On a New Symmetry Method, with Application to the Nonlinear Heat Equation

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Abstract: - We determine the group invariant solutions of the nonlinear heat equation though the linear case, using a relation that exists between the two. This is not new, but there has always been those solutions that proved difficult to evaluate through existing symmetry techniques, for both linear and nonlinear cases. We introduce what we call modified Lie symmetries to address these difficulties.

Key-Words: - Group method; Heat diffusion; Nonlinear problems; Modified symmetries.

1 Introduction

The nonlinear heat equation

$$w_{xx} - w_t = w_x^2, \tag{1}$$

reduces to the linear form

$$u_{xx} - u_t = 0, \tag{2}$$

through the transformation

$$u = e^w - 1, \tag{3}$$

or

$$w = \ln|1+u|. \tag{4}$$

There is only one known solution of (2), obtained through symmetry methods, that has practical value. This means it is the only one that will make sense for (1), after being transformed through (4). But the reality is there are many instances in applications that cannot be explained through this result. It is our objective, therefore, to unravel this mystery, and we do so through what we call modified symmetries.

Section 2 is on our theoretical basis of our modification of Lie's theory. We start first outline Lie's theory, in brief, this in subsection 2.1. Subsection 2.2 is on the challenges the pure Lie approach is faced with, as a precursor to modified symmetries, which are discussed in subsection 2.3. Section 3 is on the application of the modified symmetries to (2), and to (1) through the transformation (4). Section 4 is on the solutions, the group invariant solutions. This results in new solutions for (1), which compares favourably with what is possible through techniques, an improvement on Lie's theory.

2 The theoretical basis

2.1 The basics of Lie symmetry analysis

Lie symmetry theoretical methods, a theory first introduced by Marius Sophus Lie (1842 – 1899), a Norwegian mathematician, through his now famous 1881 paper [1], seeks to determine solutions to differential equations, by establishing their symmetries, a set

$$G = \{G_1, G_2, G_3, \ldots, G_n\}, \tag{5}$$

following from infinitesimals $\xi = \{\xi^1, \xi^2, \xi^3, \cdots, \xi^m\}$. It has to satisfy the group properties;

1. Closure property: If $G_i \in G$ and $G_j \in G$ then $G_i + G_j \in G$.
2. Identity: There exists $G_0 \in G$ such that $G_0 + G_i = G_i + G_0 = G_i \in G$.

3. Inverse: For every $G_i \in G$, there exists $G_{-i} \in G$ such that $G_{-i} + G_i = G_i + G_{-i} = G_0$.
4. Associativity: If G_i , $G_j, G_k \in G$, then $G_i + \left(G_j + G_k\right) = (G_i + G_j) + G_k \in G$.

Added to the group properties, the set may have to also be an algebra L. That is, the Jacobian multiplication satisfy and$\left[G_i, G_j\right] \in G$ for $G_i, G_j \in G$.

From the algebra L then follows the sub-algebras $L^{\{(1)\}} \subset L^{\{(2)\}} \subset \cdots \subset L^{\{(r-1)\}} \subset L^{\{(r)\}} \subset L$. That is, the entries in the sequence are all ideals, so that the algebra L is solvable.

2.2 The problem with the pure Lie theory approach

The symmetries of (2) are determined using the generator

$$G = \xi^0 \frac{\partial}{\partial t} + \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial u}, \tag{6}$$

where the infinitesimals ξ^0, ξ^1, ξ^2 depend on t, x, u, and they are

$$G_1 = \frac{\partial}{\partial x}, \tag{7}$$

$$G_2 = \frac{\partial}{\partial t}, \tag{8}$$

$$G_3 = x \frac{\partial}{\partial x} + 2t \frac{\partial}{\partial t}, \tag{9}$$

$$G_4 = ut \frac{\partial}{\partial t} + u^2 \frac{\partial}{\partial u}, \tag{10}$$

$$G_5 = t \frac{\partial}{\partial x} - \frac{x\,u}{2} \frac{\partial}{\partial u}, \tag{11}$$

$$G_6 = u \frac{\partial}{\partial u}. \tag{12}$$

The known solution follows from (4), and has the form

$$u = \left(C_1\ \frac{x}{t} + C_2\right) \frac{e^{-\frac{x^2}{4t}}}{\sqrt{t}}. \tag{13}$$

where C_1 and C_2 are integration constants. The transformation (4) then leads to

$$w = \ln \left| 1 + \left(C_1\ \frac{x}{t} + C_2\right) \frac{e^{-\frac{x^2}{4t}}}{\sqrt{t}} \right|. \tag{14}$$

The existence of other symmetries, other than the one in (4), suggests the existence of other solutions, other than the one presented in (14). Part of the reason why the other solution are obscure is that they are not easy to evaluate, in that the expressions are impossible to integrate exactly. This can be observed in the analysis [4], by Bluman and Cole.

Lie's theory has now been developed to new levels, as can be attested in [5], [6], [7], [8], [9] and [10]. Ours is just but a slight modification.

These modified symmetries that we are introducing, are simply a device for overcoming un-integrable integrals by reducing them to first principles, so that the usual integration techniques can be replaced with first principles, involving and ϵ and and δ limit theorems, and it works.

2.3 Modified symmetries

The notation we adopt for modified symmetries is not much different from the one we used for the regular Lie symmetries. For the six heat equation symmetries, we have $\tilde{G}_1, \tilde{G}_2, \tilde{G}_3, \tilde{G}_4, \tilde{G}_5$ and $\tilde{G}_6$, as the symbols representing them.

The generator for modified symmetries, too, is similar to the one used for the regular Lie symmetries. That is,

$$\tilde{G} = \tilde{\xi}^0 \frac{\partial}{\partial t} + \tilde{\xi}^1 \frac{\partial}{\partial x} + \tilde{\xi}^2 \frac{\partial}{\partial u}. \tag{15}$$

The only difference is that its infinitesimals $\tilde{\xi}^0, \tilde{\xi}^1, \tilde{\xi}^2$ depend on ω, t, x, u. The quantity ω is an infinitesimal parameter, such that $\tilde{\xi}^0 = \xi^0$, $\tilde{\xi}^1 = \xi^1, \tilde{\xi}^2 = \xi^2$ when $\omega = 0$.

The idea is that when, say ξ^0, is considered as a centre of an open neighbourhood with the radius ω, then the quantity $\tilde{\xi}^0$ is a point in this neighbourhood. This argument extends to both $\xi^1$and ξ^2.

To generate the neighbourhood we use the Lie operator £. The result is we have the points being on a parallel surface, this emanating from how the operator is defined.

The application of the operator requires

$$£\left(\tilde{\xi}^k_{tt}\right) = 0, £(\tilde{\xi}^k_{xx}) = 0 \tag{16}$$

or

$$£\left(\tilde{\xi}^k_{uu}\right) = 0,$$

subject to $\tilde{\xi}^k_{xx} = 0$, and the others. An application of a simple group classification on the middle expression leads to

$$\tilde{\xi}^k_{xx} = \omega^2 \tilde{\xi}^k, \tag{17}$$

and this can similarly be applied to the other expressions.

2.3.1 Euler's error

Texts on differential equations give the solution of (17) as

$$\tilde{\xi}^k = a\cos(\omega x) + b\sin(\omega x), \tag{18}$$

which is not exact mathematically. It follows from the formula $\tilde{\xi}^k = e^{\lambda x}$, introduced by Leonhard Euler (1707—1783), a Swiss mathematician. He did not solve the equation: He simply guessed the formula for solving it, and it is said to have been an inspired guess. Other texts phrase it as a motivated guess.

The error Euler's formula, and the solutions that result from it, is very difficult to detect in most practical situations, which is probably the reason why it continues to survive, going for three centuries now. The act of applying the formula essentially obscures it from view.

The correct solution, which we obtained through quadrature, is

$$\tilde{\xi}^k = A\,\frac{cos(i\,\omega x)}{i\,\omega} + B\,\frac{sin(i\,\omega x)}{i\,\omega}, \tag{19}$$

expressed here in the terse form

$$\tilde{\xi}^k = a\,\frac{\sin\left(i\,\omega\left[x + \frac{b}{a}\right]\right)}{i\,\omega}, \tag{20}$$

where ω is the infinitesimal parameter and i a complex number, a, b, A and B the integration parameters.

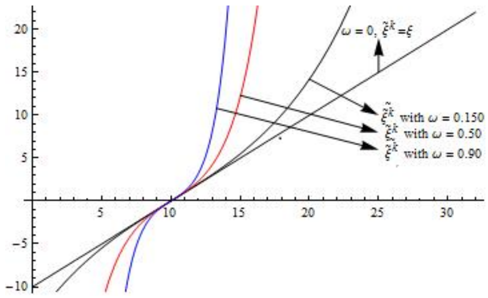

Fig. 1. A plot of (20) a=1, b=-10, with the parameter ω assuming different values. The purpose and essence of this, is to demonstrate that as the parameter reduce to zero, the other curves converge to the straight line, where $\tilde{\xi}^0 = \xi^0$, $\tilde{\xi}^1 = \xi^1, \tilde{\xi}^2 = \xi^2$.

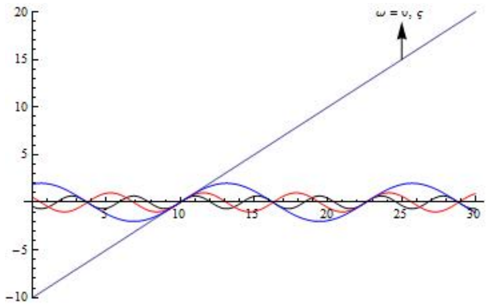

Fig. 2. A plot of (18). It is intended to demonstrate why we cannot use (18) for modified symmetries. The convergence observed in Figure 1 is absent here.

3 Modified symmetries of (2)

3.1 The symmetries

The classical Lie analysis of the heat equation is contained in many texts, and does contain a condition suitable for generating modified symmetries, which is

$$\tilde{\xi}^0_{xx} = 0. \tag{21}$$

This result is equation (4.51a) in the book by Bluman and Kumei [2], and (2.54b) in the one by Olver [3]. Olver's analysis also leads to $\tilde{\xi}^0_{uu} = 0$ and $\tilde{\xi}^1_{uu} = 0$, given in his (2.54a) and (2.54g), respectively.

A solution (20) for

$$\tilde{\xi}^0_{xx} = \omega^2 \tilde{\xi}^0, \tag{22}$$

resulting from (21), leads to the modified symmetries

$$\tilde{G}_1 = -\frac{2\,\phi e^{\omega^2 t}}{\omega^4(\omega^2+1)} \sin\left(\frac{\omega x}{i}\right)\frac{\partial}{\partial t} + \frac{i\,\phi e^{\omega^2 t}}{\omega^3(\omega^2+1)} \cos\left(\frac{\omega x}{i}\right)\frac{\partial}{\partial x} - \frac{\phi e^{\omega^2 t}}{2} sin\left(\frac{\omega x}{i}\right) u\, \frac{\partial}{\partial u}, \tag{23}$$

$$\tilde{G}_2 = \frac{2e^{\omega^2 t}}{\omega^4(\omega^2-1)} \cos\left(\frac{\omega x}{i}\right)\frac{\partial}{\partial t} + \frac{ie^{\omega^2 t}}{\omega^3(\omega^2-1)} \sin\left(\frac{\omega x}{i}\right)\frac{\partial}{\partial x} - \frac{e^{\omega^2 t}}{2} \cos\left(\frac{\omega x}{i}\right) u\, \frac{\partial}{\partial u}, \tag{24}$$

$$\tilde{G}_3 = -2\,\phi\, t \sin\left(\frac{\omega x}{i}\right)\frac{\partial}{\partial t} + \frac{i\,\phi}{\omega} \cos\left(\frac{\omega x}{i}\right)\frac{\partial}{\partial x}, \tag{25}$$

$$\tilde{G}_4 = 2\,t \cos\left(\frac{\omega x}{i}\right)\frac{\partial}{\partial t} + \frac{i}{\omega} \sin\left(\frac{\omega x}{i}\right)\frac{\partial}{\partial x}, \tag{26}$$

$$\tilde{G}_5 = 2\,\phi\, e^{-t} \sin\left(\frac{\omega x}{i}\right)\frac{\partial}{\partial t} + \frac{i\,\phi}{\omega} e^{-t} \cos\left(\frac{\omega x}{i}\right)\frac{\partial}{\partial x}, \tag{27}$$

$$\tilde{G}_6 = 2\,e^{t} \cos\left(\frac{\omega x}{i}\right)\frac{\partial}{\partial t} + \frac{i}{\omega} e^{t} \sin\left(\frac{\omega x}{i}\right)\frac{\partial}{\partial x}, \tag{28}$$

$$\tilde{G}_7 = \frac{\partial}{\partial t}, \tag{29}$$

$$\tilde{G}_8 = u\frac{\partial}{\partial u}. \tag{30}$$

4 Group invariant solutions

The characteristic equations that arise from the symmetry $\tilde{G}_1$ is

$$\frac{dt}{-\frac{2\,\phi e^{\omega^2 t}}{\omega^4(\omega^2+1)} \sin\left(\frac{\omega x}{i}\right)} = \frac{dx}{\frac{i\,\phi e^{\omega^2 t}}{\omega^3(\omega^2+1)} \cos\left(\frac{\omega x}{i}\right)} = \frac{du}{-\frac{\phi e^{\omega^2 t}}{2} sin\left(\frac{\omega x}{i}\right) u}. \tag{32}$$

That is,

$$\frac{dt}{-\frac{2\,\phi e^{\omega^2 t}}{\omega^4(\omega^2+1)} \sin\left(\frac{\omega x}{i}\right)} = \frac{dx}{\frac{i\,\phi e^{\omega^2 t}}{\omega^3(\omega^2+1)} \cos\left(\frac{\omega x}{i}\right)} = \frac{du}{-\frac{\phi e^{\omega^2 t}}{2} sin\left(\frac{\omega x}{i}\right) u}. \tag{33}$$

The first pairing gives

$$\frac{dt}{-\frac{2\,\phi e^{\omega^2 t}}{\omega^4(\omega^2+1)} \sin\left(\frac{\omega x}{i}\right)} = \frac{dx}{\frac{i\,\phi e^{\omega^2 t}}{\omega^3(\omega^2+1)} \cos\left(\frac{\omega x}{i}\right)}, \tag{34}$$

so that

$$\frac{dt}{2}=\frac{-\frac{1}{i\,\omega}\sin(\frac{\omega x}{i})dx}{\cos(\frac{\omega x}{i})}. \tag{35}$$

That is,

$$\frac{t}{2}=\frac{-\ln(\cos(\frac{\omega x}{i}))}{\omega^2}+C_1. \tag{36}$$

The second pairing gives

$$\frac{dt}{-\frac{2\,\phi e^{\omega^2 t}}{\omega^4(\omega^2+1)}\sin\,(\frac{\omega x}{i})}=\frac{du}{-\frac{\phi e^{\omega^2 t}}{2}sin(\frac{\omega x}{i})\,u}, \tag{37}$$

so that

$$\frac{\omega^4 dt}{4}=\frac{du}{\mathrm{u}}. \tag{38}$$

That is,

$$\frac{\omega^4 t}{4}=\ln|u|+C_2(C_1)\,. \tag{39}$$

A pair of solutions result from these. The first is

$$u=F_1+\frac{A}{\mathrm{sqrt}\,(2)}e^{-\frac{x^2}{2(\mathrm{x}^2-\mathrm{t}^2)}}, \tag{40}$$

Where F_1 and A are integration constants. The solution for the nonlinear heat equation (1) is then

$$w=\ln|1+F_1+\frac{A}{\mathrm{sqrt}\,(2)}e^{-\frac{x^2}{2(\mathrm{x}^2-\mathrm{t}^2)}}|\,. \tag{41}$$

It is plotted in Figure 1.

The second solution follows from

$$\tilde{G}_4=2\,t\cos\,(\frac{\omega x}{i})\frac{\partial}{\partial t}+\frac{i}{\omega}\sin\,(\frac{\omega x}{i})\frac{\partial}{\partial x}, \tag{42}$$

so that

$$u=F_1+\frac{A}{\mathrm{sqrt}\,(2)}e^{\frac{x^2}{2(\mathrm{x}^2-\mathrm{t}^2)}}, \tag{43}$$

and

$$w=\ln|1+F_1+\frac{A}{\mathrm{sqrt}\,(2)}e^{\frac{x^2}{2(\mathrm{x}^2-\mathrm{t}^2)}}|\,. \tag{44}$$

It is plotted in Figure 2.

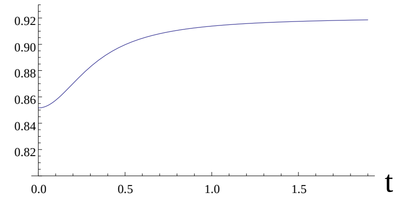

Fig. 3. A plot of the solution in (43), obtained through the techniques proposed in this contribution, and compares favourably with Figure 4.

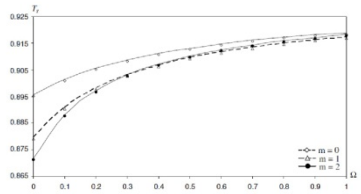

Fig. 4. A plot from Figure 5 of the work by Malek and Halel [10].

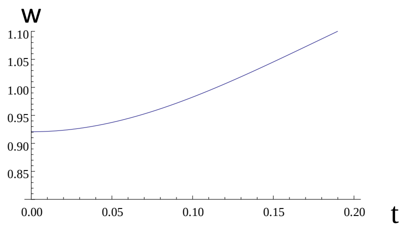

Fig. 5. A plot of the solution in (44), obtained through the techniques proposed in this contribution, and compares favorably with Figure 6.

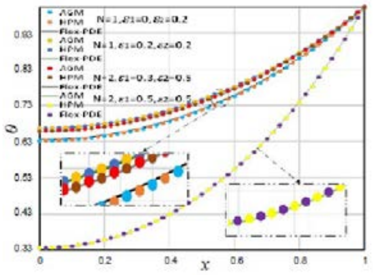

Fig. 6. A plot from Figure 2 of the work by Gholinia et. al. [11].

5 Discussion and conclusion

The primary objective of the study was to introduce our own version of Lie symmetries, which we call modified Lie symmetries, and do so by determining new solutions for the nonlinear heat equations. This we did, and did so using symmetries whose reduction to the regular cases does not yield any useful solutions. Our symmetries rendered these other symmetries usable, and we were able to deduce two new close solutions, plotted in Figures 3 and 5, and compare favourably with practical results, plotted in Figures 4 and 6.

Further applications of the modified symmetries can be found in (13), (14) and (15).

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