{2}\right),$$

i.e., that

$$\frac{h\left(\frac{1}{2}\right)}{h\left(Val_r^{\beta}\left(\&_{B\in Y}B\right)\right)}>1$$

resp.

$$\frac{h\left(\frac{1}{2}\right)}{h\left(\max\left\{Val_r^{\beta}\left(\&_{B\in Y}B\right),Val_r^{\beta}\left(\&_{C\in Z}C\right)\right\}\right)} >1.$$

By (7) resp. (8), $Val_r^{\beta}\left(\&_{A\in X}A\right)>1$.

This is not possible, however.

So, the assumption that $Z^{'}=\{A_1,A_2,...,A_n\}$, yields a contradiction.

It follows that $Z^{'}\subset\{A_1,A_2,...,A_n\}$.

We have, $\emptyset\neq Z^{'}\subset\{A_1,A_2,...,A_n\}$.

For the sake of simplicity, we shall assume that each of attributes in $\{A_1,A_2,...,A_n\}$ is defined on the same two-element set $\{p,q\}$.

Strengths of the dependencies in C are denoted by θ_2, while the strength of the dependency (X,f,Y,θ_1,F) resp. (X,m,Y,θ_1,F) is obviously θ_1.

We shall denote by $\theta^{'}\ (<1)$ the minimum of these strengths, and select some $\theta^{''}<\theta^{'}$.

Since domain of each of the attributes in $\{A_1,A_2,...,A_n\}$ is $\{p,q\}$, the similarity relation (defined on domains of the attributes in $\{A_1,A_2,...,A_n\}$), is given by $Sim\left(p,q\right)=\theta^{''}$.

Clearly, $Sim\left(p,q\right)=Sim\left(q,p\right)=1$.

Now the fact that $\emptyset\neq Z^{'}\subset\{A_1,A_2,...,A_n\}$ enables us to define the following two-element, fuzzy relation instance $r^{'}$, by putting it to have $t^{'}$, $t^{''}$ values equal p on $Z^{'}$, and $t^{'}$, $t^{''}$ values equal q on $\{A_1,A_2,...,A_n\}\setminus Z^{'}$, where $r^{'}=\left\{t^{'},t^{''}\right\}$.

We shall prove that (K,f,L,θ_2,F) is satisfied by $r^{'}$, and that (K,m,L,θ_2,F) is satisfied by $r^{'}$ for all $(K,f,L,\theta_2,F)\in C$ and all $(K,m,L,\theta_2,F)\in C$.

Furthermore, we shall prove that $r^{'}$ violates (X,f,Y,θ_1,F) resp. (X,m,Y,θ_2,F).

Let $(K,f,L,\theta_2,F)\in C$.

By our assumption, $Val_r^{\beta}\left(F_{K,L}^{f}\right)>\frac{1}{2}$.

Hence,

$$\begin{aligned}&Val_r^{\beta}\left(\left(\&_{A\in K}A\right)\rightarrow\left(\&_{B\in L}B\right)\right)\\&=h^{-1}\left(\max\left\{Val_r^{\beta}\left(\&_{A\in K}A\right)\times\right.\right.\\&\left.\left.\times\,h\left(Val_r^{\beta}\left(\&_{B\in L}B\right)\right),h\left(1\right)\right\}\right)\\&>\frac{1}{2},\end{aligned}$$

i.e.,

$$\begin{aligned}&\max\left\{Val_r^{\beta}\left(\&_{A\in K}A\right)\times\right.\\&\left.\times\,h\left(Val_r^{\beta}\left(\&_{B\in L}B\right)\right),h\left(1\right)\right\}<h\left(\frac{1}{2}\right).\end{aligned}$$

Suppose that the maximum is $h\left(1\right)$.

Then,

$$\begin{aligned}&Val_r^{\beta}\left(\&_{A\in K}A\right)h\left(Val_r^{\beta}\left(\&_{B\in L}B\right)\right)\\&\leq h\left(1\right)<h\left(\frac{1}{2}\right).\end{aligned}$$

If the maximum is

$$Val_r^{\beta}\left(\&_{A\in K}A\right)h\left(Val_r^{\beta}\left(\&_{B\in L}B\right)\right),$$

then, once again

$$Val_r^{\beta}\left(\&_{A\in K}A\right)h\left(Val_r^{\beta}\left(\&_{B\in L}B\right)\right)<h\left(\frac{1}{2}\right).$$

If $Val_r^{\beta}\left(\&_{A\in K}A\right)\leq\frac{1}{2}$, then $Val_r^{\beta}\left(A\right)\leq\frac{1}{2}$ for all $A\in K$, so $A\notin Z^{'}$, and hence $Con\left(t^{'},t^{''},A\right)=\theta^{''}$.

Thus, $Con\left(t^{'},t^{''},K\right)=\theta^{''}$.

By our definition of similarity relations on domains of attributes in $\{A_1, A_2, ..., A_n\}$, it immediately follows that $Con\left(t^{'}, t^{''}, Q\right) \geq \theta^{''}$ for any $Q \subseteq \{A_1, A_2, ..., A_n\}$.

In particular, $Con\left(t^{'}, t^{''}, L\right) \geq \theta^{''}$.

Hence,

$$\begin{aligned} &Con\left(t^{'}, t^{''}, L\right) \\ \geq&\theta^{''} = \min\left\{\theta_2, \theta^{''}\right\} = \min\left\{\theta_2, Con\left(t^{'}, t^{''}, K\right)\right\}, \end{aligned}$$

i.e., (K, f, L, θ_2, F) is satisfied by $r^{'}$, i.e., $\left(t^{'}, t^{''}, L, \theta_2, K, r^{'}\right)$ holds true.

If $Val_r^{\beta}\left(\&_{A\in K}A\right) > \frac{1}{2}$, then $Val_r^{\beta}(A) > \frac{1}{2}$ for all $A \in K$, so $A \in Z^{'}$ for all $A \in K$, and hence $Con\left(t^{'}, t^{''}, A\right) = 1$ for all $A \in K$.

Thus, $Con\left(t^{'}, t^{''}, K\right) = 1$.

If $h\left(Val_r^{\beta}\left(\&_{B\in L}B\right)\right) = 0$, then $Val_r^{\beta}\left(\&_{B\in L}B\right) = 1 > \frac{1}{2}$.

Reasoning as in the case $Val_r^{\beta}\left(\&_{A\in K}A\right) > \frac{1}{2}$, we obtain that $Con\left(t^{'}, t^{''}, L\right) = 1$.

Hence,

$$\begin{aligned} &Con\left(t^{'}, t^{''}, L\right) \\ =&1 \geq \theta_2 = \min\{\theta_2, 1\} \\ =&\min\left\{\theta_2, Con\left(t^{'}, t^{''}, K\right)\right\}, \end{aligned}$$

i.e., $\left(t^{'}, t^{''}, L, \theta_2, K, r^{'}\right)$ holds true.

Suppose that $h\left(Val_r^{\beta}\left(\&_{B\in L}B\right)\right) > 0$.

We obtain,

$$Val_r^{\beta}\left(\&_{A\in K}A\right) < \frac{h\left(\frac{1}{2}\right)}{h\left(Val_r^{\beta}\left(\&_{B\in L}B\right)\right)}.$$

Since $Val_r^{\beta}\left(\&_{A\in K}A\right) > \frac{1}{2}$, it follows that

$$\frac{h\left(\frac{1}{2}\right)}{h\left(Val_r^{\beta}\left(\&_{B\in L}B\right)\right)} > 1,$$

i.e.,

$$h\left(Val_r^{\beta}\left(\&_{B\in L}B\right)\right) < h\left(\frac{1}{2}\right),$$

i.e., $Val_r^{\beta}\left(\&_{B\in L}B\right) > \frac{1}{2}$.

Hence, as in the case $Val_r^{\beta}\left(\&_{B\in L}B\right) = 1 > \frac{1}{2}$, we conclude that $\left(t^{'}, t^{''}, L, \theta_2, K, r^{'}\right)$ holds true.

Now, let $(K, m, L, \theta_2, F) \in C$.

By our assumption, $Val_r^{\beta}\left(F_{K,L}^{m}\right) > \frac{1}{2}$.

Thus,

$$\begin{aligned} &Val_r^{\beta}\left(\left(\&_{A\in K}A\right) \to \left(\left(\&_{B\in L}B\right) \parallel \left(\&_{C\in M}C\right)\right)\right) \\ =&h^{-1}\Bigg(\max\Big\{Val_r^{\beta}\left(\&_{A\in K}A\right) \times \\ &\times h\left(Val_r^{\beta}\left(\left(\&_{B\in L}B\right) \parallel \left(\&_{C\in M}C\right)\right)\right), h(1)\Big\}\Bigg) \\ =&h^{-1}\Bigg(\max\Big\{Val_r^{\beta}\left(\&_{A\in K}A\right) \times \\ &\times h\left(\max\left\{Val_r^{\beta}\left(\&_{B\in L}B\right), Val_r^{\beta}\left(\&_{C\in M}C\right)\right\}\right), \\ &h(1)\Big\}\Bigg) \\ >&\frac{1}{2}. \end{aligned}$$

i.e.,

$$\begin{aligned} &\max\Big\{Val_r^{\beta}\left(\&_{A\in K}A\right) \times \\ &\times h\left(\max\left\{Val_r^{\beta}\left(\&_{B\in L}B\right), Val_r^{\beta}\left(\&_{C\in M}C\right)\right\}\right), \\ &h(1)\Big\} < h\left(\frac{1}{2}\right). \end{aligned}$$

Reasoning in the same way as in the fuzzy functional case, we conclude that the last inequality yields that

$$\begin{aligned} &Val_r^{\beta}\left(\&_{A\in K}A\right) \times \\ &\times h\left(\max\left\{Val_r^{\beta}\left(\&_{B\in L}B\right), Val_r^{\beta}\left(\&_{C\in M}C\right)\right\}\right) \\ <&h\left(\frac{1}{2}\right). \end{aligned}$$

If $Val_r^{\beta}\left(\&_{A\in K}A\right) \leq \frac{1}{2}$, then, as before, $Con\left(t^{'}, t^{''}, K\right) = \theta^{''}$.

Thus, there is $t^{'''} \in r^{'}$, $t^{'''} = t^{'}$, such that $\left(t^{'''}, t^{'}, t^{''}, K, L, M, \theta_2, r^{'}\right)$ holds true.

If $Val_r^{\beta}\left(\&_{A \in K} A\right) > \frac{1}{2}$, then, as earlier, $Con\left(t^{'}, t^{''}, L\right) = 1$.

If

$$h\left(\max\left\{Val_r^{\beta}\left(\&_{B \in L} B\right), Val_r^{\beta}\left(\&_{C \in M} C\right)\right\}\right) = 0,$$

then

$$\max\left\{Val_r^{\beta}\left(\&_{B \in L} B\right), Val_r^{\beta}\left(\&_{C \in M} C\right)\right\} = 1.$$

Hence, $Val_r^{\beta}\left(\&_{B \in L} B\right) = 1$ or $Val_r^{\beta}\left(\&_{C \in M} C\right) = 1$, so, as before, $Con\left(t^{'}, t^{''}, L\right) = 1$ or $Con\left(t^{'}, t^{''}, M\right) = 1$.

If $Con\left(t^{'}, t^{''}, L\right) = 1$, then, there exists $t^{'''} \in r^{'}$, $t^{'''} = t^{''}$, such that $\left(t^{'''}, t^{'}, t^{''}, K, L, M, \theta_2, r^{'}\right)$ holds true.

If $Con\left(t^{'}, t^{''}, M\right) = 1$, then, there exists $t^{'''} \in r^{'}$, $t^{'''} = t^{'}$, such that $\left(t^{'''}, t^{'}, t^{''}, K, L, M, \theta_2, r^{'}\right)$ holds true.

Suppose that

$$h\left(\max\left\{Val_r^{\beta}\left(\&_{B \in L} B\right), Val_r^{\beta}\left(\&_{C \in M} C\right)\right\}\right) > 0.$$

We obtain,

$$\frac{h\left(\frac{1}{2}\right)}{h\left(\max\left\{Val_r^{\beta}\left(\&_{B \in L} B\right), Val_r^{\beta}\left(\&_{C \in M} C\right)\right\}\right)} > Val_r^{\beta}\left(\&_{A \in K} A\right).$$

Since, $Val_r^{\beta}\left(\&_{A \in K} A\right) > \frac{1}{2}$, it follows that

$$\frac{h\left(\frac{1}{2}\right)}{h\left(\max\left\{Val_r^{\beta}\left(\&_{B \in L} B\right), Val_r^{\beta}\left(\&_{C \in M} C\right)\right\}\right)} > 1,$$

i.e.,

$$h\left(\max\left\{Val_r^{\beta}\left(\&_{B \in L} B\right), Val_r^{\beta}\left(\&_{C \in M} C\right)\right\}\right) < h\left(\frac{1}{2}\right),$$

i.e.,

$$\max\left\{Val_r^{\beta}\left(\&_{B \in L} B\right), Val_r^{\beta}\left(\&_{C \in M} C\right)\right\} > \frac{1}{2}.$$

Consequently, $Val_r^{\beta}\left(\&_{B \in L} B\right) > \frac{1}{2}$ or $Val_r^{\beta}\left(\&_{C \in M} C\right) > \frac{1}{2}$.

Hence, as in the case $Val_r^{\beta}\left(\&_{B \in L} B\right) = 1 > \frac{1}{2}$ or $Val_r^{\beta}\left(\&_{C \in M} C\right) = 1 > \frac{1}{2}$, we conclude that $\left(t^{'''}, t^{'}, t^{''}, K, L, M, \theta_2, r^{'}\right)$ holds true.

Now, we prove that $r^{'}$ satisfies (X, f, Y, θ_1, F) resp. (X, m, Y, θ_1, F).

By our assumption, we have that $Val_r^{\beta}\left(F_{X,Y}^{f}\right) \leq \frac{1}{2}$ resp. $Val_r^{\beta}\left(F_{X,Y}^{m}\right) \leq \frac{1}{2}$.

Reasoning in exactly the same way as above, we obtain that (1) resp. (2) holds true.

Without loss of generality, we may assume that (3) resp. (4) does not hold.

This actually means that (5) resp. (6) holds true.

Hence, (7) resp. (8) holds true.

If $Val_r^{\beta}\left(\&_{A \in X} A\right) \leq \frac{1}{2}$, then, (7) resp. (8) yields that

$$\frac{h\left(\frac{1}{2}\right)}{h\left(Val_r^{\beta}\left(\&_{B \in Y} B\right)\right)} = 0$$

resp.

$$\frac{h\left(\frac{1}{2}\right)}{h\left(\max\left\{Val_r^{\beta}\left(\&_{B \in Y} B\right), Val_r^{\beta}\left(\&_{C \in Z} C\right)\right\}\right)} = 0,$$

i.e., that $h\left(\frac{1}{2}\right) = 0$.

This is a contradiction, however.

Consequently, $Val_r^{\beta}\left(\&_{A \in X} A\right) > \frac{1}{2}$.

This fact, and (7) resp. (8) imply that

$$\frac{h\left(\frac{1}{2}\right)}{h\left(Val_r^{\beta}\left(\&_{B \in Y} B\right)\right)} \leq \frac{1}{2} \; (< 1)$$

resp.

$$\frac{h\left(\frac{1}{2}\right)}{h\left(\max\left\{Val_r^{\beta}\left(\&_{B \in Y} B\right), Val_r^{\beta}\left(\&_{C \in Z} C\right)\right\}\right)} \leq \frac{1}{2} \; (< 1),$$

i.e.,

$$Val_r^{\beta}\left(\&_{B\in Y}B\right)<\frac{1}{2}$$

resp.

$$\max\left\{Val_r^{\beta}\left(\&_{B\in Y}B\right),Val_r^{\beta}\left(\&_{C\in Z}C\right)\right\}<\frac{1}{2},$$

i.e., $Val_r^{\beta}\left(\&_{B\in Y}B\right)<\frac{1}{2}$ resp. $Val_r^{\beta}\left(\&_{B\in Y}B\right)<\frac{1}{2}$, $Val_r^{\beta}\left(\&_{C\in Z}C\right)<\frac{1}{2}$.

Now, in the fuzzy functional case, we have that $Val_r^{\beta}\left(\&_{A\in X}A\right)>\frac{1}{2}$, $Val_r^{\beta}\left(\&_{B\in Y}B\right)<\frac{1}{2}$, so $Con\left(t^{'},t^{''},X\right)=1$, $Con\left(t^{'},t^{''},Y\right)=\theta^{''}$.

It follows that

$$\begin{aligned}&Con\left(t^{'},t^{''},Y\right)\\&=\theta^{''}<\theta^{'}\leq\theta_1=\min\left\{\theta_1,1\right\}\\&=\min\left\{\theta_1,Con\left(t^{'},t^{''},X\right)\right\}.\end{aligned}$$

In other words, $r^{'}$ violates (X,f,Y,θ_1,F).

In the fuzzy multivalued case, we have that $Val_r^{\beta}\left(\&_{A\in X}A\right)>\frac{1}{2}$, $Val_r^{\beta}\left(\&_{B\in Y}B\right)<\frac{1}{2}$, and $Val_r^{\beta}\left(\&_{C\in Z}C\right)<\frac{1}{2}$, so $Con\left(t^{'},t^{''},X\right)=1$, $Con\left(t^{'},t^{''},Y\right)=\theta^{''}$, and $Con\left(t^{'},t^{''},Z\right)=\theta^{''}$.

Thus, if $t^{'''}\in r^{'}$, $t^{'''}=t^{'}$, then

$$\begin{aligned}&Con\left(t^{'''},t^{''},Z\right)\\&=\theta^{''}<\theta^{'}\leq\theta_1=\min\left\{\theta_1,1\right\}\\&=\min\left\{\theta_1,Con\left(t^{'},t^{''},X\right)\right\}.\end{aligned}$$

Similarly, if $t^{'''}\in r^{'}$, $t^{'''}=t^{''}$, then

$$\begin{aligned}&Con\left(t^{'''},t^{'},Y\right)\\&=\theta^{''}<\min\left\{\theta_1,Con\left(t^{'},t^{''},X\right)\right\}.\end{aligned}$$

These inequalities mean that $\left(t^{'''},t^{'},t^{''},X,Y,Z,\theta_1,r^{'}\right)$ does not hold, i.e., that $r^{'}$ violates (X,m,Y,θ_1,F).

Now, suppose that $Val_r^{\beta}\left(c^{'}\right)>\frac{1}{2}$ for every valuation Val_r^{β}, such that $Val_r^{\beta}\left(F_{K,L}^{f}\right)>\frac{1}{2}$ and $Val_r^{\beta}\left(F_{K,L}^{m}\right)>\frac{1}{2}$ for all $F_{K,L}^{f}\in C^{'}$ and all $F_{K,L}^{m}\in C^{'}$.

We shall prove that (X,f,Y,θ_1,F) resp. (X,m,Y,θ_1,F) is satisfied by r if (K,f,L,θ_2,F) is satisfied by r, and (K,m,L,θ_2,F) is satisfied by r, for all $(K,f,L,\theta_2,F)\in C$ and all $(K,m,L,\theta_2,F)\in C$, where r is a two-element fuzzy relation instance on given scheme $Sch\left(A_1,A_2,...,A_n\right)$.

Suppose that the last statement is not satisfied.

This means that there exists some $r^{'}=\left\{t^{'},t^{''}\right\}$, such that (K,f,L,θ_2,F) is satisfied by $r^{'}$, and (K,m,L,θ_2,F) is satisfied by $r^{'}$, for all $(K,f,L,\theta_2,F)\in C$, and all $(K,m,L,\theta_2,F)\in C$, but (X,f,Y,θ_1,F) resp. (X,m,Y,θ_1,F) is not satisfied by $r^{'}$.

Denote by $Z^{'}$ the set

$$\left\{A\in\left\{A_1,A_2,...,A_n\right\}\,:\,Con\left(t^{'},t^{''},A\right)=1\right\}.$$

First, suppose that $Z^{'}=\emptyset$.

It follows that $Con\left(t^{'},t^{''},A\right)=\theta^{''}$ for all $A\in\left\{A_1,A_2,...,A_n\right\}$.

Thus,

$$\begin{aligned}&Con\left(t^{'},t^{''},Q\right)\\&=\min\left\{Con\left(t^{'},t^{''},A\right)\,:\,A\in Q\right\}=\theta^{''}\end{aligned}$$

for any $Q\subseteq\left\{A_1,A_2,...,A_n\right\}$.

By our assumption, (X,f,Y,θ_1,F) resp. (X,m,Y,θ_1,F) is not satisfied by $r^{'}$, i.e., $\left(t^{'},t^{''},Y,\theta_1,X,r^{'}\right)$ resp. $\left(t^{'''},t^{'},t^{''},X,Y,Z,\theta_1,r^{'}\right)$ does not hold.

Note that $Con\left(t^{'},t^{''},X\right)=\theta^{''}$, $Con\left(t^{'},t^{''},Y\right)=\theta^{''}$, and $Con\left(t^{'},t^{''},Z\right)=\theta^{''}$.

So, the fact that $\left(t^{'},t^{''},Y,\theta_1,X,r^{'}\right)$ does not hold, yields a contradiction since $\min\left\{\theta_1,\theta^{''}\right\}=\theta^{''}$.

Similarly, $\left(t^{'''},t^{'},t^{''},X,Y,Z,\theta_1,r^{'}\right)$ does not hold, so, in particular, $\left(t^{'},t^{'},t^{''},X,Y,Z,\theta_1,r^{'}\right)$ does not hold.

This is a contradiction, since, once again

$$\theta^{''}=\min\left\{\theta_1,\theta^{''}\right\}=\min\left\{\theta_1,Con\left(t^{'},t^{''},X\right)\right\}.$$

Since the assumption $Z^{'} = \emptyset$ yields to contradiction, it follows that $Z^{'} \neq \emptyset$.

Second, suppose that $Z^{'} = \{A_1, A_2, ..., A_n\}$.

It follows that $Con\left(t^{'}, t^{''}, A\right) = 1$ for all $A \in \{A_1, A_2, ..., A_n\}$.

Thus,

$$\begin{aligned}&Con\left(t^{'}, t^{''}, Q\right)\\&= \min\left\{Con\left(t^{'}, t^{''}, A\right) \,:\, A \in Q\right\} = 1\end{aligned}$$

for any $Q \subseteq \{A_1, A_2, ..., A_n\}$.

Now, $Con\left(t^{'}, t^{''}, X\right) = 1$, $Con\left(t^{'}, t^{''}, Y\right) = 1$, and $Con\left(t^{'}, t^{''}, Z\right) = 1$.

Since (X, f, Y, θ_1, F) is not satisfied by $r^{'}$, we have that 1 is strictly smaller than $\min\{\theta_1, 1\} = \theta_1$.

This is a contradiction.

Similarly, (X, m, Y, θ_1, F) is not satisfied by $r^{'}$, so $\left(t^{'''}, t^{'}, t^{''}, X, Y, Z, \theta_1, r^{'}\right)$ does not hold, and then $\left(t^{'}, t^{'}, t^{''}, X, Y, Z, \theta_1, r^{'}\right)$ does not hold.

We obtain that 1 is strictly smaller than

$$\theta_1 = \min\{\theta_1, 1\} = \min\left\{\theta_1, Con\left(t^{'}, t^{''}, X\right)\right\}.$$

Hence, a contradiction.

So, the assumption that $Z^{'} = \{A_1, A_2, ..., A_n\}$ yields a contradiction.

It follows that $Z^{'} \subset \{A_1, A_2, ..., A_n\}$.

We have, $\emptyset \neq Z^{'} \subset \{A_1, A_2, ..., A_n\}$.

In the sequel, we shall consider the valuation $Val^1_{r'}$.

We shall prove that $Val^1_{r'}\left(F^f_{K,L}\right) > \frac{1}{2}$ and $Val^1_{r'}\left(F^m_{K,L}\right) > \frac{1}{2}$ for all $F^f_{K,L} \in C^{'}$ and all $F^m_{K,L} \in C^{'}$, but $Val^1_{r'}\left(c^{'}\right) \leq \frac{1}{2}$.

Let $F^f_{K,L} \in C^{'}$.

Suppose that $Val^1_{r'}\left(F^f_{K,L}\right) \leq \frac{1}{2}$ resp. $Val^1_{r'}\left(F^m_{K,L}\right) \leq \frac{1}{2}$.

We obtain,

$$\begin{aligned}&Val^1_{r'}\left((\&_{A\in K}A) \to (\&_{B\in L}B)\right)\\&= h^{-1}\Bigg(\max\Big\{Val^1_{r'}\left(\&_{A\in K}A\right) \times\\&\times h\left(Val^1_{r'}\left(\&_{B\in L}B\right)\right), h(1)\Big\}\Bigg)\\&\leq \frac{1}{2}\end{aligned}$$

resp.

$$\begin{aligned}&Val^1_{r'}\left((\&_{A\in K}A) \to ((\&_{B\in L}B) \parallel (\&_{C\in M}C))\right)\\&= h^{-1}\Bigg(\max\Big\{Val^1_{r'}\left(\&_{A\in K}A\right) \times\\&\times h\left(\max\left\{Val^1_{r'}\left(\&_{B\in L}B\right), Val^1_{r'}\left(\&_{C\in M}C\right)\right\}\right)\\&, h(1)\Big\}\Bigg)\\&\leq \frac{1}{2},\end{aligned}$$

i.e.,

$$\begin{aligned}&\max\Big\{Val^1_{r'}\left(\&_{A\in K}A\right) \times\\&\times h\left(Val^1_{r'}\left(\&_{B\in L}B\right)\right), h(1)\Big\} \geq h\left(\frac{1}{2}\right)\end{aligned}$$

resp.

$$\begin{aligned}&\max\Big\{Val^1_{r'}\left(\&_{A\in K}A\right) \times\\&\times h\left(\max\left\{Val^1_{r'}\left(\&_{B\in L}B\right), Val^1_{r'}\left(\&_{C\in M}C\right)\right\}\right)\\&, h(1)\Big\} \geq h\left(\frac{1}{2}\right).\end{aligned}$$

The fact that $\frac{1}{2} < 1$, implies that $h\left(\frac{1}{2}\right) > h(1)$. Therefore,

$$\begin{aligned}&Val^1_{r'}\left(\&_{A\in K}A\right) h\left(Val^1_{r'}\left(\&_{B\in L}B\right)\right)\\&= \max\Big\{Val^1_{r'}\left(\&_{A\in K}A\right) \times\\&\times h\left(Val^1_{r'}\left(\&_{B\in L}B\right)\right), h(1)\Big\} \geq h\left(\frac{1}{2}\right)\end{aligned} \tag{9}$$

resp.

$$\begin{aligned}&Val^1_{r'}\left(\&_{A\in K}A\right) h\Bigg(\max\Big\{Val^1_{r'}\left(\&_{B\in L}B\right),\\&Val^1_{r'}\left(\&_{C\in M}C\right)\Big\}\Bigg) \geq h\left(\frac{1}{2}\right).\end{aligned} \tag{10}$$

We may assume, without loss of generality, that

$$h\left(Val^1_{r'}\left(\&_{B\in L}B\right)\right)>0$$

resp.

$$h\left(\max\left\{Val^1_{r'}\left(\&_{B\in L}B\right),\right.\right.$$
$$\left.\left.Val^1_{r'}\left(\&_{C\in M}C\right)\right\}\right)>0.$$

If $Val^1_{r'}\left(\&_{A\in K}A\right)\le\frac{1}{2}$, then $Con\left(t^{'},t^{''},K\right)=\theta^{''}$, and (9) resp. (10) implies that

$$\frac{h\left(\frac{1}{2}\right)}{h\left(Val^1_{r'}\left(\&_{B\in L}B\right)\right)}=0$$

resp.

$$\frac{h\left(\frac{1}{2}\right)}{h\left(\max\left\{Val^1_{r'}\left(\&_{B\in L}B\right),Val^1_{r'}\left(\&_{C\in M}C\right)\right\}\right)}$$
$$=0,$$

i.e., that $h\left(\frac{1}{2}\right)=0$.

This is contradiction, however.

Consequently, $Val^1_{r'}\left(\&_{A\in K}A\right)>\frac{1}{2}$, so $Con\left(t^{'},t^{''},K\right)=1$, and (9) resp. (10) yields that

$$Val^1_{r'}\left(\&_{B\in L}B\right)<\frac{1}{2}$$

resp.

$$\max\left\{Val^1_{r'}\left(\&_{B\in L}B\right),Val^1_{r'}\left(\&_{C\in M}C\right)\right\}<\frac{1}{2},$$

i.e., that $Con\left(t^{'},t^{''},L\right)=\theta^{''}$ resp. $Con\left(t^{'},t^{''},L\right)=\theta^{''}$, $Con\left(t^{'},t^{''},M\right)=\theta^{''}$.

Since (K,f,L,θ_2,F) is satisfied by $r^{'}$ for all $(K,f,L,\theta_2,F)\in C$, we obtain a contradiction that $\theta_2=\min\{\theta_2,1\}$ is not larger than $\theta^{''}$.

Moreover, by our assumption, (K,m,L,θ_2,F) is satisfied by $r^{'}$ for all $(K,m,L,\theta_2,F)\in C$.

This means that $\left(t^{'''},t^{'},t^{''},K,L,M,\theta_2,r^{'}\right)$ holds true for some $t^{'''}\in r^{'}$.

This, however, is not possible since we have that $\theta_2=\min\{\theta_2,1\}$ is larger that $\theta^{''}$.

Hence, a contradiction.

Thus, our assumption that $Val^1_{r'}\left(F^f_{K,L}\right)\le\frac{1}{2}$ resp. $Val^1_{r'}\left(F^m_{K,L}\right)\le\frac{1}{2}$ is not valid.

Therefore, $Val^1_{r'}\left(F^f_{K,L}\right)>\frac{1}{2}$ and $Val^1_{r'}\left(F^m_{K,L}\right)>\frac{1}{2}$ for all $F^f_{K,L}\in C^{'}$ and all $F^m_{K,L}\in C^{'}$.

Finally, we prove that $Val^1_{r'}\left(c^{'}\right)\le\frac{1}{2}$, i.e., that $Val^1_{r'}\left(F^f_{X,Y}\right)\le\frac{1}{2}$ resp. $Val^1_{r'}\left(F^m_{X,Y}\right)\le\frac{1}{2}$.

Suppose that $Val^1_{r'}\left(F^f_{X,Y}\right)>\frac{1}{2}$ resp. $Val^1_{r'}\left(F^m_{X,Y}\right)>\frac{1}{2}$.

Reasoning in the same way as in the case $Val^{\beta}_{r}\left(F^f_{K,L}\right)>\frac{1}{2}$ resp. $Val^{\beta}_{r}\left(F^m_{K,L}\right)>\frac{1}{2}$, we obtain that

$$Val^1_{r'}\left(\&_{A\in X}A\right)h\left(Val^1_{r'}\left(\&_{B\in Y}B\right)\right)$$
$$<h\left(\frac{1}{2}\right)\tag{11}$$

resp.

$$Val^1_{r'}\left(\&_{A\in X}A\right)\times$$
$$\times\, h\left(\max\left\{Val^1_{r'}\left(\&_{B\in Y}B\right),\right.\right.\tag{12}$$
$$\left.\left.Val^1_{r'}\left(\&_{C\in Z}C\right)\right\}\right)<h\left(\frac{1}{2}\right).$$

Note that by our assumption, (X,f,Y,θ_1,F) resp. (X,m,Y,θ_1,F) is not satisfied by $r^{'}$, i.e., $\left(t^{'},t^{''},Y,\theta_1,X,r^{'}\right)$ resp. $\left(t^{'''},t^{'},t^{''},X,Y,Z,\theta_1,r^{'}\right)$ does not hold.

Without loss of generality, we shall assume that

$$h\left(Val^1_{r'}\left(\&_{B\in Y}B\right)\right)>0$$

resp.

$$h\left(\max\left\{Val^1_{r'}\left(\&_{B\in Y}B\right),Val^1_{r'}\left(\&_{C\in Z}C\right)\right\}\right)>0.$$

If $Val^1_{r'}\left(\&_{A\in X}A\right)\le\frac{1}{2}$, then $Con\left(t^{'},t^{''},X\right)=\theta^{''}$.

Since $Con\left(t^{'},t^{''},Q\right)\geq\theta^{''}$ for any $Q\subseteq\{A_1,A_2,...,A_n\}$, it follows that $Con\left(t^{'},t^{''},Y\right)\geq\theta^{''}$.

Moreover, $\min\left\{\theta_1,\theta^{''}\right\}=\theta^{''}$.

This contradicts the fact that $\left(t^{'},t^{''},Y,\theta_1,X,r^{'}\right)$ does not hold.

On the other side, in the fuzzy multivalued case, we obtain that $\left(t^{'},t^{'},t^{''},X,Y,Z,\theta_1,r^{'}\right)$ holds true.

This contradicts the fact that $\left(t^{'''},t^{'},t^{''},X,Y,Z,\theta_1,r^{'}\right)$ does not hold.

We conclude, $Val^1_{r^{'}}\left(\&_{A\in X}A\right)>\frac{1}{2}$.

Hence, $Con\left(t^{'},t^{''},X\right)=1$, and (11) resp. (12) yields that

$$h\left(Val^1_{r^{'}}\left(\&_{B\in Y}B\right)\right)<h\left(\frac{1}{2}\right)$$

resp.

$$h\left(\max\left\{Val^1_{r^{'}}\left(\&_{B\in Y}B\right),Val^1_{r^{'}}\left(\&_{C\in Z}C\right)\right\}\right)$$
$$<h\left(\frac{1}{2}\right),$$

i.e., that $Val^1_{r^{'}}\left(\&_{B\in Y}B\right)>\frac{1}{2}$ resp. $Val^1_{r^{'}}\left(\&_{B\in Y}B\right)>\frac{1}{2}$ or $Val^1_{r^{'}}\left(\&_{C\in Z}C\right)>\frac{1}{2}$.

It follows that $Con\left(t^{'},t^{''},Y\right)=1$ resp. $Con\left(t^{'},t^{''},Y\right)=1$ or $Con\left(t^{'},t^{''},Z\right)=1$.

Since $Con\left(t^{'},t^{''},Y\right)=1$, we obtain a contradiction in the fuzzy functional case with the fact that $\left(t^{'},t^{''},Y,\theta_1,X,r^{'}\right)$ does not hold.

In the fuzzy multivalued case, we have that $Con\left(t^{'},t^{''},X\right)=1$, $Con\left(t^{'},t^{''},Y\right)=1$ or $Con\left(t^{'},t^{''},X\right)=1$, $Con\left(t^{'},t^{''},Z\right)=1$.

In the first case we obtain that $\left(t^{''},t^{'},t^{''},X,Y,Z,\theta_1,r^{'}\right)$ holds true, while in the second case, we obtain that $\left(t^{'},t^{'},t^{''},X,Y,Z,\theta_1,r^{'}\right)$ holds true.

This contradicts the fact that $\left(t^{'''},t^{'},t^{''},X,Y,Z,\theta_1,r^{'}\right)$ does not hold.

Therefore, $Val^1_{r^{'}}\left(F^f_{X,Y}\right)\leq\frac{1}{2}$ resp. $Val^1_{r^{'}}\left(F^m_{X,Y}\right)\leq\frac{1}{2}$.

This contradicts our assumption that $Val^{\beta}_r\left(F^f_{X,Y}\right)>\frac{1}{2}$ resp. $Val^{\beta}_r\left(F^m_{X,Y}\right)>\frac{1}{2}$ for every valuation Val^{β}_r, such that $Val^{\beta}_r\left(F^f_{K,L}\right)>\frac{1}{2}$ and $Val^{\beta}_r\left(F^m_{K,L}\right)>\frac{1}{2}$ for all $F^f_{K,L}\in C^{'}$ and all $F^m_{K,L}\in C^{'}$.

This means that our assumption on $r^{'}$ is not valid.

Consequently, (X,f,Y,θ_1,F) resp. (X,m,Y,θ_1,F) is satisfied by r if (K,f,L,θ_2,F) is satisfied by r, and (K,m,L,θ_2,F) is satisfied by r for all $(K,f,L,\theta_2,F)\in C$ and all $(K,m,L,\theta_2,F)\in C$, where r is a two-element fuzzy relation instance on $Sch\left(A_1,A_2,...,A_n\right)$.

4 Concluding remarks

In order to complement the notation applied in the previous section, we note the following.

We say that (X,f,Y,θ,V) is satisfied by r (r is arbitrary vague relation instance on $Sch\left(A_1,A_2,...,A_n\right)$), if for t_1, $t_2\in r$,

$$Con\left(t_1,t_2,Y\right)\geq\min\left\{\theta,Con\left(t_1,t_2,X\right)\right\}.$$

We also say that (t_1,t_2,Y,θ_1,X,r) holds true (see, [16, p. 6]).

Furthermore, we say that (X,m,Y,θ,V) is satisfied by r (r is arbitrary), if for t_1, $t_2\in r$, there is $t_3\in r$, such that

$$Con\left(t_3,t_1,X\right)\geq\min\left\{\theta,Con\left(t_1,t_2,X\right)\right\},$$
$$Con\left(t_3,t_1,Y\right)\geq\min\left\{\theta,Con\left(t_1,t_2,X\right)\right\},$$
$$Con\left(t_3,t_2,Z\right)\geq\min\left\{\theta,Con\left(t_1,t_2,X\right)\right\}.$$

As in the case of fuzzy multivalued dependencies, we say that $(t_3,t_1,t_2,X,Y,Z,\theta,r)$ holds true (see, [17, p. 5]).

The concept of conformance is introduced by Definitions 3.1 and 3.2 in [15, pp. 165-166], and is applied further on in [23], and also adapted in [16] and [17].

In any case, it is based on application of appropriate similarity relation.

In this and similar researches, is is usually comfortable to select a general similarity relation which is reflexive, symmetric, and satisfies the max-min transitivity condition (this definition is quite restrictive, but also possible to occur in reality).

However, in the vague functional case, the authors are more relaxed during such selection [24], [25], [26].

Furthermore, the concept of valuation (or an interpretation) is adopted from [27], and adapted to our setting.

In short, it enables us to translate the attributes into fuzzy formulas.

Depending on the fact that $Con\left(t_1, t_2, A\right)$ is larger resp. smaller than some fixed value $\beta \in [0,1]$, we define the corresponding valuation value $Val_r^\beta\left(A\right) > \frac{1}{2}$ resp. $< \frac{1}{2}$, with $r = \{t_1, t_2\}$.

As it can be seen from the previous section, we assume that the fuzzy conjunction operator is modeled by the minimum t-norm [28, p. 17].

Furthermore, the fuzzy disjunction is modeled by the maximum t-conorm [28, p. 23] (see also, [29]).

The results derived in the previous section may be formulated in the following form.

Theorem 1. *Let I_h be an h - generated implication, and (X, m, Y, θ, F) resp. (X, m, Y, θ, V) fuzzy resp. vague multivalued dependency on $Sch\left(A_1, A_2, ..., A_n\right)$. Then, $(t_3, t_1, t_2, X, Y, Z, \theta, r)$ holds true and $Con\left(t_1, t_2, A\right) \geq \theta$ for all $A \in X$ if and only if $Con\left(t_1, t_2, X\right) \geq \theta$ and $Val_r^\theta\left(F_{X,Y}^m\right) > \frac{1}{2}$, where $r = \{t_1, t_2\}$.*

Theorem 2. *Let I_h be an h - generated implication, and (X, f, Y, θ_1, F) $((X, f, Y, \theta_1, V))$ resp. (X, m, Y, θ_1, F) $((X, m, Y, \theta_1, V))$ fuzzy (vague) functional resp. fuzzy (vague) multivalued dependency on $Sch\left(A_1, A_2, ..., A_n\right)$. Then, $(t_1, t_2, Y, \theta_1, X, r)$ resp. $(t_3, t_1, t_2, X, Y, Z, \theta_1, r)$ holds true if $(t_1, t_2, L, \theta_2, K, r)$ and $(t_3, t_1, t_2, K, L, M, \theta_2, r)$ hold true for all (K, f, L, θ_2, F) $((K, f, L, \theta_2, V))$, (K, m, L, θ_2, F) $((K, m, L, \theta_2, V)) \in C$, where $r = \{t_1, t_2\}$, if and only if $Val_r^\beta\left(F_{X,Y}^f\right) > \frac{1}{2}$ resp. $Val_r^\beta\left(F_{X,Y}^m\right) > \frac{1}{2}$ for every valuation Val_r^β, such that $Val_r^\beta\left(F_{K,L}^f\right) > \frac{1}{2}$ and $Val_r^\beta\left(F_{K,L}^m\right) > \frac{1}{2}$ for all $F_{K,L}^f \in C'$ and all $F_{K,L}^m \in C'$.*

Clearly, in Theorem 2, C denotes the set of fuzzy (vague) functional and fuzzy (vague) multivalued dependencies on $Sch\left(A_1, A_2, ..., A_n\right)$, while C' denotes the corresponding set of fuzzy formulas.

Theorems 1 and 2 are derived for an h-generated implication.

By [30, p. 179, Th. 13], I_h is an (S, N) - implication, where h is an h-generator, S is some t-conorm, and N is contiuous negation.

Moreover, by Corollaries 2 and 3 in [30], I_h is an (S, N)-implication generated from some t-conorm and some strict (strong) negation if and only if $h\left(1\right) = 0$ $(h = h^{-1})$.

Taking into account these facts, it would be natural to try to verify the aforementioned results for various individual definitions of S-implications, such as Kleene-Dienes, Reichenbach, Most Strict, Largest, Least Strict, Lukasiewicz implication, etc., or, generally, for entire family of S-implications. These ideas may be regarded as the instructions for possible future work. Regarding the limitations of this study and suggested improvements of the work, we highlight that the limiting case for the future directions would be the case of the general fuzzy implication operator definition.

Note that the results given by Theorems 1 and 2 may be applied in the way described in [21] and [20].

5 Applications

We demonstrate the practical and the engineering applications of this study by the following example.

Example 1. Consider the fuzzy (vague) functional dependencies $(\{A, B\}, f, C, \theta_1, F\left(V\right))$ and $(\{A, B\}, f, D, \theta_2, F\left(V\right))$. We shall prove that these two dependencies imply the fuzzy (vague) functional dependency $(\{A, B\}, f, \{C, D\}, \theta, F\left(V\right))$, where $\theta = \min\{\theta_1, \theta_2\}$.

Here, we assume that the dependencies are given on some scheme $Sch\left(A, B, C, D\right)$, where, clearly, $\{A, B\}$, $\{C\}$, $\{D\}$ and $\{C, D\}$ are subsets of the universal set of attributes $\{A, B, C, D\}$.

Of course, one way to derive $(\{A, B\}, f, \{C, D\}, \theta, F\left(V\right))$ is to derive it by hand. The augmentation rule for fuzzy (vague) functional dependencies states that the fuzzy (vague) functional dependency $(X, f, Y, \theta, F\left(V\right))$ yields the fuzzy (vague) functional dependency $(X \cup Z, f, Y \cup Z, \theta, F\left(V\right))$ for any subset $Z \subseteq \{A, B, C, D\}$. Moreover, the transitivity rule for fuzzy (vague) functional dependencies says that the fuzzy (vague) functional dependencies $(X, f, Y, \theta_1, F\left(V\right))$ and $(Y, f, Z, \theta_2, F\left(V\right))$ imply the fuzzy (vague) functional dependency $(X, f, Z, \theta, F\left(V\right))$, where $\theta = \min\{\theta_1, \theta_2\}$. Thus, by assumption, $(\{A, B\}, f, C, \theta_1, F\left(V\right))$ holds true. The augmentation with $\{A, B\}$ gives us the dependency $(\{A, B\}, f, \{A, B, C\}, \theta_1, F\left(V\right))$. If we augment the dependency

$(\{A,B\}, f, D, \theta_2, F(V))$ with $\{C\}$, we obtain the dependency $(\{A,B,C\}, f, \{C,D\}, \theta_2, F(V))$. Now, the transitivity rule applied on $(\{A,B\}, f, \{A,B,C\}, \theta_1, F(V))$ and $(\{A,B,C\}, f, \{C,D\}, \theta_2, F(V))$ gives us the dependency $(\{A,B\}, f, \{C,D\}, \theta, F(V))$. This completes our proof by hand, that is, the proof obtained by applying purely theoretical ingredients. This proof shows that deriving of new dependencies in this, classical way is not guaranteed to be easy task. It can be easy, of course. On the other side, it can also be very hard task to be done. Having in mind these facts, it is not so hard to conclude that this approach is not a reliable one.

On the other hand, we may apply Theorem 2 to obtain the dependency $(\{A,B\}, f, \{C,D\}, \theta, F(V))$. Indeed, suppose that r, $|r| = 2$ is a fuzzy (vague) relation instance on $Sch(A,B,C,D)$. Let $\beta \in [0,1]$ be some number. Furthermore, suppose that $Val_r^\beta\left(F^f_{\{A,B\},C}\right) = Val_r^\beta((A \wedge B) \Rightarrow C) > \frac{1}{2}$, $Val_r^\beta\left(F^f_{\{A,B\},D}\right) = Val_r^\beta((A \wedge B) \Rightarrow D) > \frac{1}{2}$. If we prove that $Val_r^\beta\left(F^f_{\{A,B\},\{C,D\}}\right) = Val_r^\beta((A \wedge B) \Rightarrow (C \wedge D)) > \frac{1}{2}$, then, in the view of Theorem 2, we shall immediately have that the dependency $(\{A,B\}, f, \{C,D\}, \theta, F(V))$ follows. In order to prove that $Val_r^\beta\left(F^f_{\{A,B\},\{C,D\}}\right) > \frac{1}{2}$ is satisfied, we assume the opposite, that $Val_r^\beta\left(F^f_{\{A,B\},\{C,D\}}\right) \leq \frac{1}{2}$ is valid. In other words, we assume that the fuzzy formulas $(A \wedge B) \Rightarrow C$, $(A \wedge B) \Rightarrow D$ and $\neg((A \wedge B) \Rightarrow (C \wedge D))$ hold true. Since $p \Rightarrow q \equiv \neg p \vee q$ for any p and q, it follows that $\neg A \vee \neg B \vee C$, $\neg A \vee \neg B \vee D$, and $A \wedge B \wedge (\neg C \vee \neg D)$ hold true. In order to obtain a contradiction, we apply the resolution principle to the conjunctive terms $\neg A \vee \neg B \vee C$, $\neg A \vee \neg B \vee D$, A, B and $\neg C \vee \neg D$ of the derived fuzzy formulas. Thus, resolving $\neg A \vee \neg B \vee C$ by A and then by B, we obtain C. Resolving $\neg A \vee \neg B \vee D$ by A and then by B, we obtain D. Finally, resolving $\neg C \vee \neg D$ by C and then by D, we obtain a contradiction. Hence, $Val_r^\beta\left(F^f_{\{A,B\},\{C,D\}}\right) > \frac{1}{2}$, so the dependency $(\{A,B\}, f, \{C,D\}, \theta, F(V))$ holds true. Note that the second proof may take more time to be done, but is also more reliable one. Namely, it always gives a solution, and it allows to be completely automated.

6 Discussion

Recall that the authors in [15] resp. [16], [17] first introduced the definitions of fuzzy functional and multivalued resp. vague functional and multivalued dependencies on the basis of conformance between attributes. With addition of the paper [18], the inference rules are listed, and are shown to be sound and complete. Based on these results, the authors in [23] proved that the set of the inference rules for fuzzy functional and fuzzy multivalued dependencies is complete set in two-element fuzzy relation instances (Theorem 7), and then, in general, in arbitrary fuzzy relation instances (Corollary 8). The corresponding results in vague setting are then derived in [19], [20], [21] and [22]. The completeness in arbitrary relations enabled the authors to leave the classical approach and its disadvantages during the process of derivation of new dependencies, and to automate the process by applying the resolution principle (see, [23], [20], [21]). The advantages of the automated approach are clearly demonstrated by Example 1. On the other side, the disadvantages of the classical approach are more than obvious: large number of inference rules, matter of their choice, unknown number of steps, uncertainty of outcome, etc. In order to obtain the completeness in arbitrary relations, the authors applied a number of various fuzzy implication operators, like: Kleene-Dienes, Reichenbach (Kleene-Dienes-Lukasiewicz), Yager, f-generated, g-generated, Lukasiewicz, Klir-Yuan [31], etc. In this paper we use the h-generated implications. This means that Theorems 1 and 2, as well as the analogous theorems, do not depend on the selection of the fuzzy implication operator. Thus, the choice of the fuzzy implication operator is auxiliary in its nature in such research. More precisely, the results derived in this paper do not affect the selected fuzzy implication at all (the h-generated implication in particular). Quite opposite, the h-generated implication is exploited to help us complete the investigation. Regarding the soundness of the inference rules for vague functional and multivalued dependencies, we also refer to [32] and [33].

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