Proof: Since,

$$\begin{aligned}\sum_{p=2}^{\infty}\left(p\cdot x^p\right) &= 2x^2+3x^3+4x^4+\cdots \\ &= x+2x^2+3x^3+4x^4+\cdots-x \\ &= x(1+2x+3x^2+4x^3+\cdots)-x \\ &= x\frac{d}{dx}(1+x+x^2+x^3+x^4+\cdots)-x\end{aligned} \quad \text{(A8)}$$

Writing the term in brackets as a su mmation produces,

$$\sum_{p=2}^{\infty}\left(p\cdot x^p\right)= x\frac{d}{dx}\left(\sum_{p=0}^{\infty}x^p\right)-x \quad \text{(A9)}$$

Recognizing the summation appearing on the right-hand side of this equation as (A1) produces,

$$\begin{aligned}\sum_{p=2}^{\infty}\left(p\cdot x^p\right) &= x\frac{d}{dx}\left(\frac{1}{1-x}\right)-x \\ &= x\left(\frac{1}{(x-1)^2}\right)-x\end{aligned} \quad \text{(A10)}$$

This finally produces,

$$\sum_{p=2}^{\infty}\left(p\cdot x^p\right)=\frac{x}{(x-1)^2}-x \qquad |x|<1 \quad \text{(A7)}$$

□

A.3 The sum of an infinite series of the form p^2x^p starting at $p = 2$

The sum of an infinite series of the form p^2x^p starting at $p = 2$ is,

$$\sum_{p=2}^{\infty}\left(p^2\cdot x^p\right)= x\left(\frac{1+x}{(1-x)^3}-1\right) \qquad |x|<1 \quad \text{(A11)}$$

Proof: Since,

$$\begin{aligned}\sum_{p=2}^{\infty}\left(p^2\cdot x^p\right) &= 4x^2+9x^3+16x^4+\cdots \\ &= (x+4x^2+9x^3+16x^4+\cdots)-x\end{aligned} \quad \text{(A12)}$$

Let the first term on the RHS be S, then,

$$S = x+4x^2+9x^3+16x^4+\cdots \quad \text{(A13)}$$

$$xS = x^2+4x^3+9x^4+16x^5+\cdots \quad \text{(A14)}$$

Subtracting these two equations and factoring out x produces,

$$\begin{aligned}(1-x)S &= x+3x^2+5x^3+7x^4+\cdots \\ &= x(1+3x+5x^2+7x^3+\cdots)\end{aligned} \quad \text{(A15)}$$

Dividing both sides by x,

$$\frac{(1-x)}{x}S = 1+3x+5x^2+7x^3+\cdots \quad \text{(A16)}$$

This can be rewritten as,

$$\begin{aligned}\frac{(1-x)}{x}S &= (2+4x+6x^2+8x^3+\cdots) \\ &\quad -(1+x+x^2+x^3+\cdots)\end{aligned} \quad \text{(A17)}$$

By algebraic manipulation, calculus and (A1), the sum of the first term of (A17) becomes,

$$\begin{aligned}2+4x+6x^2+8x^3+\cdots &= 2(1+2x+3x^2+4x^3+\cdots) \\ &= 2\frac{d}{dx}(1+x+x^2+x^3+\cdots) \\ &= \frac{2}{(1-x)^2}\end{aligned} \quad \text{(A18)}$$

The second term of (A17) is given by (A1). Hence,

$$\frac{(1-x)}{x}S = \frac{2}{(1-x)^2}-\frac{1}{1-x} \quad \text{(A19)}$$

$$S = \frac{x}{1-x}\left(\frac{2}{(1-x)^2}-\frac{1}{1-x}\right) \quad \text{(A20)}$$

which simplifies to,

$$S = \frac{x(1+x)}{(1-x)^3} \quad \text{(A21)}$$

Finally substituting this equation back into (A12) produces,

$$\sum_{p=2}^{\infty}\left(p^2\cdot x^p\right)= \frac{x(1+x)}{(1-x)^3}-x \quad \text{(A22)}$$

This finally produces,

$$\sum_{p=2}^{\infty}\left(p^2\cdot x^p\right)= x\left(\frac{1+x}{(1-x)^3}-1\right) \quad \text{(A11)}$$

□