Continuous Maps in Fuzzy Relations

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Abstract: In this paper we generalize our most recent results that are related to the algorithm that has been developed to automatically derive a fuzzy functional or a fuzzy multivalued dependency from a given set of fuzzy functional and fuzzy multivalued dependencies. Fuzzy dependencies are considered as fuzzy formulas. The first result states that a two-element fuzzy relation instance actively satisfies a fuzzy multivalued dependency if and only if the tuples of the instance are conformant on some known set of attributes with degree of conformance larger than some known constant, and the corresponding fuzzy formula is valid in appropriate interpretations. The second result states that a fuzzy functional or a fuzzy multivalued dependency follows from a set of fuzzy functional and fuzzy multivalued dependencies in two-element fuzzy relation instances if and only if the corresponding fuzzy formula is a logical consequence of the corresponding set of fuzzy formulas. Our earlier research in this direction consisted in an application of some individual fuzzy implication operator, such as Yager, Reichenbach, Kleene-Dienes fuzzy implication operator. The main purpose of this paper is to prove that the aforementioned results remain valid for a wider class of fuzzy implication operators, in particular for the family of f-generated fuzzy implication operators.

Key–Words: Strictly decreasing continuous functions, f-generated implications, f-generators, fuzzy relation instances, fuzzy dependencies

1 Introduction

In [6], the authors offered an algorithm that automatically proves that some fuzzy functional or fuzzy multivalued dependency follows from a set of fuzzy functional and fuzzy multivalued dependencies. The idea behind the method they presented, lies in the fact that fuzzy functional and fuzzy multivalued dependencies are considered as fuzzy formulas. In particular, they proved the following theorem.

Theorem A. [6, Cor. 8] *Let C be a set of fuzzy functional and fuzzy multivalued dependencies on some universal set of attributes U. Suppose that c is some fuzzy functional or fuzzy multivalued dependency on U. Denote by $C^{'}$ resp. $c^{'}$ the set of fuzzy formulas resp. the fuzzy formula associated to C resp. c. Then, the following two conditions are equivalent:*

(*a*) *Any fuzzy relation instance on scheme $R(U)$ which satisfies all dependencies in C, satisfies the dependency c.*

(*b*) $i_{r,\beta}\left(c^{'}\right) > \frac{1}{2}$ *for every* $i_{r,\beta}$ *such that* $i_{r,\beta}(\mathcal{K}) > \frac{1}{2}$ *for all* $\mathcal{K} \in C^{'}$.

Here, $i_{r,\beta}$ denotes a valuation joined to r and β, where r is a two-element fuzzy relation instance on $R(U)$, and $\beta \in [0,1]$.

Let us explain this into more details.

$R(U) = R(A_1, A_2, ..., A_n)$ is a scheme on domains D_1, D_2,..., D_n. U is the set of all attributes A_1, A_2,..., A_n on D_1, D_2,..., D_n, respectively, i.e., U is the universal set of attributes. We assume that the domain D_i of A_i is a finite set for all $i \in \{1, 2, ..., n\}$.

Fuzzy relation instance r on $R(U)$ is a subset of the cross product $2^{D_1} \times 2^{D_2} \times ... \times 2^{D_n}$.

Hence, if $t \in r$, t is of the form $(d_1, d_2, ..., d_n)$, where $d_i \subseteq D_i$ for $i \in \{1, 2, ..., n\}$. Furthermore, we consider d_i as the value of A_i on tuple t.

Let $X \subseteq U$ and $Y \subseteq U$.

Fuzzy relation instance r is said to satisfy the fuzzy functional dependency $X \overset{\theta}{\to}_F Y$ if for every pair of tuples t_1 and t_2 in r,

$$\varphi(Y[t_1, t_2]) \geq \min(\theta, \varphi(X[t_1, t_2])).$$

Fuzzy relation instance r is said to satisfy the fuzzy multivalued dependency $X \to\overset{\theta}{\to}_F Y$ if for every pair of tuples t_1 and t_2 in r, there exists a tuple t_3 in r such that

$$\varphi(X[t_3, t_1]) \geq \min(\theta, \varphi(X[t_1, t_2])),$$
$$\varphi(Y[t_3, t_1]) \geq \min(\theta, \varphi(X[t_1, t_2])),$$
$$\varphi(Z[t_3, t_2]) \geq \min(\theta, \varphi(X[t_1, t_2])),$$

where $Z = U \setminus (X \cup Y)$.

Here, $\theta \in [0, 1]$ denotes the linguistic strength of the dependency (see, [17]). If $\theta = 1$, we omit to write it in the dependency notation.

Furthermore,

$$\varphi(X[t_1, t_2]) = \min_{A_i \in X} \{\varphi(A_i[t_1, t_2])\}$$

denotes the conformance of the attribute set X on tuples t_1 and t_2, where

$$\varphi(A_i[t_1, t_2]) = \min\left\{ \min_{x \in d_1}\left\{\max_{y \in d_2}\{s_i(x, y)\}\right\}, \min_{x \in d_2}\left\{\max_{y \in d_1}\{s_i(x, y)\}\right\}\right\},$$

denotes the conformance of the attribute A_i on t_1 and t_2.

Moreover, d_1 resp. d_2 denotes the value of A_i on t_1 resp. t_2.

$s_i : D_i \times D_i \to [0, 1]$ is a similarity relation on D_i. The following conditions determine s_i:

$$s_i(x, x) = 1,$$
$$s_i(x, y) = s_i(y, x),$$
$$s_i(x, z) \geq \max_{q \in D_i}(\min(s_i(x, q), s_i(q, z))),$$

where $x, y, z \in D_i$.

In order to associate fuzzy formulas to fuzzy functional and fuzzy multivalued dependencies, the authors in [6] first consider attributes as fuzzy formulas by introducing a valuation.

If $r = \{t_1, t_2\}$ is a two-element fuzzy relation instance on $R(A_1, A_2, ..., A_n)$, and $\beta \in [0, 1]$, a valuation joined to r and β is a mapping $i_{r,\beta} : \{A_1, A_2, ..., A_n\} \to [0, 1]$, such that

$$i_{r,\beta}(A_k) > \frac{1}{2} \quad \text{if} \quad \varphi(A_k[t_1, t_2]) \geq \beta,$$
$$i_{r,\beta}(A_k) \leq \frac{1}{2} \quad \text{if} \quad \varphi(A_k[t_1, t_2]) < \beta,$$

$k \in \{1, 2, ..., n\}$.

In this way attributes become fuzzy formulas with respect to $i_{r,\beta}$.

Apart from it, fuzzy operators: conjunction, disjunction, and implication are chosen and fixed.

Requiring that $i_{r,\beta}(A_i \wedge A_j)$, $i_{r,\beta}(A_i \vee A_j)$, and $i_{r,\beta}(A_i \Rightarrow A_j)$ structurally agree with fixed fuzzy operators, $A_i \wedge A_j$, $A_i \vee A_j$, and $A_i \Rightarrow A_j$ become fuzzy formulas with respect to $i_{r,\beta}$.
Consequently, $(\wedge_{A \in X} A) \Rightarrow (\wedge_{B \in Y} B)$, $(\wedge_{A \in X} A) \Rightarrow ((\wedge_{B \in Y} B) \vee (\wedge_{C \in Z} C))$, etc., where X, Y, $Z \subseteq U$, become fuzzy formulas with respect to $i_{r,\beta}$ as well.

Accordingly, for example, the fuzzy formula (with respect to $i_{r,\beta}$)

$$(\wedge_{A \in X} A) \Rightarrow (\wedge_{B \in Y} B)$$

is joined to fuzzy functional dependency $X \xrightarrow{\theta_1}_F Y$.

Similarly, the fuzzy formula (with respect to $i_{r,\beta}$)

$$(\wedge_{A \in X} A) \Rightarrow ((\wedge_{B \in Y} B) \vee (\wedge_{C \in Z} C)),$$

where $Z = U \setminus (X \cup Y)$, is joined to fuzzy multivalued dependency $X \to\overset{\theta_2}{\to}_F Y$.

In order to illustrate what we said at the beginning of the section, consider the following example.

Example 1. If the fuzzy functional and the fuzzy multivalued dependencies:

$$A_1A_2A_4 \to\overset{\theta_1}{\to}_F A_3A_5A_7A_8,$$
$$A_1A_2A_4 \to\overset{\theta_2}{\to}_F A_5A_6A_8,$$
$$A_1A_2A_4A_5A_8 \xrightarrow{\theta_3}_F A_3A_5A_6A_7$$

hold true, where $U = \{A_i \mid 1 \leq i \leq 8\}$ is the universal set of attributes, then the fuzzy functional dependency $A_1A_2A_4 \overset{\min(\theta_1,\theta_2,\theta_3)}{\to}_F A_5A_6A_7$ holds also true.

Proof. I (applying inference rules IR1–IR17, see next section)

We have:

1) $A_1A_2A_4 \to\xrightarrow{\theta_1}_F A_3A_5A_7A_8$ (input)

2) $A_1A_2A_4 \to\xrightarrow{\theta_1}_F A_1A_2A_3A_4A_5A_7A_8$ (IR7, 1) augment with $A_1A_2A_4$)

3) $A_1A_2A_4 \to\xrightarrow{\theta_2}_F A_5A_6A_8$ (input)

4) $A_1A_2A_3A_4A_5A_7A_8 \to\xrightarrow{\theta_2}_F A_3A_5A_6A_7A_8$ (IR7, 3) augment with $A_3A_5A_7A_8$)

5) $A_1A_2A_4 \overset{\min(\theta_1,\theta_2)}{\to\to}_F A_6$ (IR8, 2), 4))

6) $A_1A_2A_4 \overset{\min(\theta_1,\theta_2)}{\to\to}_F A_1A_2A_4A_6$ (IR7, 5) augment with $A_1A_2A_4$)

7) $A_1A_2A_4A_6 \to\xrightarrow{\theta_2}_F A_5A_6A_8$ (IR7, 3) augment with A_6)

8) $A_1A_2A_4 \overset{\min(\theta_1,\theta_2)}{\to\to}_F A_5A_8$ (IR8, 6), 7))

9) $A_1A_2A_4A_5A_8 \xrightarrow{\theta_3}_F A_3A_5A_6A_7$ (input)

10) $A_1A_2A_4 \overset{\min(\theta_1,\theta_2,\theta_3)}{\to}_F A_3A_6A_7$ (IR17, 8), 9))

Therefore, the claim holds true. □

Proof. II (applying Theorem A and resolution principle)

First, we join the fuzzy formulas:

$$\mathcal{K}_1 \equiv (A_1 \wedge A_2 \wedge A_4) \Rightarrow ((A_3 \wedge A_5 \wedge A_7 \wedge A_8) \vee A_6),$$
$$\mathcal{K}_2 \equiv (A_1 \wedge A_2 \wedge A_4) \Rightarrow ((A_5 \wedge A_6 \wedge A_8) \vee (A_3 \wedge A_7)),$$
$$\mathcal{K}_3 \equiv (A_1 \wedge A_2 \wedge A_4 \wedge A_5 \wedge A_8) \Rightarrow (A_3 \wedge A_5 \wedge A_6 \wedge A_7),$$
$$c' \equiv (A_1 \wedge A_2 \wedge A_4) \Rightarrow (A_3 \wedge A_6 \wedge A_7)$$

to given set of fuzzy dependencies.

Second, we find conjunctive normal forms of the formulas: $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$ and $\neg c'$. We obtain:

$$\begin{aligned}
\mathcal{K}_1 \equiv\ & (\neg A_1 \vee \neg A_2 \vee A_3 \vee \neg A_4 \vee A_6) \wedge \\
& (\neg A_1 \vee \neg A_2 \vee \neg A_4 \vee A_5 \vee A_6) \wedge \\
& (\neg A_1 \vee \neg A_2 \vee \neg A_4 \vee A_6 \vee A_7) \wedge \\
& (\neg A_1 \vee \neg A_2 \vee \neg A_4 \vee A_6 \vee A_8), \\
\mathcal{K}_2 \equiv\ & (\neg A_1 \vee \neg A_2 \vee A_3 \vee \neg A_4 \vee A_5) \wedge \\
& (\neg A_1 \vee \neg A_2 \vee \neg A_4 \vee A_5 \vee A_7) \wedge \\
& (\neg A_1 \vee \neg A_2 \vee A_3 \vee \neg A_4 \vee A_6) \wedge \\
& (\neg A_1 \vee \neg A_2 \vee \neg A_4 \vee A_6 \vee A_7) \wedge \\
& (\neg A_1 \vee \neg A_2 \vee A_3 \vee \neg A_4 \vee A_8) \wedge \\
& (\neg A_1 \vee \neg A_2 \vee \neg A_4 \vee A_7 \vee A_8), \\
\mathcal{K}_3 \equiv\ & (\neg A_1 \vee \neg A_2 \vee A_3 \vee \neg A_4 \vee \neg A_5 \vee \neg A_8) \wedge \\
& (\neg A_1 \vee \neg A_2 \vee \neg A_4 \vee \neg A_5 \vee A_6 \vee \neg A_8) \wedge \\
& (\neg A_1 \vee \neg A_2 \vee \neg A_4 \vee \neg A_5 \vee A_7 \vee \neg A_8), \\
\neg c' \equiv\ & A_1 \wedge A_2 \wedge A_4 \wedge (\neg A_3 \vee \neg A_6 \vee \neg A_7).
\end{aligned}$$

Third, we apply the resolution principle to conjunctive terms that appear within conjunctive normal forms of the formulas: $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$ and $\neg c'$. We get:

1) $\neg A_1 \vee \neg A_2 \vee \neg A_4 \vee A_6 \vee A_7$ (input)

2) A_1 (input)

3) $\neg A_2 \vee \neg A_4 \vee A_6 \vee A_7$ (resolvent from 1) and 2))

4) A_2 (input)

5) $\neg A_4 \vee A_6 \vee A_7$ (resolvent from 3) and 4))

6) A_4 (input)

7) $A_6 \vee A_7$ (resolvent from 5) and 6))

8) $\neg A_3 \vee \neg A_6 \vee \neg A_7$ (input)

9) $\neg A_3$ (resolvent from 7) and 8))

10) $\neg A_1 \vee \neg A_2 \vee A_3 \vee \neg A_4 \vee A_5$ (input)

11) $\neg A_2 \vee A_3 \vee \neg A_4 \vee A_5$ (resolvent from 2) and 10))

12) $A_3 \vee \neg A_4 \vee A_5$ (resolvent from 4) and 11))

13) $\neg A_4 \vee A_5$ (resolvent from 9) and 12))

14) A_5 (resolvent from 6) and 13))

15) $\neg A_1 \vee \neg A_2 \vee A_3 \vee \neg A_4 \vee A_8$ (input)

16) $\neg A_2 \vee A_3 \vee \neg A_4 \vee A_8$ (resolvent from 2) and 15))

17) $A_3 \vee \neg A_4 \vee A_8$ (resolvent from 4) and 16))

18) $\neg A_4 \vee A_8$ (resolvent from 9) and 17))

19) A_8 (resolvent from 6) and 18))

20) $\neg A_1 \vee \neg A_2 \vee A_3 \vee \neg A_4 \vee \neg A_5 \vee \neg A_8$ (input)

21) $\neg A_2 \vee A_3 \vee \neg A_4 \vee \neg A_5 \vee \neg A_8$ (resolvent from 2) and 20))

22) $A_3 \vee \neg A_4 \vee \neg A_5 \vee \neg A_8$ (resolvent from 4) and 21))

23) $\neg A_4 \vee \neg A_5 \vee \neg A_8$ (resolvent from 9) and 22))

24) $\neg A_5 \vee \neg A_8$ (resolvent from 6) and 23))

25) $\neg A_8$ (resolvent from 14) and 24))

Resolving 19) and 25), we obtain a refutation of $\neg c^{'}$. In other words, the formulas: $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$ and $\neg c^{'}$ cannot be valid simultaneously. This means that the assertion (b) of Theorem A holds true. Now, the assertion (a) of Theorem A yields that the fuzzy functional dependency $A_1A_2A_4 \overset{\min(\theta_1,\theta_2,\theta_3)}{\rightarrow}_F A_5A_6A_7$ follows from the given set of fuzzy dependencies. □

Obviously, the steps suggested in the second proof of Example 1 could be fully automated.

As we already noted, the authors in [6] applied fuzzy operators: conjunction, disjunction and implication to associate fuzzy formulas to fuzzy functional and fuzzy multivalued dependencies. In particular, they applied minimum t-norm, maximum t-co-norm and Kleene-Dienes fuzzy implication operator, respectively.

The structure of these fuzzy logic operators is explicitly applied in the following theorems.

Theorem B. [6, Th. 2] *Let $r = \{t_1, t_2\}$ be any two − element, fuzzy relation instance on scheme $R(A_1, A_2, ..., A_n)$, U be the universal set of attributes A_1, A_2,..., A_n and X, Y be subsets of U. Let $Z = U \setminus (X \cup Y)$. Then, r satisfies the fuzzy multivalued dependency $X \rightarrow\overset{\theta}{\rightarrow}_F Y$, θ-actively if and only if $\varphi(X[t_1, t_2]) \geq \theta$ and $i_{r,\theta}(\mathcal{H}) > \frac{1}{2}$, where $\mathcal{H}$ denotes the fuzzy formula $(\wedge_{A \in X} A) \Rightarrow ((\wedge_{B \in Y} B) \vee (\wedge_{C \in Z} C))$ associated to $X \rightarrow\overset{\theta}{\rightarrow}_F Y$.*

Theorem C. [6, Th. 7] *Let C be a set of fuzzy functional and fuzzy multivalued dependencies on some universal set of attributes U. Suppose that c is some fuzzy functional or fuzzy multivalued dependency on U. Denote by $C^{'}$ resp. $c^{'}$ the set of fuzzy formulas resp. the fuzzy formula associated to C resp. c. Then, the following two conditions are equivalent:*

(a) *Any two − element, fuzzy relation instance on scheme $R(U)$ which satisfies all dependencies in C, satisfies the dependency c.*

(b) *$i_{r,\beta}\left(c^{'}\right) > \frac{1}{2}$ for every $i_{r,\beta}$ such that $i_{r,\beta}(\mathcal{K}) > \frac{1}{2}$ for all $\mathcal{K} \in C^{'}$.*

Theorem C yields Theorem A in [6].

Note that the same theorem is proved in [4, pp. 38-42] for Yager's fuzzy implication operator, as well as in [5, pp. 293-296] for Kleene-Dienes-Lukasiewicz fuzzy implication operator.

Fuzzy relation instance r is said to satisfy the fuzzy multivalued dependency $X \rightarrow\overset{\theta}{\rightarrow}_F Y$, θ - actively if r satisfies $X \rightarrow\overset{\theta}{\rightarrow}_F Y$ and $\varphi(A[t_1, t_2]) \geq \theta$ for all $A \in X$ and all $t_1, t_2 \in r$.

Clearly, r satisfies $X \rightarrow\overset{\theta}{\rightarrow}_F Y$, θ-actively if and only if r satisfies $X \rightarrow\overset{\theta}{\rightarrow}_F Y$ and $\varphi(X[t_1, t_2]) \geq \theta$ for all $t_1, t_2 \in r$.

The following theorem holds true (see, [6, Th. 1]).

Theorem D. *Let $r = \{t_1, t_2\}$ be any two-element, fuzzy relation instance on scheme $R(A_1, A_2, ..., A_n)$, U be the universal set of attributes A_1, A_2,..., A_n and X, Y be subsets of U. Let $Z = U \setminus (X \cup Y)$. Then, r satisfies the fuzzy multivalued dependency $X \rightarrow\overset{\theta}{\rightarrow}_F Y$, θ-actively if and only if*

$$\varphi(X[t_1, t_2]) \geq \theta, \quad \varphi(Y[t_1, t_2]) \geq \theta \quad \text{or}$$
$$\varphi(X[t_1, t_2]) \geq \theta, \quad \varphi(Z[t_1, t_2]) \geq \theta.$$

Notice that the proof of Theorem D does not depend on the choice of fuzzy implication operator.

The authors in [6] applied the θ-active concept to derive the main result of the paper. The main result, i.e., Theorem 5, also yields Theorem A in [6].

Theorem B is an auxiliary result in [6].

The same theorem is proved in [4, pp. 37-38] for Yager's fuzzy implication operator, as well as in [5,

p. 288] for Kleene-Dienes-Lukasiewicz fuzzy implication operator.

Yager [22], has introduced two families of fuzzy implication operators, called the f-generated and g-generated implications, respectively (see also, [1, pp. 109f]). In this paper we research the concept of f-generated implications.

Let $f : [0,1] \to [0,+\infty]$ be a strictly decreasing and continuous function with $f(1) = 0$. The function $I_f : [0,1]^2 \to [0,1]$ defined by

$$I_f(x,y) = f^{-1}(xf(y)),$$

x, $y \in [0,1]$, with the understanding $0 \cdot \infty = 0$, is called an f-generated implication. The function f itself is called an f-generator (or an f-generator of I_f).

Since $f : [0,1] \to [0,+\infty]$ is strictly decreasing function, and $f(1) = 0$, we conclude that $f(x) \in [0, f(0)]$ for all $x \in [0,1]$. Hence, I_f is correctly defined if $xf(y) \in [0, f(0)]$ for all x, $y \in [0,1]$.

Since $x \in [0,1]$, and $f(y) \in [0,+\infty]$ for $y \in [0,1]$, we have that $xf(y) \geq 0$ for x, $y \in [0,1]$.

On the other side, $x \leq 1$ for $x \in [0,1]$, yields that

$$xf(y) \leq f(y) \leq f(0)$$

for $y \in [0,1]$.

Thus, $xf(y) \in [0, f(0)]$ for all x, $y \in [0,1]$, i.e., I_f is correctly defined.

By [1, p. 112, Th. 3.1.4.], $I_{f_1} = I_{f_2}$ if and only if there is a constant $c \in (0,+\infty)$ such that $f_2(x) = cf_1(x)$ for all $x \in [0,1]$, where f_1, $f_2 : [0,1] \to [0,+\infty]$ are any two f-generators.

We point out the following facts.

If f is an f-generator, then $f(0) = +\infty$ or $f(0) < +\infty$. If $f(0) < +\infty$, then the function $f_1 : [0,1] \to [0,1]$ defined by

$$f_1(x) = \frac{f(x)}{f(0)},$$

$x \in [0,1]$ is an f-generator. Namely, f_1 is a strictly decreasing and continuous function with $f_1(1) = 0$. Since $f(x) = f(0) \cdot f_1(x)$ for all $x \in [0,1]$, where f, f_1 are f-generators, and $f(0) \in (0,+\infty)$, we conclude that $I_f = I_{f_1}$. In other words, if f is an f-generator, then we may assume that either $f(0) = +\infty$ or $f(0) = 1$.

For example, if we take the f-generator $f(x) = -\log x$, $x \in [0,1]$, then the corresponding f-generated implication with $f(0) = +\infty$ is Yager's fuzzy implication (see, [21])

$$I_Y(x,y) = y^x,$$

x, $y \in [0,1]$, with the understanding $0^0 = 1$.

In concluding remarks of [4], [5] and [6], the authors noted that the set of applicable fuzzy implication operators to Theorems B and C could be possibly widen to include not only those covered by the above mentioned papers. The goal of this paper is to give a positive answer to this query, and to prove that the aforementioned theorems remain valid for arbitrary f-generated fuzzy implication operator (with either $f(0) = +\infty$ or $f(0) = 1$).

If $i_{r,\beta}$ is a valuation joined to r and β, then we assume that

$$i_{r,\beta}(A \wedge B) = \min(i_{r,\beta}(A), i_{r,\beta}(B)),$$
$$i_{r,\beta}(A \vee B) = \max(i_{r,\beta}(A), i_{r,\beta}(B)),$$
$$i_{r,\beta}(A \Rightarrow B) = f^{-1}(i_{r,\beta}(A) f(i_{r,\beta}(B)))$$

for A, $B \in U$.

2 Inference rules

The authors in [17] derived the following inference rules for fuzzy functional dependencies (shorter $FFDs$) and fuzzy multivalued dependencies (shorter $FMVDs$):

IR1 Inclusive rule for $FFDs$: If $X \xrightarrow{\theta_1}_F Y$ holds, and $\theta_1 \geq \theta_2$, then $X \xrightarrow{\theta_2}_F Y$ holds.

IR2 Reflexive rule for $FFDs$: If $X \supseteq Y$, then $X \to_F Y$ holds.

IR3 Augmentation rule for $FFDs$: If $X \xrightarrow{\theta}_F Y$ holds, then $XZ \xrightarrow{\theta}_F YZ$ holds.

IR4 Transitivity rule for $FFDs$: If $X \xrightarrow{\theta_1}_F Y$ holds and $Y \xrightarrow{\theta_2}_F Z$ holds, then $X \xrightarrow{\min(\theta_1,\theta_2)}_F Z$ holds.

IR5 Inclusive rule for $FMVDs$: If $X \to\xrightarrow{\theta_1}_F Y$ holds, and $\theta_1 \geq \theta_2$, then $X \to\xrightarrow{\theta_2}_F Y$ holds.

IR6 Complementation rule for $FMVDs$: If $X \to\xrightarrow{\theta}_F Y$ holds, then $X \to\xrightarrow{\theta}_F U - XY$ holds.

IR7 Augmentation rule for $FMVDs$: If $X \to\!\!\xrightarrow{\theta}_F Y$ holds, and $W \supseteq Z$, then $WX \to\!\!\xrightarrow{\theta}_F YZ$ holds.

IR8 Transitivity rule for $FMVDs$: If $X \to\!\!\xrightarrow{\theta_1}_F Y$ holds and $Y \to\!\!\xrightarrow{\theta_2}_F Z$ holds, then $X \stackrel{\min(\theta_1,\theta_2)}{\to\!\!\to}_F Z - Y$ holds.

IR9 Replication rule: If $X \xrightarrow{\theta}_F Y$ holds, then $X \to\!\!\xrightarrow{\theta}_F Y$ holds.

IR10 Coalescence rule for $FFDs$ and $FMVDs$: If $X \to\!\!\xrightarrow{\theta_1}_F Y$ holds, $Z \subseteq Y$, and for some W disjoint from Y we have $W \xrightarrow{\theta_2}_F Z$, then $X \stackrel{\min(\theta_1,\theta_2)}{\to}_F Z$ holds.

IR11 Union rule for $FFDs$: If $X \xrightarrow{\theta_1}_F Y$ holds and $X \xrightarrow{\theta_2}_F Z$ holds, then $X \stackrel{\min(\theta_1,\theta_2)}{\to}_F YZ$ holds.

IR12 Pseudotransitivity rule for $FFDs$: If $X \xrightarrow{\theta_1}_F Y$ holds and $WY \xrightarrow{\theta_2}_F Z$ holds, then $WX \stackrel{\min(\theta_1,\theta_2)}{\to}_F Z$ holds.

IR13 Decomposition rule for $FFDs$: If $X \xrightarrow{\theta}_F Y$ holds and $Z \subseteq Y$, then $X \xrightarrow{\theta}_F Z$ holds.

IR14 Union rule for $FMVDs$: If $X \to\!\!\xrightarrow{\theta_1}_F Y$ holds and $X \to\!\!\xrightarrow{\theta_2}_F Z$ holds, then $X \stackrel{\min(\theta_1,\theta_2)}{\to\!\!\to}_F YZ$ holds.

IR15 Pseudotransitivity rule for $FMVDs$: If $X \to\!\!\xrightarrow{\theta_1}_F Y$ holds and $WY \to\!\!\xrightarrow{\theta_2}_F Z$ holds, then $WX \stackrel{\min(\theta_1,\theta_2)}{\to\!\!\to}_F Z - WY$ holds.

IR16 Decomposition rule for $FMVDs$: If $X \to\!\!\xrightarrow{\theta_1}_F Y$ holds and $X \to\!\!\xrightarrow{\theta_2}_F Z$ holds, then $X \stackrel{\min(\theta_1,\theta_2)}{\to\!\!\to}_F Y \cap Z$, $X \stackrel{\min(\theta_1,\theta_2)}{\to\!\!\to}_F Y - Z$, $X \stackrel{\min(\theta_1,\theta_2)}{\to\!\!\to}_F Z - Y$ hold.

IR17 Mixed pseudotransitivity rule: If $X \to\!\!\xrightarrow{\theta_1}_F Y$ holds and $XY \xrightarrow{\theta_2}_F Z$ holds, then $X \stackrel{\min(\theta_1,\theta_2)}{\to}_F Z - Y$ holds.

Here, U is some universal set of attributes and X, Y, Z, $W \subseteq U$. Moreover, $U - XY$, for example, means $U \setminus (X \cup Y)$.

Note that these rules are consistent (see, [17]), i.e., they reduce to the classical ones [2] when crisp attributes are included.

3 Proofs

Proof. (of Theorem B)

Since $r = \{t_1, t_2\}$ and $\theta \in [0,1]$ are given, a valuation $i_{r,\theta}$ is determined.

($\Rightarrow$) Suppose that r satisfies $X \to\!\!\xrightarrow{\theta}_F Y$, θ-actively.

By Theorem D,

$$\varphi(X[t_1,t_2]) \geq \theta, \;\; \varphi(Y[t_1,t_2]) \geq \theta \quad \text{or}$$
$$\varphi(X[t_1,t_2]) \geq \theta, \;\; \varphi(Z[t_1,t_2]) \geq \theta.$$

Let $\varphi(X[t_1,t_2]) \geq \theta$ and $\varphi(Y[t_1,t_2]) \geq \theta$ hold true.

Now,

$$\min_{A \in X} \{\varphi(A[t_1,t_2])\} = \varphi(X[t_1,t_2]) \geq \theta.$$

Hence, $\varphi(A[t_1,t_2]) \geq \theta$ for all $A \in X$, i.e., $i_{r,\theta}(A) > \frac{1}{2}$ for all $A \in X$. Therefore,

$$i_{r,\theta}(\wedge_{A \in X} A) = \min\{i_{r,\theta}(A) \mid A \in X\} > \frac{1}{2}.$$

Similarly, $\varphi(Y[t_1,t_2]) \geq \theta$ yields that $i_{r,\theta}(\wedge_{B \in Y} B) > \frac{1}{2}$.

We have,

$$\begin{aligned}
& i_{r,\theta}(\mathcal{H}) \\
&= i_{r,\theta}((\wedge_{A \in X} A) \Rightarrow ((\wedge_{B \in Y} B) \vee (\wedge_{C \in Z} C))) \\
&= f^{-1}\Big(i_{r,\theta}(\wedge_{A \in X} A) \\
&\quad f(i_{r,\theta}((\wedge_{B \in Y} B) \vee (\wedge_{C \in Z} C)))\Big) \\
&= f^{-1}\Big(i_{r,\theta}(\wedge_{A \in X} A) \cdot \\
&\quad f(\max(i_{r,\theta}(\wedge_{B \in Y} B), i_{r,\theta}(\wedge_{C \in Z} C)))\Big).
\end{aligned}$$

Put

$$\begin{aligned}
a &= i_{r,\theta}(\wedge_{A \in X} A), \\
b &= \max(i_{r,\theta}(\wedge_{B \in Y} B), i_{r,\theta}(\wedge_{C \in Z} C)).
\end{aligned}$$

Hence,

$$i_{r,\theta}(\mathcal{H}) = f^{-1}(af(b)).$$

Since $i_{r,\theta}\left(\wedge_{A\in X}A\right) > \frac{1}{2}$, it follows that $a > \frac{1}{2}$. Similarly, $i_{r,\theta}\left(\wedge_{B\in Y}B\right) > \frac{1}{2}$, yields that $b > \frac{1}{2}$.

If $b = 1$, then $f(b) = 0$. Therefore,

$$f^{-1}\left(af(b)\right) = f^{-1}(0) = 1.$$

We conclude, $i_{r,\theta}(\mathcal{H}) > \frac{1}{2}$.

Suppose that $b < 1$. Now, $f(b) > 0$.

If $f^{-1}\left(af(b)\right) \leq \frac{1}{2}$, then $af(b) \geq f\left(\frac{1}{2}\right)$, i.e.,

$$a \geq \frac{f\left(\frac{1}{2}\right)}{f(b)}.$$

Since $b > \frac{1}{2}$, it follows that $f(b) < f\left(\frac{1}{2}\right)$, i.e.,

$$\frac{f\left(\frac{1}{2}\right)}{f(b)} > 1.$$

Consequently,

$$a \geq \frac{f\left(\frac{1}{2}\right)}{f(b)} > 1.$$

This is not possible, however. Therefore, $f^{-1}\left(af(b)\right) > \frac{1}{2}$, i.e., $i_{r,\theta}(\mathcal{H}) > \frac{1}{2}$.

Now, suppose that $\varphi\left(X\left[t_1, t_2\right]\right) \geq \theta$ and $\varphi\left(Z\left[t_1, t_2\right]\right) \geq \theta$ hold true. Reasoning as in the previous case, we conclude that $i_{r,\theta}\left(\wedge_{A\in X}A\right) > \frac{1}{2}$ and $i_{r,\theta}\left(\wedge_{C\in Z}C\right) > \frac{1}{2}$. Therefore, $a > \frac{1}{2}$ and $b > \frac{1}{2}$.

Now, reasoning in exactly the same way as in the previous case, we obtain that $i_{r,\theta}(\mathcal{H}) > \frac{1}{2}$.

($\Leftarrow$) Suppose that $\varphi\left(X\left[t_1, t_2\right]\right) \geq \theta$ and $i_{r,\theta}(\mathcal{H}) > \frac{1}{2}$ hold true.

We have,

$$i_{r,\theta}(\mathcal{H}) = f^{-1}\left(af(b)\right) > \frac{1}{2},$$

where

$$a = i_{r,\theta}\left(\wedge_{A\in X}A\right),$$
$$b = \max\left(i_{r,\theta}\left(\wedge_{B\in Y}B\right), i_{r,\theta}\left(\wedge_{C\in Z}C\right)\right).$$

Since $\varphi\left(X\left[t_1, t_2\right]\right) \geq \theta$, we have that $a > \frac{1}{2}$.

We shall prove that $b > \frac{1}{2}$. Namely, $b > \frac{1}{2}$ implies that $i_{r,\theta}\left(\wedge_{B\in Y}B\right) > \frac{1}{2}$ or $i_{r,\theta}\left(\wedge_{C\in Z}C\right) > \frac{1}{2}$.

If $i_{r,\theta}\left(\wedge_{B\in Y}B\right) > \frac{1}{2}$, for example, then

$$\min\left\{i_{r,\theta}(B) \mid B \in Y\right\} = i_{r,\theta}\left(\wedge_{B\in Y}B\right) > \frac{1}{2}$$

yields that $i_{r,\theta}(B) > \frac{1}{2}$ for all $B \in Y$, i.e., $\varphi\left(B\left[t_1, t_2\right]\right) \geq \theta$ for all $B \in Y$. Hence,

$$\varphi\left(Y\left[t_1, t_2\right]\right) = \min_{B\in Y}\left\{\varphi\left(B\left[t_1, t_2\right]\right)\right\} \geq \theta.$$

Similarly, if $i_{r,\theta}\left(\wedge_{C\in Z}C\right) > \frac{1}{2}$, then $\varphi\left(Z\left[t_1, t_2\right]\right) \geq \theta$.

If $f^{-1}\left(af(b)\right) = 1$, then $af(b) = 0$. Since $a > \frac{1}{2}$, we conclude that $f(b) = 0$, i.e., $b = 1$. Hence, $b > \frac{1}{2}$.

Suppose that $f^{-1}\left(af(b)\right) < 1$. Then, $af(b) > f(1) = 0$. Since $a > \frac{1}{2}$, we obtain that $f(b) > 0$. Furthermore, since $f^{-1}\left(af(b)\right) > \frac{1}{2}$, it follows that $af(b) < f\left(\frac{1}{2}\right)$, i.e.,

$$a < \frac{f\left(\frac{1}{2}\right)}{f(b)}.$$

Assume that $b \leq \frac{1}{2}$.

If $b = \frac{1}{2}$, then

$$\frac{f\left(\frac{1}{2}\right)}{f(b)} = 1.$$

Since,

$$a < \frac{f\left(\frac{1}{2}\right)}{f(b)},$$

it follows that $a < 1$. This is not necessarily true, however. Namely, the case $a = 1$ may occur as well.

Suppose that $b < \frac{1}{2}$. Now, $f(b) > f\left(\frac{1}{2}\right)$, i.e.,

$$\frac{f\left(\frac{1}{2}\right)}{f(b)} < 1.$$

Therefore,

$$a < \frac{f\left(\frac{1}{2}\right)}{f(b)}$$

is not necessarily satisfied since $a > \frac{1}{2}$ may happen to be larger than

$$\frac{f\left(\frac{1}{2}\right)}{f(b)}.$$

We conclude,

$$a < \frac{f\left(\frac{1}{2}\right)}{f\left(b\right)}$$

holds true for $a > \frac{1}{2}$ only if $b > \frac{1}{2}$. Namely, if $b > \frac{1}{2}$, then $f\left(b\right) < f\left(\frac{1}{2}\right)$, i.e.,

$$\frac{f\left(\frac{1}{2}\right)}{f\left(b\right)} > 1.$$

Consequently,

$$a < \frac{f\left(\frac{1}{2}\right)}{f\left(b\right)}$$

holds true for every $a > \frac{1}{2}$.

Thus, $b > \frac{1}{2}$. Now, bearing in mind what we said earlier, we have that $\varphi\left(Y\left[t_1, t_2\right]\right) \geq \theta$ or $\varphi\left(Z\left[t_1, t_2\right]\right) \geq \theta$. Hence,

$$\varphi\left(X\left[t_1, t_2\right]\right) \geq \theta, \quad \varphi\left(Y\left[t_1, t_2\right]\right) \geq \theta \quad \text{or}$$
$$\varphi\left(X\left[t_1, t_2\right]\right) \geq \theta, \quad \varphi\left(Z\left[t_1, t_2\right]\right) \geq \theta.$$

By Theorem D, r satisfies $X \rightarrow\overset{\theta}{\rightarrow}_F Y$, θ-actively. This completes the proof. □

Proof. (of Theorem C)

We write $X \overset{\theta_1}{\longrightarrow}_F Y$ resp. $X \rightarrow\overset{\theta_1}{\longrightarrow}_F Y$ instead of c if c is a fuzzy functional resp. fuzzy multivalued dependency. Consequently, we write

$$\left(\wedge_{A\in X} A\right) \Rightarrow \left(\wedge_{B\in Y} B\right)$$

resp.

$$\left(\wedge_{A\in X} A\right) \Rightarrow \left(\left(\wedge_{B\in Y} B\right) \vee \left(\wedge_{D\in Z} D\right)\right)$$

instead of $c^{'}$, where $Z = U \setminus \left(X \cup Y\right)$.

We choose the set $\{a, b\}$ to be the domain of each of the attributes in U.

Fix some $\theta^{''} \in \left[0, \theta^{'}\right)$, where $\theta^{'}$ is the minimum of the strengths of all dependencies that appear in $C \cup \{c\}$. We may assume that $\theta^{'} < 1$. Otherwise, if $\theta^{'} = 1$, the claim of the theorem reduces to the non-interesting case where each $c_1 \in C \cup \{c\}$ has the strength 1.

Let $s\left(a, b\right) = \theta^{''}$.

(a) ⇒ (b) Assume that (b) does not hold.

Then, there exists some $i_{r,\beta}$ such that $i_{r,\beta}\left(\mathcal{K}\right) > \frac{1}{2}$ for all $\mathcal{K} \in C^{'}$ and $i_{r,\beta}\left(c^{'}\right) \leq \frac{1}{2}$.

Note that $i_{r,\beta}$ is joined to some two-element, fuzzy relation instance $r = \{t_1, t_2\}$ on $R\left(U\right)$ and some $\beta \in [0, 1]$.

Denote, $Z^{'} = \left\{A \in U \mid i_{r,\beta}\left(A\right) > \frac{1}{2}\right\}$.

Suppose that $Z^{'} = \varnothing$.

Then, $i_{r,\beta}\left(A\right) \leq \frac{1}{2}$ for all $A \in U$.

Since $i_{r,\beta}\left(c^{'}\right) \leq \frac{1}{2}$, we have that

$$i_{r,\beta}\left(\left(\wedge_{A\in X} A\right) \Rightarrow \left(\wedge_{B\in Y} B\right)\right) \leq \frac{1}{2}$$

resp.

$$i_{r,\beta}\left(\left(\wedge_{A\in X} A\right) \Rightarrow \left(\left(\wedge_{B\in Y} B\right) \vee \left(\wedge_{D\in Z} D\right)\right)\right) \leq \frac{1}{2},$$

i.e.,

$$f^{-1}\left(i_{r,\beta}\left(\wedge_{A\in X} A\right) f\left(i_{r,\beta}\left(\wedge_{B\in Y} B\right)\right)\right) \leq \frac{1}{2}$$

resp.

$$f^{-1}\Big(i_{r,\beta}\left(\wedge_{A\in X} A\right) \cdot f\left(i_{r,\beta}\left(\left(\wedge_{B\in Y} B\right) \vee \left(\wedge_{D\in Z} D\right)\right)\right)\Big) \leq \frac{1}{2},$$

i.e.,

$$f^{-1}\left(i_{r,\beta}\left(\wedge_{A\in X} A\right) f\left(i_{r,\beta}\left(\wedge_{B\in Y} B\right)\right)\right) \leq \frac{1}{2}$$

resp.

$$f^{-1}\Big(i_{r,\beta}\left(\wedge_{A\in X} A\right) \cdot f\left(\max\left(i_{r,\beta}\left(\wedge_{B\in Y} B\right), i_{r,\beta}\left(\wedge_{D\in Z} D\right)\right)\right)\Big) \leq \frac{1}{2},$$

i.e.,

$$i_{r,\beta}\left(\wedge_{A\in X} A\right) f\left(i_{r,\beta}\left(\wedge_{B\in Y} B\right)\right) \geq f\left(\frac{1}{2}\right)$$

resp.

$$i_{r,\beta}\left(\wedge_{A\in X} A\right) \cdot f\left(\max\left(i_{r,\beta}\left(\wedge_{B\in Y} B\right), i_{r,\beta}\left(\wedge_{D\in Z} D\right)\right)\right) \geq f\left(\frac{1}{2}\right).$$

If

$$f\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right)\right)=0$$

resp.

$$f\left(\max\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right),i_{r,\beta}\left(\wedge_{D\in Z}D\right)\right)\right)=0$$

then,

$$i_{r,\beta}\left(\wedge_{B\in Y}B\right)=1$$

resp.

$$\max\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right),i_{r,\beta}\left(\wedge_{D\in Z}D\right)\right)=1,$$

i.e.,

$$i_{r,\beta}\left(\wedge_{B\in Y}B\right)=1$$

resp.

$$i_{r,\beta}\left(\wedge_{B\in Y}B\right)=1 \quad \text{or} \quad i_{r,\beta}\left(\wedge_{D\in Z}D\right)=1.$$

Hence, in any case, there is $A\in U$ such that $i_{r,\beta}(A)$ $=1>\frac{1}{2}$. This is a contradiction. We conclude,

$$f\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right)\right)>0$$

resp.

$$f\left(\max\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right),i_{r,\beta}\left(\wedge_{D\in Z}D\right)\right)\right)>0.$$

Thus,

$$i_{r,\beta}\left(\wedge_{A\in X}A\right)\geq\frac{f\left(\frac{1}{2}\right)}{f\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right)\right)} \tag{1}$$

resp.

$$i_{r,\beta}\left(\wedge_{A\in X}A\right) \geq\frac{f\left(\frac{1}{2}\right)}{f\left(\max\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right),i_{r,\beta}\left(\wedge_{D\in Z}D\right)\right)\right)}. \tag{2}$$

Since $i_{r,\beta}(A)\leq\frac{1}{2}$ for all $A\in U$, we obtain that

$$i_{r,\beta}\left(\wedge_{B\in Y}B\right)=\min\left\{i_{r,\beta}(B)\mid B\in Y\right\}\leq\frac{1}{2}$$

resp.

$$\max\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right),i_{r,\beta}\left(\wedge_{D\in Z}D\right)\right)\leq\frac{1}{2},$$

i.e.,

$$f\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right)\right)\geq f\left(\frac{1}{2}\right)$$

resp.

$$f\left(\max\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right),i_{r,\beta}\left(\wedge_{D\in Z}D\right)\right)\right)\geq f\left(\frac{1}{2}\right),$$

i.e.,

$$1\geq\frac{f\left(\frac{1}{2}\right)}{f\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right)\right)}$$

resp.

$$1\geq\frac{f\left(\frac{1}{2}\right)}{f\left(\max\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right),i_{r,\beta}\left(\wedge_{D\in Z}D\right)\right)\right)}.$$

Having in mind these inequalities, we conclude that (1) resp. (2) holds true only if $i_{r,\beta}\left(\wedge_{A\in X}A\right)=1$. Therefore, $i_{r,\beta}\left(\wedge_{A\in X}A\right)=1$ yields that there exists $A\in U$ such that $i_{r,\beta}(A)=1>\frac{1}{2}$. This is a contradiction.

Consequently, $Z'\neq\varnothing$.

Suppose that $Z'=U$.

Now, $i_{r,\beta}(A)>\frac{1}{2}$ for all $A\in U$.

Since $i_{r,\beta}\left(c'\right)\leq\frac{1}{2}$, we have that

$$i_{r,\beta}\left(\wedge_{A\in X}A\right)f\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right)\right)\geq f\left(\frac{1}{2}\right)$$

resp.

$$i_{r,\beta}\left(\wedge_{A\in X}A\right)\cdot f\left(\max\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right),i_{r,\beta}\left(\wedge_{D\in Z}D\right)\right)\right)\geq f\left(\frac{1}{2}\right).$$

If

$$f\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right)\right)=0$$

resp.

$$f\left(\max\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right),i_{r,\beta}\left(\wedge_{D\in Z}D\right)\right)\right)=0$$

then, $0\geq f\left(\frac{1}{2}\right)$. This is impossible, however. Namely, $\frac{1}{2}<1$ yields that $f\left(\frac{1}{2}\right)>f(1)=0$. Hence,

$$f\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right)\right)>0$$

resp.

$$f\left(\max\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right), i_{r,\beta}\left(\wedge_{D\in Z}D\right)\right)\right) > 0.$$

Therefore, (1) resp. (2) holds true.

Since $i_{r,\beta}(A) > \frac{1}{2}$ for all $A \in U$, we have that

$$i_{r,\beta}\left(\wedge_{B\in Y}B\right) = \min\left\{i_{r,\beta}(B) \mid B \in Y\right\} > \frac{1}{2}$$

resp.

$$\max\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right), i_{r,\beta}\left(\wedge_{D\in Z}D\right)\right) > \frac{1}{2},$$

i.e.,

$$f\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right)\right) < f\left(\frac{1}{2}\right)$$

resp.

$$f\left(\max\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right), i_{r,\beta}\left(\wedge_{D\in Z}D\right)\right)\right) < f\left(\frac{1}{2}\right),$$

i.e.,

$$\frac{f\left(\frac{1}{2}\right)}{f\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right)\right)} > 1$$

resp.

$$\frac{f\left(\frac{1}{2}\right)}{f\left(\max\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right), i_{r,\beta}\left(\wedge_{D\in Z}D\right)\right)\right)} > 1.$$

Therefore, by (1) resp. (2), $i_{r,\beta}\left(\wedge_{A\in X}A\right) > 1$. This is a contradiction.

We conclude, $Z' \neq U$.

Let $r' = \left\{t', t''\right\}$ be the two element, fuzzy relation instance on $R(U)$, given by table 1.

Table 1:

	attributes of Z'	other attributes
t'	$a, a, \ldots, a$	$a, a, \ldots, a$
t''	$a, a, \ldots, a$	$b, b, \ldots, b$

We shall prove that r' satisfies all dependencies from the set C, and violates the dependency c.

Let $K \xrightarrow{\theta_2}_F L$ be any fuzzy functional dependency from the set C. We have,

$$f^{-1}\left(i_{r,\beta}\left(\wedge_{A\in K}A\right) f\left(i_{r,\beta}\left(\wedge_{B\in L}B\right)\right)\right)$$
$$= i_{r,\beta}\left(\left(\wedge_{A\in K}A\right) \Rightarrow \left(\wedge_{B\in L}B\right)\right) > \frac{1}{2},$$

i.e.,

$$i_{r,\beta}\left(\wedge_{A\in K}A\right) f\left(i_{r,\beta}\left(\wedge_{B\in L}B\right)\right) < f\left(\frac{1}{2}\right).$$

If $f\left(i_{r,\beta}\left(\wedge_{B\in L}B\right)\right) = 0$, then $i_{r,\beta}\left(\wedge_{B\in L}B\right) = 1$, i.e.,

$$\min\left\{i_{r,\beta}(B) \mid B \in L\right\} = 1.$$

Hence, $i_{r,\beta}(B) = 1$ for all $B \in L$, i.e., $B \in Z'$ for all $B \in L$, i.e., $L \subseteq Z'$.

We obtain, $\varphi\left(L\left[t', t''\right]\right) = 1$. Consequently,

$$\varphi\left(L\left[t', t''\right]\right) = 1 \geq \min\left(\theta_2, \varphi\left(K\left[t', t''\right]\right)\right). \quad (3)$$

This means that r' satisfies the dependency $K \xrightarrow{\theta_2}_F L$.

Suppose that $f\left(i_{r,\beta}\left(\wedge_{B\in L}B\right)\right) > 0$. Now,

$$i_{r,\beta}\left(\wedge_{A\in K}A\right) < \frac{f\left(\frac{1}{2}\right)}{f\left(i_{r,\beta}\left(\wedge_{B\in L}B\right)\right)}. \quad (4)$$

If $i_{r,\beta}\left(\wedge_{A\in K}A\right) \leq \frac{1}{2}$, then

$$\min\left\{i_{r,\beta}(A) \mid A \in K\right\} \leq \frac{1}{2}.$$

Now, there exists $A \in K$ such that $i_{r,\beta}(A) \leq \frac{1}{2}$. This means that $A \notin Z'$, i.e., that $\varphi\left(A\left[t', t''\right]\right) = \theta''$. Consequently, $\varphi\left(K\left[t', t''\right]\right) = \theta''$.

Since $s(a, b) = \theta''$, we have that $\varphi\left(Q\left[t', t''\right]\right) \geq \theta''$ for any attribute set $Q \subseteq U$. In particular, $\varphi\left(L\left[t', t''\right]\right) \geq \theta''$.

We conclude,

$$\varphi\left(L\left[t', t''\right]\right) \geq \theta'' = \min\left(\theta_2, \varphi\left(K\left[t', t''\right]\right)\right).$$

In other words, r' satisfies $K \xrightarrow{\theta_2}_F L$.

Let $i_{r,\beta}\left(\wedge_{A\in K}A\right) > \frac{1}{2}$. Since (4) holds true, this inequality yields that

$$\frac{f\left(\frac{1}{2}\right)}{f\left(i_{r,\beta}\left(\wedge_{B\in L}B\right)\right)} > 1.$$

Hence,

$$f\left(\frac{1}{2}\right) > f\left(i_{r,\beta}\left(\wedge_{B\in L}B\right)\right),$$

i.e.,

$$\frac{1}{2} < i_{r,\beta}\left(\wedge_{B\in L}B\right),$$

i.e.,

$$\frac{1}{2} < \min\left\{i_{r,\beta}\left(B\right) \mid B\in L\right\}.$$

Now, $i_{r,\beta}\left(B\right) > \frac{1}{2}$ for all $B\in L$, i.e., $B\in Z^{'}$ for all $B\in L$, i.e., $L\subseteq Z^{'}$.

Hence, $\varphi\left(L\left[t^{'},t^{''}\right]\right) = 1$.

Consequently, (3) holds true. This means that $r^{'}$ satisfies the dependency $K \xrightarrow{\theta_2}_F L$.

Let $K \to\xrightarrow{\theta_2}_F L$ be any fuzzy multivalued dependency from the set C.

We have,

$$\begin{aligned}
&f^{-1}\Big(i_{r,\beta}\left(\wedge_{A\in K}A\right)\cdot\\
&\quad f\left(\max\left(i_{r,\beta}\left(\wedge_{B\in L}B\right), i_{r,\beta}\left(\wedge_{D\in M}D\right)\right)\right)\Big)\\
&= f^{-1}\Big(i_{r,\beta}\left(\wedge_{A\in K}A\right)\cdot\\
&\quad f\left(i_{r,\beta}\left(\left(\wedge_{B\in L}B\right)\vee\left(\wedge_{D\in M}D\right)\right)\right)\Big)\\
&= i_{r,\beta}\left(\left(\wedge_{A\in K}A\right)\Rightarrow\left(\left(\wedge_{B\in L}B\right)\vee\left(\wedge_{D\in M}D\right)\right)\right)\\
&> \frac{1}{2},
\end{aligned}$$

i.e.,

$$\begin{aligned}
&i_{r,\beta}\left(\wedge_{A\in K}A\right)\cdot\\
&\quad f\left(\max\left(i_{r,\beta}\left(\wedge_{B\in L}B\right), i_{r,\beta}\left(\wedge_{D\in M}D\right)\right)\right)\\
&< f\left(\frac{1}{2}\right),
\end{aligned}$$

where $M = U\setminus\left(K\cup L\right)$.

Suppose that $i_{r,\beta}\left(\wedge_{A\in K}A\right) \le \frac{1}{2}$.

Reasoning as in the case of fuzzy functional dependency $K \xrightarrow{\theta_2}_F L$, we conclude that $\varphi\left(K\left[t^{'},t^{''}\right]\right) = \theta^{''}$.

Now, there is $t^{'''}\in r^{'}$, $t^{'''} = t^{'}$ such that

$$\begin{aligned}
\varphi\left(K\left[t^{'''},t^{'}\right]\right) = 1 \ge \min\left(\theta_2, \varphi\left(K\left[t^{'},t^{''}\right]\right)\right),\\
\varphi\left(L\left[t^{'''},t^{'}\right]\right) = 1 \ge \min\left(\theta_2, \varphi\left(K\left[t^{'},t^{''}\right]\right)\right),\\
\varphi\left(M\left[t^{'''},t^{''}\right]\right) \ge \theta^{''} \ge \min\left(\theta_2, \varphi\left(K\left[t^{'},t^{''}\right]\right)\right).
\end{aligned}$$

This means that $r^{'}$ satisfies $K \to\xrightarrow{\theta_2}_F L$.

Let $i_{r,\beta}\left(\wedge_{A\in K}A\right) > \frac{1}{2}$. Now,

$$\min\left\{i_{r,\beta}\left(A\right) \mid A\in K\right\} > \frac{1}{2}.$$

Hence, $i_{r,\beta}\left(A\right) > \frac{1}{2}$ for all $A\in K$, i.e., $A\in Z^{'}$ for all $A\in K$, i.e., $K\subseteq Z^{'}$.

Therefore, $\varphi\left(K\left[t^{'},t^{''}\right]\right) = 1$.

Assume that

$$f\left(\max\left(i_{r,\beta}\left(\wedge_{B\in L}B\right), i_{r,\beta}\left(\wedge_{D\in M}D\right)\right)\right) = 0.$$

Hence,

$$\max\left(i_{r,\beta}\left(\wedge_{B\in L}B\right), i_{r,\beta}\left(\wedge_{D\in M}D\right)\right) = 1,$$

i.e., $i_{r,\beta}\left(\wedge_{B\in L}B\right) = 1$ or $i_{r,\beta}\left(\wedge_{D\in M}D\right) = 1$, i.e.,

$$\min\left\{i_{r,\beta}\left(B\right) \mid B\in L\right\} = 1$$

or

$$\min\left\{i_{r,\beta}\left(D\right) \mid D\in M\right\} = 1,$$

i.e., $i_{r,\beta}\left(B\right) = 1$ for all $B\in L$ or $i_{r,\beta}\left(D\right) = 1$ for all $D\in M$, i.e., $B\in Z^{'}$ for all $B\in L$ or $D\in Z^{'}$ for all $D\in M$, i.e., $L\subseteq Z^{'}$ or $M\subseteq Z^{'}$.

In other words, $\varphi\left(L\left[t^{'},t^{''}\right]\right) = 1$ or $\varphi\left(M\left[t^{'},t^{''}\right]\right) = 1$.

Suppose that $\varphi\left(L\left[t^{'},t^{''}\right]\right) = 1$.

Now, there exists $t^{'''}\in r^{'}$, $t^{'''} = t^{''}$ such that

$$\begin{aligned}\varphi\left(K\left[t^{'''},t^{'}\right]\right)&=1\\&\geq \min\left(\theta_2,\varphi\left(K\left[t^{'},t^{''}\right]\right)\right),\\ \varphi\left(L\left[t^{'''},t^{'}\right]\right)&=1\\&\geq \min\left(\theta_2,\varphi\left(K\left[t^{'},t^{''}\right]\right)\right),\\ \varphi\left(M\left[t^{'''},t^{''}\right]\right)&=1\\&\geq \min\left(\theta_2,\varphi\left(K\left[t^{'},t^{''}\right]\right)\right).\end{aligned} \tag{5}$$

This means that $r^{'}$ satisfies $K \to\xrightarrow{\theta_2}_F L$.

Suppose that $\varphi\left(M\left[t^{'},t^{''}\right]\right)=1$

Now, there is $t^{'''}\in r^{'}$, $t^{'''}=t^{'}$ such that (5) holds true. Therefore, $r^{'}$ satisfies $K \to\xrightarrow{\theta_2}_F L$.

Assume that,

$$f\left(\max\left(i_{r,\beta}\left(\wedge_{B\in L}B\right),i_{r,\beta}\left(\wedge_{D\in M}D\right)\right)\right)>0.$$

Now,

$$\begin{aligned}&i_{r,\beta}\left(\wedge_{A\in K}A\right)\\&<\frac{f\left(\frac{1}{2}\right)}{f\left(\max\left(i_{r,\beta}\left(\wedge_{B\in L}B\right),i_{r,\beta}\left(\wedge_{D\in M}D\right)\right)\right)}.\end{aligned}$$

Since, $i_{r,\beta}\left(\wedge_{A\in K}A\right)>\frac{1}{2}$, the previous inequality yields that

$$\frac{f\left(\frac{1}{2}\right)}{f\left(\max\left(i_{r,\beta}\left(\wedge_{B\in L}B\right),i_{r,\beta}\left(\wedge_{D\in M}D\right)\right)\right)}>1,$$

i.e.,

$$f\left(\frac{1}{2}\right)>f\left(\max\left(i_{r,\beta}\left(\wedge_{B\in L}B\right),i_{r,\beta}\left(\wedge_{D\in M}D\right)\right)\right),$$

i.e.,

$$\frac{1}{2}<\max\left(i_{r,\beta}\left(\wedge_{B\in L}B\right),i_{r,\beta}\left(\wedge_{D\in M}D\right)\right).$$

Hence, $i_{r,\beta}\left(\wedge_{B\in L}B\right)>\frac{1}{2}$ or $i_{r,\beta}\left(\wedge_{D\in M}D\right)>\frac{1}{2}$.

Reasoning as earlier, we obtain that $\varphi\left(L\left[t^{'},t^{''}\right]\right)=1$ or $\varphi\left(M\left[t^{'},t^{''}\right]\right)=1$.

If $\varphi\left(L\left[t^{'},t^{''}\right]\right)=1$, then there is $t^{'''}\in r^{'}$, $t^{'''}=t^{''}$ such that (5) holds true.

If $\varphi\left(M\left[t^{'},t^{''}\right]\right)=1$, then there is $t^{'''}\in r^{'}$, $t^{'''}=t^{'}$ such that (5) holds true.

We conclude, $r^{'}$ satisfies the dependency $K \to\xrightarrow{\theta_2}_F L$.

It remains to prove that $r^{'}$ violates the dependency $X \xrightarrow{\theta_1}_F Y$ resp. $X \to\xrightarrow{\theta_1}_F Y$.

Let

$$\begin{aligned}&f^{-1}\left(i_{r,\beta}\left(\wedge_{A\in X}A\right)f\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right)\right)\right)\\&=i_{r,\beta}\left(\left(\wedge_{A\in X}A\right)\Rightarrow\left(\wedge_{B\in Y}B\right)\right)\\&=i_{r,\beta}\left(c^{'}\right)\leq\frac{1}{2}.\end{aligned}$$

Now,

$$f\left(\frac{1}{2}\right)\leq i_{r,\beta}\left(\wedge_{A\in X}A\right)f\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right)\right).$$

Assume that $f\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right)\right)\neq 0,+\infty$.

We obtain,

$$\frac{f\left(\frac{1}{2}\right)}{f\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right)\right)}\leq i_{r,\beta}\left(\wedge_{A\in X}A\right).$$

If we assume that $i_{r,\beta}\left(\wedge_{A\in X}A\right)\leq\frac{1}{2}$, then the previous inequality yields that

$$\frac{f\left(\frac{1}{2}\right)}{f\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right)\right)}=0,$$

i.e., that $f\left(\frac{1}{2}\right)=0$. Since $\frac{1}{2}\neq 1$, this is not possible.

Hence, $i_{r,\beta}\left(\wedge_{A\in X}A\right)>\frac{1}{2}$.

As we already seen, this implies that $\varphi\left(X\left[t^{'},t^{''}\right]\right)=1$.

Moreover,

$$\frac{f\left(\frac{1}{2}\right)}{f\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right)\right)}\leq i_{r,\beta}\left(\wedge_{A\in X}A\right)$$

yields that

$$\frac{f\left(\frac{1}{2}\right)}{f\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right)\right)}\leq\frac{1}{2}<1.$$

We obtain,

$$f\left(\frac{1}{2}\right)<f\left(i_{r,\beta}\left(\wedge_{B\in Y}B\right)\right),$$

i.e.,

$$\frac{1}{2} > i_{r,\beta}\left(\wedge_{B\in Y} B\right).$$

As earlier, we conclude that there exists $B \in Y$ such that $i_{r,\beta}(B) < \frac{1}{2}$, i.e., that $B \notin Z'$.

Therefore, $\varphi\left(B\left[t', t''\right]\right) = \theta''$, and then $\varphi\left(Y\left[t', t''\right]\right) = \theta''$.

We have,

$$\begin{aligned}\varphi\left(Y\left[t', t''\right]\right) &= \theta'' < \theta' \leq \theta_1 \\ &= \min\left(\theta_1, \varphi\left(X\left[t', t''\right]\right)\right).\end{aligned}$$

This means that r' violates $X \xrightarrow{\theta_1}_F Y$.

Let

$$\begin{aligned}&f^{-1}\Big(i_{r,\beta}\left(\wedge_{A\in X} A\right)\cdot \\ &\quad f\left(\max\left(i_{r,\beta}\left(\wedge_{B\in Y} B\right), i_{r,\beta}\left(\wedge_{D\in Z} D\right)\right)\right)\Big) \\ &= i_{r,\beta}\left(\left(\wedge_{A\in X} A\right) \Rightarrow \left(\left(\wedge_{B\in Y} B\right) \vee \left(\wedge_{D\in Z} D\right)\right)\right) \\ &= i_{r,\beta}\left(c'\right) \leq \frac{1}{2}.\end{aligned}$$

Now,

$$\begin{aligned}f\left(\frac{1}{2}\right) &\leq i_{r,\beta}\left(\wedge_{A\in X} A\right)\cdot \\ &\quad f\left(\max\left(i_{r,\beta}\left(\wedge_{B\in Y} B\right), i_{r,\beta}\left(\wedge_{D\in Z} D\right)\right)\right).\end{aligned}$$

Suppose that,

$$f\left(\max\left(i_{r,\beta}\left(\wedge_{B\in Y} B\right), i_{r,\beta}\left(\wedge_{D\in Z} D\right)\right)\right) \neq 0, +\infty.$$

We have,

$$\begin{aligned}&\frac{f\left(\frac{1}{2}\right)}{f\left(\max\left(i_{r,\beta}\left(\wedge_{B\in Y} B\right), i_{r,\beta}\left(\wedge_{D\in Z} D\right)\right)\right)} \\ &\leq i_{r,\beta}\left(\wedge_{A\in X} A\right).\end{aligned}$$

If we assume that $i_{r,\beta}\left(\wedge_{A\in X} A\right) \leq \frac{1}{2}$, then, reasoning in the same way as in the case of fuzzy functional dependency $X \xrightarrow{\theta_1}_F Y$, we obtain that $f\left(\frac{1}{2}\right) = 0$, i.e., $\frac{1}{2} = 1$. This is a contradiction.

Hence, $i_{r,\beta}\left(\wedge_{A\in X} A\right) > \frac{1}{2}$.

As before, this inequality yields that $\varphi\left(X\left[t', t''\right]\right) = 1$.

Moreover,

$$\begin{aligned}&\frac{f\left(\frac{1}{2}\right)}{f\left(\max\left(i_{r,\beta}\left(\wedge_{B\in Y} B\right), i_{r,\beta}\left(\wedge_{D\in Z} D\right)\right)\right)} \\ &\leq i_{r,\beta}\left(\wedge_{A\in X} A\right)\end{aligned}$$

yields that

$$\begin{aligned}&\frac{f\left(\frac{1}{2}\right)}{f\left(\max\left(i_{r,\beta}\left(\wedge_{B\in Y} B\right), i_{r,\beta}\left(\wedge_{D\in Z} D\right)\right)\right)} \\ &\leq \frac{1}{2} < 1,\end{aligned}$$

i.e.,

$$f\left(\frac{1}{2}\right) < f\left(\max\left(i_{r,\beta}\left(\wedge_{B\in Y} B\right), i_{r,\beta}\left(\wedge_{D\in Z} D\right)\right)\right),$$

i.e.,

$$\frac{1}{2} > \max\left(i_{r,\beta}\left(\wedge_{B\in Y} B\right), i_{r,\beta}\left(\wedge_{D\in Z} D\right)\right).$$

This means that $i_{r,\beta}\left(\wedge_{B\in Y} B\right) < \frac{1}{2}$ and $i_{r,\beta}\left(\wedge_{D\in Z} D\right) < \frac{1}{2}$.

Hence, $\varphi\left(Y\left[t', t''\right]\right) = \theta''$ and $\varphi\left(Z\left[t', t''\right]\right) = \theta''$.

If $t''' \in r'$ and $t''' = t'$, we have

$$\begin{aligned}\varphi\left(X\left[t''', t'\right]\right) &= 1 \geq \min\left(\theta_1, \varphi\left(X\left[t', t''\right]\right)\right), \\ \varphi\left(Y\left[t''', t'\right]\right) &= 1 \geq \min\left(\theta_1, \varphi\left(X\left[t', t''\right]\right)\right), \\ \varphi\left(Z\left[t''', t''\right]\right) &= \theta'' < \theta' \leq \theta_1 \\ &= \min\left(\theta_1, \varphi\left(X\left[t', t''\right]\right)\right).\end{aligned}$$

Similarly, if $t''' \in r'$ and $t''' = t''$, we obtain

$$\begin{aligned}\varphi\left(X\left[t''', t'\right]\right) &= 1 \geq \min\left(\theta_1, \varphi\left(X\left[t', t''\right]\right)\right), \\ \varphi\left(Y\left[t''', t'\right]\right) &= \theta'' < \theta' \leq \theta_1\end{aligned}$$

$$= \min\left(\theta_1, \varphi\left(X\left[t', t''\right]\right)\right),$$
$$\varphi\left(Z\left[t''', t''\right]\right) = 1 \geq \min\left(\theta_1, \varphi\left(X\left[t', t''\right]\right)\right).$$

Consequently, r' does not satisfy the dependency $X \rightarrow\xrightarrow{\theta_1}_F Y$.

(*b*) ⇒ (*a*) Assume that (*a*) does not hold.

Then, there exists some two-element fuzzy relation instance $r' = \left\{t', t''\right\}$ on $R(U)$ such that r' satisfies all dependencies from the set C, and violates the dependency c.

Thus, r' violates $X \xrightarrow{\theta_1}_F Y$ resp. $X \rightarrow\xrightarrow{\theta_1}_F Y$.

Let $Z' = \left\{A \in U \mid \varphi\left(A\left[t', t''\right]\right) = 1\right\}$.

Suppose that $Z' = \emptyset$.

Then, $\varphi\left(A\left[t', t''\right]\right) = \theta''$ for every $A \in U$.

Hence, $\varphi\left(Q\left[t', t''\right]\right) = \theta''$ for every $Q \subseteq U$.

If r' violates $X \xrightarrow{\theta_1}_F Y$, then, we have

$$\varphi\left(Y\left[t', t''\right]\right) < \min\left(\theta_1, \varphi\left(X\left[t', t''\right]\right)\right).$$

We obtain,

$$\theta'' < \min\left(\theta_1, \theta''\right) = \theta''.$$

This is a contradiction.

Suppose that r' violates $X \rightarrow\xrightarrow{\theta_1}_F Y$.

Then, the conditions

$$\begin{aligned} \varphi\left(X\left[t', t'\right]\right) &\geq \min\left(\theta_1, \varphi\left(X\left[t', t''\right]\right)\right), \\ \varphi\left(Y\left[t', t'\right]\right) &\geq \min\left(\theta_1, \varphi\left(X\left[t', t''\right]\right)\right), \\ \varphi\left(Z\left[t', t''\right]\right) &\geq \min\left(\theta_1, \varphi\left(X\left[t', t''\right]\right)\right) \end{aligned} \quad (6)$$

are not all satisfied at the same time.

The first and the second condition in (6) are always satisfied. Hence, it must be

$$\begin{aligned} \theta'' = \varphi\left(Z\left[t', t''\right]\right) &< \min\left(\theta_1, \varphi\left(X\left[t', t''\right]\right)\right) \\ &= \min\left(\theta_1, \theta''\right) = \theta''. \end{aligned}$$

This is a contradiction.

We conclude, $Z' \neq \emptyset$.

Now, suppose that $Z' = U$.

Then, $\varphi\left(A\left[t', t''\right]\right) = 1$ for all $A \in U$.

Hence, $\varphi\left(Q\left[t', t''\right]\right) = 1$ for all $Q \subseteq U$.

If r' violates $X \xrightarrow{\theta_1}_F Y$, we obtain

$$\begin{aligned} 1 = \varphi\left(Y\left[t', t''\right]\right) &< \min\left(\theta_1, \varphi\left(X\left[t', t''\right]\right)\right) \\ &= \min\left(\theta_1, 1\right) = \theta_1. \end{aligned}$$

This is a contradiction.

If r' violates $X \rightarrow\xrightarrow{\theta_1}_F Y$, then the conditions given by (6) are not all satisfied at the same time.

Since the first and the second conditions in (6) always hold true, we conclude that

$$\begin{aligned} 1 = \varphi\left(Z\left[t', t''\right]\right) &< \min\left(\theta_1, \varphi\left(X\left[t', t''\right]\right)\right) \\ &= \min\left(\theta_1, 1\right) = \theta_1. \end{aligned}$$

This is a contradiction.

Hence, $Z' \neq U$.

We join $i_{r',1}$ to r' and $1 \in [0, 1]$.

We have,

$$\begin{aligned} i_{r',1}(A) &\in \left(\frac{1}{2}, 1\right] \text{ if } \varphi\left(A\left[t', t''\right]\right) = 1, \\ i_{r',1}(A) &\in \left[0, \frac{1}{2}\right] \text{ if } \varphi\left(A\left[t', t''\right]\right) < 1. \end{aligned}$$

In other words,

$$\begin{aligned} i_{r',1}(A) &\in \left(\frac{1}{2}, 1\right] \text{ if } A \in Z', \\ i_{r',1}(A) &\in \left[0, \frac{1}{2}\right] \text{ if } A \in U \backslash Z'. \end{aligned}$$

We shall prove that $i_{r',1}(\mathcal{K}) > \frac{1}{2}$ for all $\mathcal{K} \in C'$ and $i_{r',1}\left(c'\right) \leq \frac{1}{2}$.

First, let $\mathcal{K} \in C'$ be of the form

$$\left(\wedge_{A \in K} A\right) \Rightarrow \left(\wedge_{B \in L} B\right).$$

This implication corresponds to some fuzzy functional dependency $K \xrightarrow{\theta_2}_F L$ from the set C.

Since r' satisfies $K \xrightarrow{\theta_2}_F L$, we have that

$$\varphi\left(L\left[t', t''\right]\right) \geq \min\left(\theta_2, \varphi\left(K\left[t', t''\right]\right)\right). \quad (7)$$

Suppose that

$$i_{r',1}\left(\left(\wedge_{A\in K}A\right)\Rightarrow\left(\wedge_{B\in L}B\right)\right)\leq\frac{1}{2}.$$

Now,

$$\frac{1}{2}\geq f^{-1}\left(i_{r',1}\left(\wedge_{A\in K}A\right)f\left(i_{r',1}\left(\wedge_{B\in L}B\right)\right)\right),$$

i.e.,

$$f\left(\frac{1}{2}\right)\leq i_{r',1}\left(\wedge_{A\in K}A\right)f\left(i_{r',1}\left(\wedge_{B\in L}B\right)\right).$$

Assume that,

$$f\left(i_{r',1}\left(\wedge_{B\in L}B\right)\right)\neq 0,+\infty.$$

We obtain,

$$\frac{f\left(\frac{1}{2}\right)}{f\left(i_{r',1}\left(\wedge_{B\in L}B\right)\right)}\leq i_{r',1}\left(\wedge_{A\in K}A\right).$$

As earlier, the assumption $i_{r',1}\left(\wedge_{A\in K}A\right)\leq\frac{1}{2}$ yields the contradiction $f\left(\frac{1}{2}\right)=0$.

Hence, $i_{r',1}\left(\wedge_{A\in K}A\right)>\frac{1}{2}$.

Therefore, $i_{r',1}\left(A\right)>\frac{1}{2}$ for all $A\in K$, i.e., $\varphi\left(A\left[t',t''\right]\right)=1$ for all $A\in K$, i.e., $\varphi\left(K\left[t',t''\right]\right)=1$.

Moreover, the inequality

$$\frac{f\left(\frac{1}{2}\right)}{f\left(i_{r',1}\left(\wedge_{B\in L}B\right)\right)}\leq i_{r',1}\left(\wedge_{A\in K}A\right).$$

yields that

$$\frac{f\left(\frac{1}{2}\right)}{f\left(i_{r',1}\left(\wedge_{B\in L}B\right)\right)}\leq\frac{1}{2}<1.$$

Now,

$$f\left(\frac{1}{2}\right)<f\left(i_{r',1}\left(\wedge_{B\in L}B\right)\right),$$

i.e.,

$$\frac{1}{2}>i_{r',1}\left(\wedge_{B\in L}B\right).$$

This means that there is $B\in L$ such that $i_{r',1}\left(B\right)<\frac{1}{2}$, i.e., $\varphi\left(B\left[t',t''\right]\right)<1$, i.e., $\varphi\left(B\left[t',t''\right]\right)=\theta''$.

Consequently, $\varphi\left(L\left[t',t''\right]\right)=\theta''$.

Now, by (7),

$$\begin{aligned}\theta''=\varphi\left(L\left[t',t''\right]\right)&\geq\min\left(\theta_2,\varphi\left(K\left[t',t''\right]\right)\right)\\&=\min\left(\theta_2,1\right)=\theta_2.\end{aligned}$$

This is a contradiction.

We condlude,

$$i_{r',1}\left(\left(\wedge_{A\in K}A\right)\Rightarrow\left(\wedge_{B\in L}B\right)\right)>\frac{1}{2}.$$

Second, let $\mathcal{K}\in C'$ be of the form

$$\left(\wedge_{A\in K}A\right)\Rightarrow\left(\left(\wedge_{B\in L}B\right)\vee\left(\wedge_{D\in M}D\right)\right),$$

where $M=U\setminus\left(K\cup L\right)$.

Note that $\mathcal{K}$ corresponds to some fuzzy multivalued dependency $K\rightarrow\xrightarrow{\theta_2}_F L$ from the set C.

Since r' satisfies $K\rightarrow\xrightarrow{\theta_2}_F L$, we know that there exists $t'''\in r'$ such that

$$\begin{aligned}\varphi\left(K\left[t''',t'\right]\right)&\geq\min\left(\theta_2,\varphi\left(K\left[t',t''\right]\right)\right),\\\varphi\left(L\left[t''',t'\right]\right)&\geq\min\left(\theta_2,\varphi\left(K\left[t',t''\right]\right)\right),\\\varphi\left(M\left[t''',t''\right]\right)&\geq\min\left(\theta_2,\varphi\left(K\left[t',t''\right]\right)\right).\end{aligned}$$

Suppose that,

$$i_{r',1}\left(\left(\wedge_{A\in K}A\right)\Rightarrow\left(\left(\wedge_{B\in L}B\right)\vee\left(\wedge_{D\in M}D\right)\right)\right)\leq\frac{1}{2}.$$

Now,

$$\begin{aligned}\frac{1}{2}\geq&f^{-1}\Big(i_{r',1}\left(\wedge_{A\in K}A\right)\cdot\\&f\left(\max\left(i_{r',1}\left(\wedge_{B\in L}B\right),i_{r',1}\left(\wedge_{D\in M}D\right)\right)\right)\Big),\end{aligned}$$

i.e.,

$$\begin{aligned}f\left(\frac{1}{2}\right)\leq&i_{r',1}\left(\wedge_{A\in K}A\right)\cdot\\&f\Big(\max\Big(i_{r',1}\left(\wedge_{B\in L}B\right),\\&i_{r',1}\left(\wedge_{D\in M}D\right)\Big)\Big).\end{aligned}$$

Assume that,

$$f\left(\max\left(i_{r',1}\left(\wedge_{B\in L}B\right), i_{r',1}\left(\wedge_{D\in M}D\right)\right)\right) \neq 0, +\infty.$$

We have,

$$\frac{f\left(\frac{1}{2}\right)}{f\left(\max\left(i_{r',1}\left(\wedge_{B\in L}B\right), i_{r',1}\left(\wedge_{D\in M}D\right)\right)\right)} \leq i_{r',1}\left(\wedge_{A\in K}A\right).$$

If $i_{r',1}\left(\wedge_{A\in K}A\right) \leq \frac{1}{2}$, then $f\left(\frac{1}{2}\right) = 0$, i.e., $\frac{1}{2} = 1$.

This is a contradiction.

Hence, $i_{r',1}\left(\wedge_{A\in K}A\right) > \frac{1}{2}$.

Therefore, $i_{r',1}(A) > \frac{1}{2}$ for all $A \in K$, i.e., $\varphi\left(A\left[t', t''\right]\right) = 1$ for all $A \in K$, i.e., $\varphi\left(K\left[t', t''\right]\right) = 1$.

Moreover, the inequality

$$\frac{f\left(\frac{1}{2}\right)}{f\left(\max\left(i_{r',1}\left(\wedge_{B\in L}B\right), i_{r',1}\left(\wedge_{D\in M}D\right)\right)\right)} \leq i_{r',1}\left(\wedge_{A\in K}A\right).$$

yields that

$$\frac{f\left(\frac{1}{2}\right)}{f\left(\max\left(i_{r',1}\left(\wedge_{B\in L}B\right), i_{r',1}\left(\wedge_{D\in M}D\right)\right)\right)} \leq \frac{1}{2} < 1.$$

Hence,

$$f\left(\frac{1}{2}\right) < f\left(\max\left(i_{r',1}\left(\wedge_{B\in L}B\right), i_{r',1}\left(\wedge_{D\in M}D\right)\right)\right),$$

i.e.,

$$\frac{1}{2} > \max\left(i_{r',1}\left(\wedge_{B\in L}B\right), i_{r',1}\left(\wedge_{D\in M}D\right)\right).$$

We conclude, $i_{r',1}\left(\wedge_{B\in L}B\right) < \frac{1}{2}$ and $i_{r',1}\left(\wedge_{D\in M}D\right) < \frac{1}{2}$.

This means that there exist $B \in L$ and $D \in M$ such that $i_{r',1}(B) < \frac{1}{2}$ and $i_{r',1}(D) < \frac{1}{2}$, i.e.,

$$\varphi\left(B\left[t', t''\right]\right) < 1 \text{ and } \varphi\left(D\left[t', t''\right]\right) < 1,$$

i.e.,

$$\varphi\left(B\left[t', t''\right]\right) = \theta'' \text{ and } \varphi\left(D\left[t', t''\right]\right) = \theta'',$$

i.e.,

$$\varphi\left(L\left[t', t''\right]\right) = \theta'' \text{ and } \varphi\left(M\left[t', t''\right]\right) = \theta''.$$

Now the third condition of the conditions

$$\begin{aligned}\varphi\left(K\left[t', t'\right]\right) &\geq \min\left(\theta_2, \varphi\left(K\left[t', t''\right]\right)\right),\\ \varphi\left(L\left[t', t'\right]\right) &\geq \min\left(\theta_2, \varphi\left(K\left[t', t''\right]\right)\right),\\ \varphi\left(M\left[t', t''\right]\right) &\geq \min\left(\theta_2, \varphi\left(K\left[t', t''\right]\right)\right)\end{aligned}$$

is not satisfied.

Similarly, the second condition of the conditions

$$\begin{aligned}\varphi\left(K\left[t'', t'\right]\right) &\geq \min\left(\theta_2, \varphi\left(K\left[t', t''\right]\right)\right),\\ \varphi\left(L\left[t'', t'\right]\right) &\geq \min\left(\theta_2, \varphi\left(K\left[t', t''\right]\right)\right),\\ \varphi\left(M\left[t'', t''\right]\right) &\geq \min\left(\theta_2, \varphi\left(K\left[t', t''\right]\right)\right)\end{aligned}$$

is not satisfied.

This means that r' does not satisfy the dependency $K \to\xrightarrow{\theta_2}_F L$.

Hence, a contradiction.

We conclude,

$$i_{r',1}\left(\left(\wedge_{A\in K}A\right) \Rightarrow \left(\left(\wedge_{B\in L}B\right) \vee \left(\wedge_{D\in M}D\right)\right)\right) > \frac{1}{2}.$$

It remains to prove that $i_{r',1}\left(c'\right) \leq \frac{1}{2}$.

Let r' violates $X \xrightarrow{\theta_1}_F Y$.

We have,

$$\varphi\left(Y\left[t', t''\right]\right) < \min\left(\theta_1, \varphi\left(X\left[t', t''\right]\right)\right). \quad (8)$$

Suppose that,

$$f^{-1}\left(i_{r',1}\left(\wedge_{A\in X}A\right)f\left(i_{r',1}\left(\wedge_{B\in Y}B\right)\right)\right)$$
$$=i_{r',1}\left(\left(\wedge_{A\in X}A\right)\Rightarrow\left(\wedge_{B\in Y}B\right)\right)$$
$$=i_{r',1}\left(c'\right)>\frac{1}{2}.$$

Now,

$$f\left(\frac{1}{2}\right)>i_{r',1}\left(\wedge_{A\in X}A\right)f\left(i_{r',1}\left(\wedge_{B\in Y}B\right)\right).$$

If $f\left(i_{r',1}\left(\wedge_{B\in Y}B\right)\right)=0$, then $i_{r',1}\left(\wedge_{B\in Y}B\right)=1$.

Hence, $i_{r',1}\left(B\right)=1$ for all $B\in Y$, i.e., $\varphi\left(B\left[t',t''\right]\right)=1$ for all $B\in Y$, i.e., $\varphi\left(Y\left[t',t''\right]\right)=1$.

This contradicts (8).

We conclude, $f\left(i_{r',1}\left(\wedge_{B\in Y}B\right)\right)>0$, i.e., $i_{r',1}\left(\wedge_{B\in Y}B\right)<1$.

Suppose that $f\left(i_{r',1}\left(\wedge_{B\in Y}B\right)\right)\neq 0,+\infty$.

If $i_{r',1}\left(\wedge_{B\in Y}B\right)>\frac{1}{2}$, then $i_{r',1}\left(B\right)>\frac{1}{2}$ for all $B\in Y$, i.e., $\varphi\left(B\left[t',t''\right]\right)=1$ for all $B\in Y$, i.e., $\varphi\left(Y\left[t',t''\right]\right)=1$.

This contradicts (8).

Therefore, $i_{r',1}\left(\wedge_{B\in Y}B\right)\leq\frac{1}{2}$.

Now, $f\left(i_{r',1}\left(\wedge_{B\in Y}B\right)\right)\geq f\left(\frac{1}{2}\right)$, i.e.,

$$\frac{\left(\frac{1}{2}\right)}{f\left(i_{r',1}\left(\wedge_{B\in Y}B\right)\right)}\leq 1.$$

Since,

$$i_{r',1}\left(\wedge_{A\in X}A\right)<\frac{\left(\frac{1}{2}\right)}{f\left(i_{r',1}\left(\wedge_{B\in Y}B\right)\right)},$$

and $f\left(i_{r',1}\left(\wedge_{B\in Y}B\right)\right)\neq+\infty$, we obtain that $i_{r',1}\left(\wedge_{A\in X}A\right)=0$.

Hence, there is $A\in X$ such that $i_{r',1}\left(A\right)=0$, i.e., $\varphi\left(A\left[t',t''\right]\right)<1$, i.e., $\varphi\left(A\left[t',t''\right]\right)=\theta''$, i.e., $\varphi\left(X\left[t',t''\right]\right)=\theta''$.

Similarly, $i_{r',1}\left(\wedge_{B\in Y}B\right)\leq\frac{1}{2}$ yields that $\varphi\left(Y\left[t',t''\right]\right)=\theta''$.

Now, by (8)

$$\theta''=\varphi\left(Y\left[t',t''\right]\right)<\min\left(\theta_1,\varphi\left(X\left[t',t''\right]\right)\right)$$
$$=\min\left(\theta_1,\theta''\right)=\theta''.$$

This is a contradiction.

Consequently,

$$i_{r',1}\left(\left(\wedge_{A\in X}A\right)\Rightarrow\left(\wedge_{B\in Y}B\right)\right)$$
$$=i_{r',1}\left(c'\right)\leq\frac{1}{2}.$$

Let r' violates $X\xrightarrow{\theta_1}_F Y$.

Now, the third condition in (6) is not satisfied, i.e., the following inequality holds true.

$$\varphi\left(Z\left[t',t''\right]\right)<\min\left(\theta_1,\varphi\left(X\left[t',t''\right]\right)\right). \quad (9)$$

Similarly, the second condition of the conditions

$$\varphi\left(X\left[t'',t'\right]\right)\geq\min\left(\theta_1,\varphi\left(X\left[t',t''\right]\right)\right),$$
$$\varphi\left(Y\left[t'',t'\right]\right)\geq\min\left(\theta_1,\varphi\left(X\left[t',t''\right]\right)\right),$$
$$\varphi\left(Z\left[t'',t'\right]\right)\geq\min\left(\theta_1,\varphi\left(X\left[t',t''\right]\right)\right)$$

is not satisfied, i.e., the following inequality holds true.

$$\varphi\left(Y\left[t'',t'\right]\right)<\min\left(\theta_1,\varphi\left(X\left[t',t''\right]\right)\right). \quad (10)$$

Suppose that,

$$f^{-1}\Big(i_{r',1}\left(\wedge_{A\in X}A\right)\cdot$$
$$f\left(\max\left(i_{r',1}\left(\wedge_{B\in Y}B\right),i_{r',1}\left(\wedge_{D\in Z}D\right)\right)\right)\Big)$$
$$=i_{r',1}\left(\left(\wedge_{A\in X}A\right)\Rightarrow\left(\left(\wedge_{B\in Y}B\right)\vee\left(\wedge_{D\in Z}D\right)\right)\right)$$
$$=i_{r',1}\left(c'\right)>\frac{1}{2}.$$

Now,

$$f\left(\frac{1}{2}\right)>i_{r',1}\left(\wedge_{A\in X}A\right)\cdot$$
$$f\Big(\max\Big(i_{r',1}\left(\wedge_{B\in Y}B\right),$$
$$i_{r',1}\left(\wedge_{D\in Z}D\right)\Big)\Big).$$

If

$$f\left(\max\left(i_{r',1}\left(\wedge_{B\in Y}B\right),i_{r',1}\left(\wedge_{D\in Z}D\right)\right)\right)=0$$

then

$$\max\left(i_{r',1}\left(\wedge_{B\in Y}B\right),i_{r',1}\left(\wedge_{D\in Z}D\right)\right)=1,$$

i.e., $i_{r',1}\left(\wedge_{B\in Y}B\right)=1$ or $i_{r',1}\left(\wedge_{D\in Z}D\right)=1$.

If $i_{r',1}\left(\wedge_{B\in Y}B\right)=1$, then reasoning as before, we obtain that $\varphi\left(Y\left[t',t''\right]\right)=1$.

This contradicts (10).

If $i_{r',1}\left(\wedge_{D\in Z}D\right)=1$, then $\varphi\left(Z\left[t',t''\right]\right)=1$.

This contradicts (9).

Therefore,

$$f\left(\max\left(i_{r',1}\left(\wedge_{B\in Y}B\right),i_{r',1}\left(\wedge_{D\in Z}D\right)\right)\right)>0,$$

i.e.,

$$\max\left(i_{r',1}\left(\wedge_{B\in Y}B\right),i_{r',1}\left(\wedge_{D\in Z}D\right)\right)<1.$$

Suppose that,

$$f\left(\max\left(i_{r',1}\left(\wedge_{B\in Y}B\right),i_{r',1}\left(\wedge_{D\in Z}D\right)\right)\right)\neq+\infty.$$

If

$$\max\left(i_{r',1}\left(\wedge_{B\in Y}B\right),i_{r',1}\left(\wedge_{D\in Z}D\right)\right)>\frac{1}{2}$$

then $i_{r',1}\left(\wedge_{B\in Y}B\right)>\frac{1}{2}$ or $i_{r',1}\left(\wedge_{D\in Z}D\right)>\frac{1}{2}$.

In the first case, we have that $\varphi\left(Y\left[t',t''\right]\right)=1$.

This contradicts (10).

In the second case, we have that $\varphi\left(Z\left[t',t''\right]\right)=1$.

This contradicts (9).

Therefore,

$$\max\left(i_{r',1}\left(\wedge_{B\in Y}B\right),i_{r',1}\left(\wedge_{D\in Z}D\right)\right)\leq\frac{1}{2}.$$

Now,

$$\begin{aligned}&f\left(\max\left(i_{r',1}\left(\wedge_{B\in Y}B\right),i_{r',1}\left(\wedge_{D\in Z}D\right)\right)\right)\\&\geq f\left(\frac{1}{2}\right),\end{aligned}$$

i.e.,

$$\frac{f\left(\frac{1}{2}\right)}{f\left(\max\left(i_{r',1}\left(\wedge_{B\in Y}B\right),i_{r',1}\left(\wedge_{D\in Z}D\right)\right)\right)}\leq1.$$

Since,

$$\begin{aligned}&i_{r',1}\left(\wedge_{A\in X}A\right)\\&<\frac{f\left(\frac{1}{2}\right)}{f\left(\max\left(i_{r',1}\left(\wedge_{B\in Y}B\right),i_{r',1}\left(\wedge_{D\in Z}D\right)\right)\right)}\end{aligned}$$

and

$$f\left(\max\left(i_{r',1}\left(\wedge_{B\in Y}B\right),i_{r',1}\left(\wedge_{D\in Z}D\right)\right)\right)\neq+\infty,$$

we conclude that $i_{r',1}\left(\wedge_{A\in X}A\right)=0$.

Hence, reasoning as earlier, we obtain that $\varphi\left(X\left[t',t''\right]\right)=\theta''$.

Similarly,

$$\max\left(i_{r',1}\left(\wedge_{B\in Y}B\right),i_{r',1}\left(\wedge_{D\in Z}D\right)\right)\leq\frac{1}{2}$$

yields that $i_{r',1}\left(\wedge_{B\in Y}B\right)\leq\frac{1}{2}$ and $i_{r',1}\left(\wedge_{D\in Z}D\right)\leq\frac{1}{2}$.

Hence, $\varphi\left(Y\left[t',t''\right]\right)=\theta''$ and $\varphi\left(Z\left[t',t''\right]\right)=\theta''$.

Now,

$$\begin{aligned}\theta''=\varphi\left(Z\left[t',t''\right]\right)&<\min\left(\theta_1,\varphi\left(X\left[t',t''\right]\right)\right)\\&=\min\left(\theta_1,\theta''\right)=\theta''\end{aligned}$$

contradicts (9), and

$$\begin{aligned}\theta''=\varphi\left(Y\left[t',t''\right]\right)&<\min\left(\theta_1,\varphi\left(X\left[t',t''\right]\right)\right)\\&=\min\left(\theta_1,\theta''\right)=\theta''\end{aligned}$$

contradicts (10).

Consequently,

$$\begin{aligned}&i_{r',1}\left(\left(\wedge_{A\in X}A\right)\Rightarrow\left(\left(\wedge_{B\in Y}B\right)\vee\left(\wedge_{D\in Z}D\right)\right)\right)\\&=i_{r',1}\left(c'\right)\leq\frac{1}{2}.\end{aligned}$$

This completes the proof. □

4 Applications of continuous maps in fuzzy relations and concluding remarks

In [22], Yager applied continuous maps to introduce f-generated and g-generated implications (see also, [20]). From his work, it appears that the motivation to define these two families stems from a desire to study and exploit the role of fuzzy implications (and hence continuous maps) in approximate reasoning. Yager's fuzzy implications are obtained from continuous maps that are either strictly decreasing or increasing with the unit interval $[0,1]$ as their domain and $[0,+\infty]$ as their codomain. Analogously, one can try to obtain fuzzy implications from such maps whose codomain is also $[0,1]$. One such attempt was made by Balasubramaniam [9], [10], where a new class of fuzzy implications, called $h-$implications has been proposed.

Continuous maps have been extensively used in the theory of fuzzy transforms (firstly proposed by Perfilieva [11]). The fuzzy transform is a method that can be applied to a continuous function on a bounded domain or to a discrete function on a finite domain. In particular, the domain $[a,b]$ (which is an interval of real numbers with nods $a = x_0 = x_1 < ... < x_n = x_{n+1} = b$, $n \geq 2$) is supposed to be partitioned by fuzzy sets A_1, A_2,..., A_n identified with their membership functions $A_1(x)$, $A_2(x)$,..., $A_n(x) : [a,b] \to [0,1]$. More precisely, A_1, A_2,..., A_n form a partition of $[a,b]$, i.e., $A_1(x)$, $A_2(x)$,..., $A_n(x)$ are basic functions, if for $k \in \{1,2,...,n\}$: (1) $A_k(x_k) = 1$, (2) $A_k(x) = 0$ for $x \notin (x_{k-1}, x_{k+1})$, (3) A_k is continuous, (4) A_k is strictly increasing (decreasing) on $[x_{k-1}, x_k]$ ($[x_k, x_{k+1}]$) for $k \in \{2,3,...,n\}$ ($k \in \{1,2,...,n-1\}$), (5) $\sum_{k=1}^{n} A_k(x) = 1$ for all $x \in [a,b]$. The F-transform $\mathbb{F}_n[f]$ of a function $f \in C[a,b]$ with respect to basic functions $A_1(x)$, $A_2(x)$,..., $A_n(x)$ is the n-tuple $[F_1, F_2, ..., F_n] \in \mathbb{R}^n$, where

$$F_k = \frac{\int_a^b f(x) A_k(x)\, dx}{\int_a^b A_k(x)\, dx},$$

$k \in \{1,2,...,n\}$. The elements F_1, F_2,..., $F_n \in \mathbb{R}$ are called the components of the F-transform $\mathbb{F}_n[f]$. The F-transform $\mathbb{F}_n[f]$ (with respect to $A_1(x)$, $A_2(x)$,..., $A_n(x)$) is a linear map, i.e.,

$$\mathbb{F}_n[\alpha f + \beta g] = \alpha \mathbb{F}_n[f] + \beta \mathbb{F}_n[g]$$

for α, $\beta \in \mathbb{R}$ and f, $g \in C[a,b]$. On the basis of knowledge of the components, one defines the function

$$f_{F,n}(x) = \sum_{i=1}^{n} F_i A_i(x),$$

$x \in [a,b]$. This function is called the inverse F-transform of f with respect to $A_1(x)$, $A_2(x)$,..., $A_n(x)$. It approximates f with arbitrary precision in the sense that for every $\varepsilon > 0$, there exists an integer $n(\varepsilon)$, and a fuzzy partition A_1, A_2,..., $A_{n(\varepsilon)}$ of $[a,b]$ such that for $x \in [a,b]$

$$\left| f(x) - f_{F,n(\varepsilon)}(x) \right| < \varepsilon.$$

The theory of F-transform is successfully applied in signal and image processing (see, e.g., [7]). In [7], the authors applied the coding-decoding method of image processing based on an application of the direct and inverse fuzzy transform, An image is considered as a fuzzy relation which is divided into blocks. Each block is compressed by discrete fuzzy transform (of a function in two variables), and successively decompressed by inverse fuzzy transform. The decompressed blocks are recomposed for the reconstruction of the image, whose quality is evaluated by calculating the peak signal to noise ratio (with respect to the original image).

F-transforms are applied in compression [8]. There, the authors compress certain areas (neighbourhoods of edges). The method is based on similarity between various blocks and the compression of only one representative.

Further application of F-transform is in denoising [13]. A particular interest is paid to the problem of removing noise, i.e., to a method based on nonlinear signal processing.

F-transforms are successfully applied in numerical solutions of partial differential equations (see, e.g., [19]). In [19], three types of partial differential equations (with physical background) are considered: heat equation, wave equation and Poisson's equation. These equations (on a domain $D = X \times X$) have the same form

$$\mathbb{L}\left(\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = q(x,y),$$

where $\mathbb{L}$ is a linear form. Continuous functions within differential equations are replaced by their discrete representations. The obtained systems of algebraic equations are solvable by existing numerical methods. Finally, the numerical solutions are transformed back into continuous ones.

In [14], F-transform is applied in data analysis. The method is used for detection and characterization of dependencies between attributes.

In [18] however, F-transform is applied within neural networks.

The concepts of fuzzy rough sets, and fuzzy topologies are associated with fuzzy transform in [12]. In [12] it is shown that the fuzzy transform can be represented as a fuzzy approximation operator (studied in the operator-oriented view of fuzzy rough set theory). Moreover, it is shown that the use of fuzzy rough sets results reduces efforts in proving fuzzy transform theoretic results. As noted in [12], the fact that the concept of fuzzy topology can be naturally associated with the theory of fuzzy rough sets, yields that there is a possibility of establishing a connection between the theory of fuzzy topology and the theory of F-transform.

Galois connection (within category theory) plays an important role in establishing relationships between various spatial structures. In [15], the authors proved that there are several Galois connections between the category of Alexandroff L-fuzzy topological spaces and the category of reflexive L-fuzzy relations.

In [3], the authors derived various properties of C-metric spaces: convergence properties, a canonical decomposition, a C-fixed point theorem, where C-metric on X is a real map on $X \times X$ which satisfies only two metric axioms: symmetry and triangular inequality (this means that C-metric is an approximate metric, i.e., a real map on $X \times X$ which satisfies only a part of the metric axioms).

The results derived in this paper, i.e., Theorems B and C, generalize the corresponding results obtained in [4] for Yager fuzzy implication operator, as well as the results obtained in [5] for Reichenbach fuzzy implication operator (see also, [6] for Kleene-Dienes fuzzy implication approach).

Namely, if we take the f-generator $f(x) = -\log x$, $x \in [0,1]$ (with $f(0) = +\infty$), then the corresponding f-generated implication is the Yager fuzzy implication [21].

Moreover, if we take the f-generator $f(x) = 1 - x$, $x \in [0,1]$ (with $f(0) = 1$), then the corresponding f-generated implication is the Reichenbach fuzzy implication (or Kleene-Dienes-Lukasiewicz fuzzy implication) [16].

The author is convinced that Theorems B and C could be verified for the family of g-generated implications (see, [1]).

For the family of h-generated implications see [9] and [10].

For detailed study on fuzzy implications we refer to [1].

For detailed study on fuzzy functional and fuzzy multivalued dependencies that we apply in this paper, we refer to [17] (see also [2] when crisp attributes are included).

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