The $\mathcal{H}_\infty$ Model Following Control: An LMI Approach

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Abstract: The aim of this paper is to develop a new approach for a solution of the model following control (MFC) problem with a dynamic compensator by using linear matrix inequalities (LMIs). The $\mathcal{H}_\infty$ model following control problem is derived following LMI formulation. First, the $\mathcal{H}_\infty$ optimal control problem is revisited by referring to Lemmas assuring all admissible controllers minimizing the $\mathcal{H}_\infty$ norm of the transfer function between the exogenous inputs and the outputs. Then, the solvability condition and a design procedure for a two degrees of freedom (2 DOF) dynamic feedback control law is introduced. The existence of a 2 DOF dynamic output feedback controller for the model following control is proven and the stability of the closed-loop system is satisfied by assuring the Hurwitz condition. The benchmark thermal process (PT-326) as the first order process with time-delay is regulated by the presented 2 DOF dynamic output feedback controller. The simulation results illustrate that the presented controller regulates a system with dead-time as a large set of generic industrial systems and the $\mathcal{H}_\infty$ norm of the closed-loop system is assured less than the $\mathcal{H}_\infty$ norm of the desired model system.

Key–Words: Model Following Control, Linear Matrix Inequalities, Two Degrees of Freedom $\mathcal{H}_\infty$ Control, Dead-Time Systems.

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1 Introduction

The model following control problem (MFC) is one of the most familiar problems in the control theory [9, 18]. Let $G_m(s)$ and $G(s)$ be proper transfer matrices of a model system and the given system, respectively. The model following control is to minimize the error in a certain sense between the outputs of the given system and a model system so that the dynamical behavior of the given system approximates one of the model system in Figure 1.

The main application of the MFC approach is in the era of flight simulation. The aim of flight simulation is to impose the characteristics of a flight vehicle to be simulated on airborne simulators. Furthermore, the MFC concepts have been realised in several experimental helicopter simulator programs [9].

Recently, many problems in the control theory have been examined and parameterized via LMIs [1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 14, 15]. In this study, we first show that the MFC problem can be considered as a special case of the standard $\mathcal{H}_\infty$ optimal control problem (OCP). Then, the solvability conditions of the problem which are based on the solutions of three LMIs in [12], are reduced to the solution of only one LMI. Finally, the $\mathcal{H}_\infty$ MFC approach is used to control a system with dead-time.

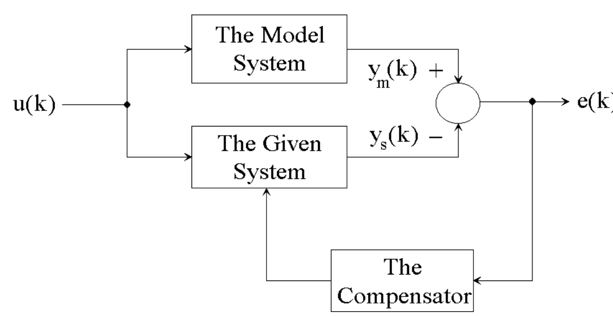

Figure 1: The model following control block diagram.

In this paper, we will mainly follow the terminology of [16]. There feedback structures are categorized depending on the degree of the freedom in the structure. And for a high performance, we choose the two degrees of freedom (2 DOF) structure. Although there are some papers on the $\mathcal{H}_\infty$ design of 2 DOF controller [8, 15], to the best of our knowledge, the MFC problem has not been treated in a $\mathcal{H}_\infty$-settings in the literature.

The following notation will be used throught the paper: $dim(S)$ denotes the dimension of the linear space S. $Ker(M)$ and $Im(M)$ are the null space and the range of the linear operator M, respectively. The rank of a matrix A is defined by

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$$rank(A) = dim(ImA). \tag{1}$$

N^* is the complex-conjugate transpose of the complex matrix N. The $\mathcal{H}_\infty$ norm of a continuous-time transfer matrix $G(s)$ is defined by:

$$\|G(s)\|_\infty = \sup_{\omega\in\mathbb{R}} \sigma_{max}(G(j\omega)). \tag{2}$$

The $\mathcal{H}_\infty$ norm of a discrete-time transfer matrix $G(z)$ is defined by

$$\|G(z)\|_\infty = \sup_{\omega\in[0,2\pi]} \sigma_{max}(G(e^{j\omega})). \tag{3}$$

Here σ_{max} is the largest singular value, i.e.

$$\sigma_{max}(A) = \sqrt{\lambda_{max}(A^*A)} \tag{4}$$

where the matrix $A\in\mathbb{C}^{n\times m}$. λ_{max} is also the largest eigenvalue. Moreover $P > 0$ denotes that the matrix P is positive definite.

2 Preliminaries

Consider a causal discrete linear time-invariant (LTI) generalized plant $P(z)$ described by the state-space equations:

$$\begin{aligned} \underline{x}(k+1) &= \underline{A}\underline{x}(k) + B_1 w(k) + B_2 u(k) && (5)\\ z(k) &= C_1\underline{x}(k) + D_{11}w(k) + D_{12}u(k) && (6)\\ y(k) &= C_2\underline{x}(k) + D_{21}w(k) + D_{22}u(k) && (7) \end{aligned}$$

where $\underline{x}(k)\in\mathbb{R}^n$ is the state vector, $u(k)\in\mathbb{R}^{m_2}$ is the vector of control inputs, $w(k)\in\mathbb{R}^{m_1}$ is the vector of exogenous inputs, i.e. reference signals, disturbance signals, sensor noise, etc., $y(k)\in\mathbb{R}^{p_2}$ is the vector of measurements and $z(k)\in\mathbb{R}^{p_1}$ is the vector of output signals, whose are used to illustrate the performance of the control system. The closed-loop system with the controller $K(z)$ is shown in Figure 2:

It is obvious that the plant $P(z)$ shown in Figure 2 is given by,

$$\begin{aligned} P(z) &= \begin{bmatrix} P_{11}(z) & P_{12}(z) \\ P_{21}(z) & P_{22}(z) \end{bmatrix} && (8)\\ &= \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \\ &+ \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (zI - \underline{A})^{-1} \begin{bmatrix} B_1 & B_2 \end{bmatrix}. && (9) \end{aligned}$$

And the closed-loop transfer matrix from $w(k)$ to $z(k)$ is derived by,

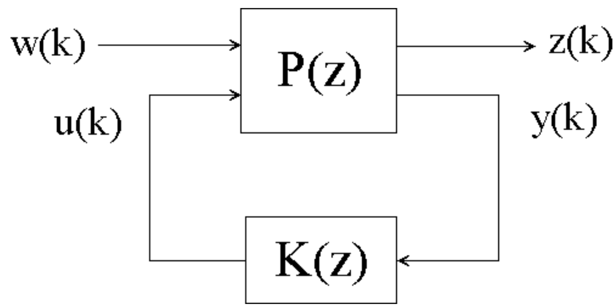

Figure 2: The closed-loop system with the controller $K(z)$.

$$\begin{aligned} T_{zw}(z) &= P_{11}(z) \\ &+ P_{12}(z)K(z)(I - P_{22}(z)K(z))^{-1}P_{21}(z). && (10) \end{aligned}$$

The discrete-time $\mathcal{H}_\infty$ OCP is to find all admissible controllers $K(z)$ such that $\|T_{zw}(z)\|_\infty$ is minimized. The following lemma is well known as the synthesis theorem for the discrete-time $\mathcal{H}_\infty$ OCP in LMI formulation:

Lemma 1 *A controller of order $n_K\geq n$, which holds $\|T_{zw}(z)\|_\infty < \gamma$ exists and the closed-loop system in Figure 2 is internally stable if and only if there exist the matrices $X > 0$ and $Y > 0$ such that,*

$$\begin{bmatrix} N_o & 0 \\ 0 & I_{p_1} \end{bmatrix}^* \begin{bmatrix} \underline{A}^*X\underline{A} - X & \underline{A}^*XB_1 & C_1^* \\ B_1^*X\underline{A} & -\gamma I_{m_1} + B_1^*XB_1 & D_{11}^* \\ C_1 & D_{11} & -\gamma I_{p_1} \end{bmatrix} \cdot \begin{bmatrix} N_o & 0 \\ 0 & I_{p_1} \end{bmatrix} < 0 \tag{11}$$

$$\begin{bmatrix} N_c & 0 \\ 0 & I_{m_1} \end{bmatrix}^* \begin{bmatrix} \underline{A}Y\underline{A}^* - Y & \underline{A}YC_1^* & B_1 \\ C_1Y\underline{A}^* & -\gamma I_{p_1} + C_1YC_1^* & D_{11} \\ B_1^* & D_{11}^* & -\gamma I_{m_1} \end{bmatrix} \cdot \begin{bmatrix} N_c & 0 \\ 0 & I_{m_1} \end{bmatrix} < 0 \tag{12}$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0 \tag{13}$$

where N_o and N_c are full rank matrices with,

$$\begin{aligned} ImN_o &= Ker\begin{bmatrix} C_2 & D_{21} \end{bmatrix} && (14)\\ ImN_c &= Ker\begin{bmatrix} B_2^* & D_{12}^* \end{bmatrix} && (15) \end{aligned}$$

and $(\underline{A}, B_2, C_2)$ is stabilizable and detectable, the matrix $D_{22} = 0$.

Proof: See [12]. ■

In order to present the synthesis theorems of the $\mathcal{H}_\infty$ MFC problem, let us give the following lemmas. They will be used to prove the theorems, which will be presented later. The first lemma is well known as **The Bounded Real Lemma** and can be used to turn the $\mathcal{H}_\infty$ OCP into an LMI:

Lemma 2 *Consider a discrete-time transfer matrix $T(z)$ of (not necessarily minimal) realization $T(z) = D + C(zI - A)^{-1}B$. The following statements are equivalent:*
i) $\|D + C(zI - A)^{-1}B\|_\infty < \gamma$ *and the matrix A is Hurwitz,*
ii) *there exists a solution $X > 0$ to the LMI:*

$$\begin{bmatrix} -X^{-1} & A & B & 0 \\ A^* & -X & 0 & C^* \\ B^* & 0 & -\gamma I & D^* \\ 0 & C & D & -\gamma I \end{bmatrix} < 0, \tag{16}$$

iii) *there exists a solution $Y > 0$ to the LMI:*

$$\begin{bmatrix} AYA^* - Y & AYC^* & B \\ CYA^* & -\gamma I + CYC^* & D \\ B^* & D^* & -\gamma I \end{bmatrix} < 0. \tag{17}$$

Proof: See [12]. ■

It is **Dual Bounded Real Lemma** in the part iii of above Theorems.

Lemma 3 *The block matrix*

$$\begin{bmatrix} P & M \\ M^* & N \end{bmatrix} < 0 \tag{18}$$

if and only if

$$N < 0 \qquad and \qquad P - MN^{-1}M^* < 0. \tag{19}$$

In the sequel, $P - MN^{-1}M^*$ will be referred to as the **Schur complement** *of N.*

Proof: See [4]. ■

Lemma 4 *In a continuous-time system, (A, C) is detectable if and only if there exists a matrix $X > 0$ such that,*

$$N^*(A^*X + XA)N < 0 \tag{20}$$

where N is a full column rank matrix with

$$ImN = KerC. \tag{21}$$

Proof: See [10]. ■

Lemma 5 *In a discrete-time system, (A, C) is detectable if and only if there exists a matrix $X > 0$ such that,*

$$N^*(A^*XA - X)N < 0 \tag{22}$$

where N is a full column rank matrix with

$$ImN = KerC. \tag{23}$$

Proof: This one is the discrete-time form of the Lemma 4. ■

3 The $\mathcal{H}_\infty$ Model Following Control Problem with The Two Degrees of Freedom Dynamic Output Feedback in LMI Formulation

In order to solve the $\mathcal{H}_\infty$ MFC problem via LMI approach, the problem should be formulated as a standard $\mathcal{H}_\infty$ OCP in the state-space equations. For this aim, we will take any realizations of the given system $G(z)$ and the model system $G_m(z)$ as follows:

$$G(z): \quad x(k+1) = Ax(k) + Bu(k) \tag{24}$$
$$y_s(k) = Cx(k) \tag{25}$$

$$G_m(k): \quad q(k+1) = Fq(k) + Gw(k) \tag{26}$$
$$y_m(k) = Hq(k) + Jw(k) \tag{27}$$

where $x(k)\in\mathbb{R}^{n_s}$, $q(k)\in\mathbb{R}^{n_m}$, $u(k)\in\mathbb{R}^m$, $w(k)\in\mathbb{R}^m$, $y_s(k)\in\mathbb{R}^p$ and $y_m(k)\in\mathbb{R}^p$. We take that the given system is strictly proper because of the assumption $D_{22} = 0$ in Lemma 1. But there is no loss of generality, [12]. The control input $u(k)$ can be generated by a two degrees of freedom dynamic output feedback controller:

$$U(z) = L(z)Z(z) + M(z)W(z). \tag{28}$$

And the plant $P(z)$ shown in Figure 3 can be given as follows:

$$\begin{bmatrix} x(k+1) \\ q(k+1) \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix} \begin{bmatrix} x(k) \\ q(k) \end{bmatrix} + \begin{bmatrix} 0 \\ G \end{bmatrix} w(k) + \begin{bmatrix} B \\ 0 \end{bmatrix} u(k) \tag{29}$$

$$z(k) = \begin{bmatrix} -C & H \end{bmatrix} \begin{bmatrix} x(k) \\ q(k) \end{bmatrix} + Jw(k) \tag{30}$$

$$y(k) = \begin{bmatrix} z(k) \\ w(k) \end{bmatrix} = \begin{bmatrix} -C & H \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ q(k) \end{bmatrix} + \begin{bmatrix} J \\ I \end{bmatrix} w(k). \tag{31}$$

The 2 DOF dynamic feedback controller transfer matrix

$$K(z) = \begin{bmatrix} L(z) & M(z) \end{bmatrix} \tag{32}$$

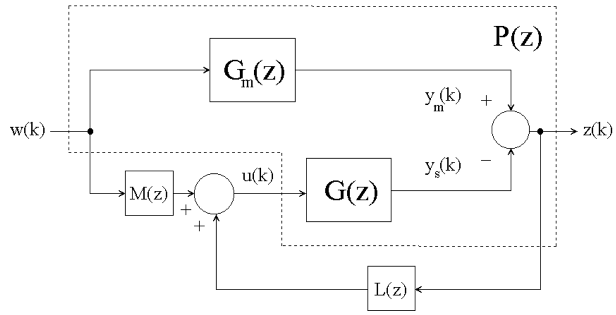

Figure 3: A two degrees of freedom dynamic output feedback controller for the model following control

can be determined from the $\mathcal{H}_\infty$ OCP explained in the previous section, then the controller minimizes the $\mathcal{H}_\infty$ norm of the closed-loop transfer matrix $T_{zw}(z)$. As a result, the following Remark can be given:

Remark: *The $\mathcal{H}_\infty$ MFC problem with the two degrees of freedom dynamic output compensator is equivalent to the $\mathcal{H}_\infty$ OCP in Figure 3 and 4.* ■

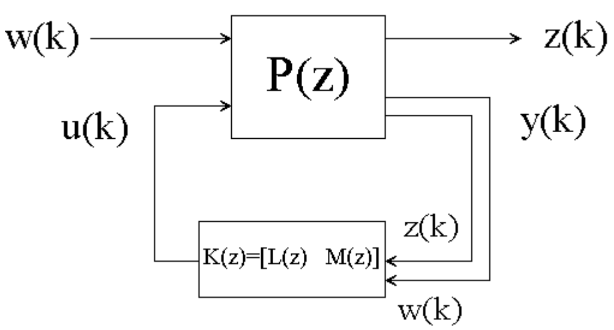

Figure 4: The closed-loop system with the two degrees of freedom dynamic output feedback

However, since Lemma 1 will be used, $(\underline{A}, B_2, C_2)$ must be stabilizable and detectable. Therefore, the following lemma is given for the internal stability of the closed-loop system in the $\mathcal{H}_\infty$ MFC problem:

Lemma 6 *There exists a solution of the $\mathcal{H}_\infty$ MFC problem in Figure 3 if and only if (A, B, C) is stabilizable and detectable, the matrix F is Hurwitz.*

Proof: $(\underline{A}, B_2)$ is stabilizable if and only if there exists a matrix V such that $\underline{A} + B_2V$ is Hurwitz, [17]. When the equations (29), (30) and (31) are used,

$$\begin{aligned} \underline{A} + B_2V &= \begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix} \\ &= \begin{bmatrix} A + BV_1 & BV_2 \\ 0 & F \end{bmatrix} \end{aligned} \tag{33}$$

is written. So (A, B) is stabilizable and the matrix F is Hurwitz.

On the other hand, $(\underline{A}, C_2)$ is detectable if and only if there exists a matrix W such that $\underline{A} + WC_2$ is Hurwitz, [17]. When the equations (29), (30) and (31) are used,

$$\begin{aligned} \underline{A} + WC_2 &= \begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix} + \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \begin{bmatrix} -C & H \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} A - W_{11}C & W_{11}H \\ -W_{21}C & F + W_{21}H \end{bmatrix} \end{aligned} \tag{34}$$

is obtained. Since, the matrix F was taken as Hurwitz above, $W_{21} = 0$ can be written. Therefore,

$$\underline{A} + WC_2 = \begin{bmatrix} A - W_{11}C & W_{11}H \\ 0 & F \end{bmatrix} \tag{35}$$

is found. So, (A, C) is detectable and the matrix F is Hurwitz. Finally, $(\underline{A}, B_2, C_2)$ is stabilizable and detectable if and only if (A, B, C) is stabilizable and detectable and the matrix F is Hurwitz. ■

In order to guarantee the existence of a 2 DOF dynamic feedback controller, i.e. the closed-loop system in Figure 3 is internally stable, throughout the paper, we assume that (A, B, C) of the given system is stabilizable and detectable, the matrix F of the model system is Hurwitz.

4 Main Results

We want to give two lemmas to simplify the synthesis theorems:

Lemma 7 *Suppose (A, C) is detectable in a discrete-time system. For every the matrix $Y > 0$, there always exists a matrix $X > 0$ such that,*

$$N^*(A^*XA - X)N < 0 \tag{36}$$

$$X \geq Y^{-1} \tag{37}$$

where N is a full column rank matrix with

$$ImN = KerC. \tag{38}$$

Proof: From Lemma 5, (A, C) is detectable if and only if there exists a matrix $X_0 > 0$ such that,

$$N^*(A^*X_0A - X_0)N < 0 \tag{39}$$

where $ImN = KerC$. The matrix

$$X = \epsilon X_0 > 0 \tag{40}$$

also satisfies to the LMI (36) for an arbitrary number $\epsilon \in \mathcal{R}^+$. Since, the matrix X_0 is positive definite,

$X_0 = P^* P$ can be written such that the matrix P is nonsingular. When

$$\begin{aligned} \epsilon X_0 \geq Y^{-1} \quad & \Longleftrightarrow \\ \epsilon \geq \lambda_{max}[(P^*)^{-1} Y^{-1} P^{-1}] = \lambda_{max}[(PYP^*)^{-1}] \end{aligned} \quad (41)$$

is written, the proof is completed. ■

We can now present the synthesis theorems on the LMI-based solution of the problem:

Theorem 8 *A two degrees of freedom dynamic feedback controller* $K(z) = \begin{bmatrix} L(z) & M(z) \end{bmatrix}$ *exists for the* $\mathcal{H}_\infty$ *MFC problem if and only if there exists a matrix* $Y > 0$ *such that,*

$$\begin{bmatrix} N_c & 0 \\ 0 & I_m \end{bmatrix}^* \begin{bmatrix} \begin{pmatrix} A & 0 \\ 0 & F \end{pmatrix} Y \begin{pmatrix} A & 0 \\ 0 & F \end{pmatrix}^* - Y & \begin{pmatrix} A & 0 \\ 0 & F \end{pmatrix} Y \begin{pmatrix} -C^* \\ H^* \end{pmatrix} & \begin{pmatrix} 0 \\ G \end{pmatrix} \\ \begin{pmatrix} -C & H \end{pmatrix} Y \begin{pmatrix} A & 0 \\ 0 & F \end{pmatrix}^* & -\gamma I_p + \begin{pmatrix} -C & H \end{pmatrix} Y \begin{pmatrix} -C^* \\ H^* \end{pmatrix} & J \\ \begin{pmatrix} 0 & G^* \end{pmatrix} & J^* & -\gamma I_m \end{bmatrix} \cdot \begin{bmatrix} N_c & 0 \\ 0 & I_m \end{bmatrix} < 0 \quad (42)$$

where N_c is a full rank matrix with

$$ImN_c = Ker \begin{bmatrix} B^* & 0_{m \times n_m} & 0_{m \times p} \end{bmatrix}. \quad (43)$$

Proof: Let use the equations (29), (30) and (31) in Lemma 1,

$$\begin{aligned} ImN_o &= Ker \begin{bmatrix} C_2 & D_{21} \end{bmatrix} \\ &= Ker \begin{bmatrix} -C & H & J \\ 0_{m \times n_s} & 0_{m \times n_m} & I_m \end{bmatrix}. \end{aligned} \quad (44)$$

Thus

$$N_o = \begin{bmatrix} N \\ 0_{m \times r} \end{bmatrix} \quad (45)$$

is written where

$$ImN = Ker \begin{bmatrix} -C & H \end{bmatrix} \quad (46)$$

and

$$r = dim \left(Ker \begin{bmatrix} -C & H \end{bmatrix} \right). \quad (47)$$

So, the LMI (11) can be derived as follows:

$$\begin{bmatrix} N & 0 \\ 0_{m \times r} & 0 \\ 0 & I_p \end{bmatrix}^* \begin{bmatrix} \underline{A}^* X \underline{A} - X & \underline{A}^* X B_1 & C_1^* \\ B_1^* X \underline{A} & -\gamma I_m + B_1^* X B_1 & D_{11}^* \\ C_1 & D_{11} & -\gamma I_p \end{bmatrix} \cdot \begin{bmatrix} N & 0 \\ 0_{m \times r} & 0 \\ 0 & I_p \end{bmatrix} < 0 \quad (48)$$

and

$$\begin{bmatrix} N & 0 \\ 0 & I_p \end{bmatrix}^* \begin{bmatrix} \underline{A}^* X \underline{A} - X & C_1^* \\ C_1 & -\gamma I_p \end{bmatrix} \begin{bmatrix} N & 0 \\ 0 & I_p \end{bmatrix} < 0 \quad (49)$$

or

$$N^* \left(\begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix}^* X \begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix} - X \right) N < 0. \quad (50)$$

From the Schur complement argument, the inequality (13) is reduced to

$$X \geq Y^{-1}. \quad (51)$$

Because of Lemma 6, $\left(\begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix}, \begin{bmatrix} -C & H \end{bmatrix} \right)$ is easily seen as detectable. Thus, there exists a matrix $X > 0$ such that the inequalities (50) and (51) are both satisfied according to Lemma 7. The LMI (42) is also obtained when the equations (29), (30) and (31) are used in (12). ■

On the other hand, if the given system is stable, a theorem can be obtained about the beginning value of γ iteration.

Theorem 9 *In the discrete-time* $\mathcal{H}_\infty$ *MFC problem, if the given system is stable, we have* $\|T_{zw}(z)\|_\infty < \|G_m(z)\|_\infty$.

Proof: From Dual Bounded Real Lemma, a matrix $Y > 0$ exists such that,

$$\begin{bmatrix} \begin{pmatrix} A & 0 \\ 0 & F \end{pmatrix} Y \begin{pmatrix} A & 0 \\ 0 & F \end{pmatrix}^* - Y & \begin{pmatrix} A & 0 \\ 0 & F \end{pmatrix} Y \begin{pmatrix} -C^* \\ H^* \end{pmatrix} & \begin{pmatrix} 0 \\ G \end{pmatrix} \\ \begin{pmatrix} -C & H \end{pmatrix} Y \begin{pmatrix} A & 0 \\ 0 & F \end{pmatrix}^* & -\gamma I_p + \begin{pmatrix} -C & H \end{pmatrix} Y \begin{pmatrix} -C^* \\ H^* \end{pmatrix} & J \\ \begin{pmatrix} 0 & G^* \end{pmatrix} & J^* & -\gamma I_m \end{bmatrix} < 0 \quad (52)$$

if and only if the matrix $\begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix}$ is Hurwitz and

$$\left\| J + \begin{pmatrix} -C & H \end{pmatrix} \begin{pmatrix} zI - A & 0 \\ 0 & zI - F \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ G \end{pmatrix} \right\|_\infty < \gamma. \quad (53)$$

(53) is equivalent to

$$\|J + H(zI - F)^{-1} G\|_\infty = \|G_m(z)\|_\infty < \gamma. \quad (54)$$

Therefore, for γ which is greater than $\mathcal{H}_\infty$ norm of the model system $G_m(z)$, (52) and so (42) is satisfied. That is $\|T_{zw}(z)\|_\infty < \|G_m(z)\|_\infty$. ■

4.1 Controller Construction

When the controller design procedure in [12] and the Theorem 8 are used, a construction procedure of the $\mathcal{H}_\infty$ MFC problem with the 2 DOF dynamic feedback are obtained as follows. Furthermore, some optimization softwares [13] should be used to solve LMIs:

Step 1: Find a solution matrix $Y > 0$ of the LMI (42) for the minimum of γ. (If the given system is stable, calculate $\|G_m(z)\|_\infty$ for the beginning value of γ iteration because of Theorem 9.)

Step 2: Obtain a matrix $X_0 > 0$ such that,

$$N^* \left(\begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix}^* X_0 \begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix} - X_0 \right) N < 0 \tag{55}$$

where

$$ImN = Ker \begin{bmatrix} -C & H \end{bmatrix}. \tag{56}$$

Step 3: Find a nonsingular matrix P from $X_0 = P^* P$.

Step 4: Choose a number $\epsilon \in \mathcal{R}^+$ such that,

$$\epsilon \geq \lambda_{max}[(PYP^*)^{-1}]. \tag{57}$$

Then we have $X = \epsilon X_0$.

Step 5: We can construct the matrix $X_{cl} > 0$ by finding a matrix $X_2 \in \mathcal{R}^{n \times n_K}$ such that,

$$X_2 X_2^* = X - Y^{-1} \geq 0 \tag{58}$$

where $n_K \geq n = n_s + n_m$. Then

$$X_{cl} = \begin{bmatrix} X & X_2 \\ X_2^* & I \end{bmatrix}. \tag{59}$$

Step 6: Obtain the following matrices,

$$P = \begin{bmatrix} 0 & 0 & I_{n_K} & 0 \\ B^* & 0_{m \times n_m} & 0 & 0_{m \times (n+n_K+m+p)} \end{bmatrix} \tag{60}$$

$$Q = \begin{bmatrix} 0 & 0 & 0 & I_{n_K} & 0 & 0 \\ 0 & -C & H & 0 & J & 0_p \\ 0_{m \times (n+n_K)} & 0 & 0 & 0 & I_m & 0 \end{bmatrix} \tag{61}$$

$$H_{X_{cl}} = \begin{bmatrix} -X_{cl}^{-1} & \begin{pmatrix} A & 0 & 0 \\ 0 & F & 0 \\ 0 & 0 & 0_{n_K} \end{pmatrix} & \begin{pmatrix} 0_{n_s \times m} \\ G \\ 0_{n_K \times m} \end{pmatrix} & \begin{pmatrix} 0 \\ -C^* \\ H^* \\ 0_{n_K \times p} \end{pmatrix} \\ (\ldots)^* & -X_{cl} & 0 & 0 \\ (\ldots)^* & 0 & -\gamma I_m & J^* \\ 0 & (\ldots)^* & J & -\gamma I_p \end{bmatrix}. \tag{62}$$

Step 7: Find a solution

$$\Omega = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} \tag{63}$$

to the LMI

$$H_{X_{cl}} + Q^* \Omega^* P + P^* \Omega Q < 0. \tag{64}$$

Step 8: Obtain the 2 DOF dynamic feedback control law as

$$\begin{aligned} K(z) &= \begin{bmatrix} L(z) & M(z) \end{bmatrix} \\ &= C_K(zI - A_K)^{-1} B_K + D_K. \end{aligned} \tag{65}$$

5 Design Example

The benchmark thermal process (PT-326) [11] has a dynamic behavior. It is the first order process with time-delay that is similar to many industrials plant, such as steamboilers, furnaces and HVAC (Heating, Ventilating and Air-Conditioning) systems:

$$G(s) = \frac{K e^{-sT_d}}{\tau s + 1} \tag{66}$$

where K is the static gain, τ is the time constant and T_d is the time-delay. We take that

$$K = 0.734 \tag{67}$$
$$T_d = 200 \quad ms \tag{68}$$
$$\tau = 600 \quad ms. \tag{69}$$

If the $\mathcal{Z}$ transformation of (66) is found with the sampling period $T_s = 66$ ms, the following relation is obtained:

$$G(z) = \frac{0.076}{z^3(z - 0.896)}. \tag{70}$$

Moreover, we take the model system which is really faster and has no error in the unit step response, as follows,

$$G_m(z) = \frac{0.168}{z^3(z - 0.832)}. \tag{71}$$

The state-space equations of $G(z)$ are obtained as

$$x(k+1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0.896 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(k) \tag{72}$$

$$y_s(k) = \begin{bmatrix} 0.076 & 0 & 0 & 0 \end{bmatrix} x(k). \tag{73}$$

The state-space equations of $G_m(z)$ are obtained as

$$q(k+1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0.832 \end{bmatrix} q(k) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} w(k) \tag{74}$$

$$y_m(k) = \begin{bmatrix} 0.168 & 0 & 0 & 0 \end{bmatrix} q(k). \tag{75}$$

Since (70) is stable, we can start γ iteration from $\|G_m(z)\|_\infty = 1$. When we search for a controller in the discrete-time $\mathcal{H}_\infty$ MFC problem, γ_{min} and the dynamic output feedback controller are obtained as follows:

$$\begin{aligned}\gamma_{min} &\cong 0.00008\\ L(z) &\cong \frac{3.2478z^8 - 2.8546z^7 - 0.0043z^6 + 0.0001z^5 - 0.0009z^4}{z^8 - 0.2527z^7 - 0.0714z^6 - 0.0767z^5 - 0.2205z^4}\\ M(z) &\cong \frac{2.1973z^8 - 0.6954z^7 - 0.2381z^6 - 0.2261z^5 - 0.5216z^4}{z^8 - 0.2527z^7 - 0.0714z^6 - 0.0767z^5 - 0.2205z^4}.\end{aligned}$$

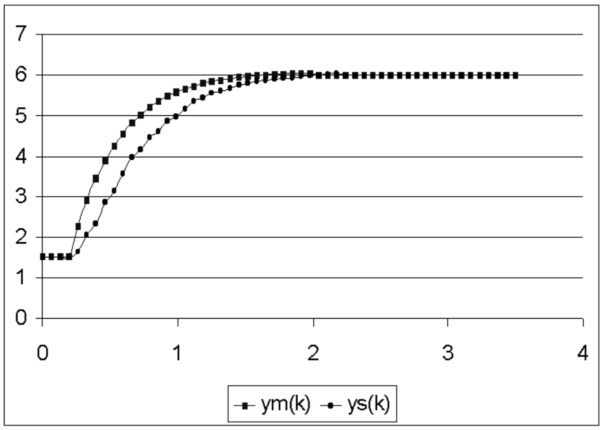

Figure 5: The response of the benchmark thermal process with the 2 DOF dynamic compensator is compared with the response of the model system output.

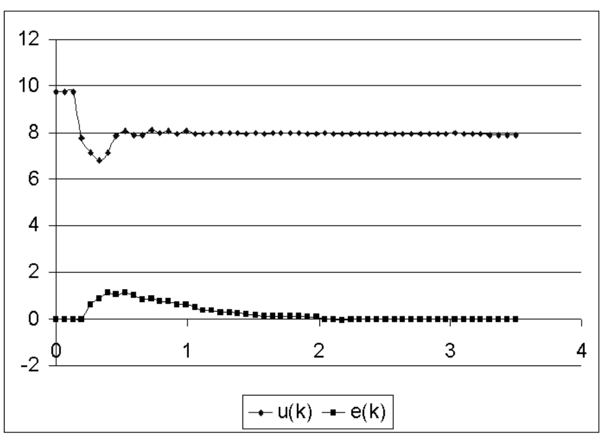

Figure 6: The responses of the error and control signals.

Figure 5 illustrates the unit step responses of the transfer matrix of the given system with the 2 DOF dynamic compensator and the transfer matrix $G_m(z)$ of the model system and Figure 6 shows the error signal $e(k)$ and the control signal $u(k)$. They are well matched over γ_{min}. Since there is a limitation $(10V)$ on the control input, the unit step responses has some error. However a system which have a dead time, is controlled by using $\mathcal{H}_\infty$ MFC.

6 Conclusions

In this paper, a LMI-based solution of the $\mathcal{H}_\infty$ MFC problem is presented. The solvability conditions of the problem are derived. It is observed that only one linear matrix inequality determines the solution for the $\mathcal{H}_\infty$ MFC problem. Moreover, if the given system is stable, a theorem for the beginning value of γ iteration is found by using the synthesis theorem. The $\mathcal{H}_\infty$ norm of the closed-loop system is assured less than the $\mathcal{H}_\infty$ norm of the desired model system. The effectiveness of the presented methodology is validated by a simulation study. A generic industrial system being the first order process with a dead-time is modeled and the presented 2 DOF dynamic feedback controller is applied. The internal stability of the closed-loop system in the $\mathcal{H}_\infty$ MFC approach is satisfied assuring the error convergence to zero with a limited control input.

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