An Optimal Controller for Time-Varying Stochastic Systems with Multiple Time Delays

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Abstract: - A flexible controller for optimal control of linear time-varying stochastic systems with multiple time delays is developed. The plants to be controlled are represented using a multi-input multi-output controlled autoregressive moving average model. The delays are described using a diagonal matrix. Input and output filters in the form of linear time-varying moving average operators are introduced into a generalized minimum variance control cost functional in order to meet the needs of various applications. The controller is applicable to a large class of linear time-varying systems.

Key-Words: - Multiple variable systems, multiple delays, generalized minimum variance control, time-varying systems.

1 Introduction

PID controllers are the most popular controllers in industry. Generalized minimum variance controllers (GMVCs) retains the integral action of the PID controllers for disturbance rejection and have been applied to single input and single output (SISO) systems for replacement of the traditional PID controllers for optimal control of DC motors [1-4]. There are many multiple input and multiple output (MIMO) control systems, for example, vehicle control systems and chemical plants. The GMVCs have the potential to make significant performance improvement when applied to replace the PID controllers because each of the individual controllers in a MIMO control system will collaborate for optimal control to minimize a single cost functional.

Koivo developed the MIMO generalized minimum variance controller for linear time-invariant (LTI) systems [5]. It extended the GMVCs from LTI SISO systems [6], [7] for MIMO LTI plants. The GMVCs have also been extended from transfer functions for SISO LTI state space models [8].

The GMVCs have the cost functional that includes both an output tracking error variance and a quadratic function of filtered input. They are flexible providing a mechanism for compromise between output tracking accuracy and plant input fluctuation. The introduction of filters for plant output and input allows more control of transient performance of the closed-loop control systems.

The GMVCs are based on transfer functions that are defined for LTI systems only. The standard LTI SISO GMVCs [6], [7] were extended from LTI SISO transfer functions for time-varying SISO transfer operators for control of linear time-varying (LTV) systems [9], where LTV moving average filters are introduced and noncommutivity of time-varying transfer operators is overcome by a pseudo commutation technique specifically developed for LTV plants. An LTV GMVC was developed recently for LTV MIMO systems with a single delay [10].

In this paper, we extend the LTV GMVC to LTV MIMO systems with multiple time delays. LTV moving average filters in standard forms are applied for optimization extending the standard LTI GMVCs from LTI MIMO systems with a single delay for LTV MIMO plants with multiple delays.

2 Control Objective

We consider LTV plants represented by a MIMO LTV controlled autoregressive moving average (CARMA) model,

$$A(k,q^{-1})D(q)Y(k) = B(k,q^{-1})U(k) + C(k,q^{-1})D(q)W(k) \quad (1)$$

where $U(k)$ and $Y(k)$ are plant input and output, and $W(k)$ is a zero mean, independent Gaussian vector. Variance of $W(k)$ is time-varying and uniformly bounded away from infinity. In CARMA model (1), q is a one-step-advance operator satisfying

$$\begin{aligned} qF(k)G(k\text{-}2) &= F(k+1)qG(k\text{-}2) \\ &= F(k+1)G(k\text{-}1) \end{aligned} \tag{2}$$

and

$$\begin{aligned} A(k,q^{-1}) &= I + A_1(k)q^{-1} + A_2(k)q^{-2} + \dots + A_n(k)q^{-n} \\ B(k,q^{-1}) &= B_0(k) + B_1(k)q^{-1} + \dots + B_m(k)q^{-m} \\ C(k,q^{-1}) &= I + C_1(k)q^{-1} + \dots + C_h(k)q^{-h} \end{aligned} \tag{3}$$

are LTV moving average operators (MAO's) in the form of time-varying polynomials in the one-step-delay operator q^{-1}, where I is an identity matrix and all the time-varying parameter matrices $A_i(k)$, $B_j(k)$ and $C_r(k)$, $i=1, 2, ..., n$, $j=0, 1, ...,m$, $r=1, 2, ..., h$, are square matrices with finite and the same dimension. They have norms that are uniformly bounded away from infinity. It is also assumed that the determinant of $B_0(k)$ is uniformly bounded away from zero. In the CARMA model $D(q)$ represents the multiple time delays between plant inputs and outputs. It has the form of the following diagonal matrix.

$$D(q) = diag(q^{d_1}, q^{d_2}, ..., q^{d_p}) \tag{4}$$

where $d_i > 0$, $i=1, 2, \dots, p$, are positive integers representing the time delay between the *ith* input and the *ith* output. Without losing generality we assume $d_i \le d_{i+1}$. The LTV CARMA model can be rewritten in the following standard form.

$$\begin{aligned} A(k,q^{-1})Y'(k) &= B(k,q^{-1})U(k) \\ &+ C(k,q^{-1})W'(k) \end{aligned} \tag{5}$$

where

$$\begin{aligned} W'(k) &= D(q)W(k) \\ Y'(k) &= D(q)Y(k) = \\ &\begin{bmatrix} y_1(k+d_1) & y_2(k+d_2) & \dots & y_p(k+d_p) \end{bmatrix}^T \end{aligned} \tag{6}$$

with the superscript T representing matrix transpose. The inverse operator of an LTV MAO, for example $A(k,q^{-1})$, is called an LTV autoregressive operator (ARO) as in the LTI cases and is denoted by $A^{-1}(k,q^{-1})$ [9].

As in the LTI case filters can be introduced to improve transient performance of the GMVC systems. Given a p-dimensional uniformly bounded reference vector $S(k)$ and LTV MIMO moving average filters

$$\begin{aligned} P(k,q^{-1}) &= I + P_1(k)q^{-1} + P_2(k)q^{-2} + \\ &\dots + P_{np}(k)q^{-np} \\ Q(k,q^{-1}) &= Q_0(k) + Q_1(k)q^{-1} + Q_2(k)q^{-2} + \\ &\dots + Q_{nq}(k)q^{-nq} \end{aligned} \tag{7}$$

the generalized output and generalized reference are defined as

$$\begin{aligned} \Psi(k) &= P(k,q^{-1})D(q)Y(k) \\ Z(k) &= Q(k,q^{-1})D(q)S(k) \end{aligned} \tag{8}$$

The cost functional of the LTV MIMO GMVC is as follows.

$$\begin{aligned} J(k) = E\{&[(\Psi(k) - Z(k))^T \Lambda(k)(\Psi(k) - Z(k)) \\ &+ (R(k,q^{-1})U(k))^T V(k) R(k,q^{-1})U(k)] / Data(k)\} \end{aligned} \tag{9}$$

where $Data(k)=\{Y(k), U(k), Y(k\text{-}1), U(k\text{-}1), \dots\}$ is the data set that has the input and output up to and including current time k representing all the available data for control on current time k, E is for the mathematical expectation conditioned on $Data(k)$ and

$$\begin{aligned} R(k,q^{-1}) &= R_0(k) + R_1(k)q^{-1} \\ &+ R_2(k)q^{-2} + \dots + R_{nr}(k)q^{-nr} \end{aligned} \tag{10}$$

is an LTV MIMO moving average filter for generalized plant input $R(k,q^{-1})U(k)$. The control objective is to find appropriate $U(k)$ such that the cost functional of the LTV MIMO GMVC is minimized and all the variables in the closed-loop control systems are uniformly bounded away from infinity. In the cost functional, $\Lambda(k)$ and $V(k)$ are uniformly positive definite time-varying weighting matrices for the generalized tracking error and filtered input. They are chosen to be uniformly bounded away from infinity. It is assumed that the LTV ARO's $P^{-1}(k,q^{-1})$ and $Q^{-1}(k,q^{-1})$ are exponentially stable and all the time-varying parameter matrices of the three LTV MIMO filters are uniformly bounded away from infinity. In addition, the determinates of $Q_0(k)$ and $R_0(k)$ are assumed to be uniformly bounded away from zero. These assumptions are not restrictive because the choice of the three filters is in our hands.

3 GMVC

We first develop a minimum variance predictor (MVP) for LTV CARMA model (1) for prediction of the filtered output $P(k,q^{-1})D(q)Y(k)$ in order for dealing with the multiple delays and the stochastic part of the system for development of the LTV GMVC. Left dividing (1) using $A(k,q^{-1})P^{-1}(k,q^{-1})$ on both sides and noting (8) we have

$$\begin{aligned}\Psi(k) &= [A(k,q^{-1})P^{-1}(k,q^{-1})]^{-1}B(k,q^{-1})U(k)\\ &+[A(k,q^{-1})P^{-1}(k,q^{-1})]^{-1}C(k,q^{-1})D(q)W(k)\end{aligned} \tag{11}$$

The first term on the right hand side of the above equation is the deterministic part of the filtered output. The second term is the stochastic part. The key for the prediction is to separate the stochastic part into two components. The first depends only on the noise to occur in the future and the second depends only on the noise up to and including current time k. The noise up to and including the current time can be estimated using *Data(k)*. The stochastic part can be rewritten as

$$\begin{aligned}&[A(k,q^{-1})P^{-1}(k,q^{-1})]^{-1}C(k,q^{-1})D(q)W(k)\\ &= P(k,q^{-1})A^{-1}(k,q^{-1})C(k,q^{-1})D(q)W(k)\end{aligned} \tag{12}$$

We apply the following long division in order to divide the above noise into two parts.

$$C(k,q^{-1})D(q) = A(k,q^{-1})F(k,q)+G(k,q^{-1}) \tag{13}$$

where

$$F(k,q) = F_0(k)q^d + F_1(k)q^{d-1}+...+F_{d-1}(k)q \tag{14}$$

is the quotient with d being the maximum time advance in $D(q)$ and

$$G(k,q^{-1}) = g_0(k)+g_1(k)q^{-1}+...+g_s(k)q^{-s} \tag{15}$$

is the remainder. Substituting (13) into (12) we have

$$\begin{aligned}&P(k,q^{-1})A^{-1}(k,q^{-1})C(k,q^{-1})D(q)W(k)\\ &= P(k,q^{-1})F(k,q)W(k)\\ &+P(k,q^{-1})A^{-1}(k,q^{-1})G(k,q^{-1})W(k)\end{aligned} \tag{16}$$

Noting (3), (7) and (15) we know that the last term on the right hand side of the above equation depends only on the noise up to and including current time k. However, noting (7) and (14) we know the first term on the right hand side depends on not only the future noise but also the noise up to and including current time k if the output filter $P(k,q^{-1})$ is not an identity matrix. We introduce the following equation in order to separate the future noise from the first term.

$$P(k,q^{-1})F(k,q) = H(k,q)+L(k,q^{-1}) \tag{17}$$

where

$$\begin{aligned}H(k,q) &= H_0(k)q^d + H_1(k)q^{d-1}+\\ &...+H_{d-1}(k)q\end{aligned} \tag{18}$$

has all the terms that have advance operator q and $L(k,q^{-1})$ has all the terms that have either the operator q^0 or the delayed operator q^{-1}. Substituting (12), (13) and (17) into (11) we have

$$\begin{aligned}\Psi(k) &= [A(k,q^{-1})P^{-1}(k,q^{-1})]^{-1}B(k,q^{-1})U(k)\\ &+P(k,q^{-1})F(k,q)W(k)\\ &+P(k,q^{-1})A^{-1}(k,q^{-1})G(k,q^{-1})W(k)\\ &= P(k,q^{-1})A^{-1}(k,q^{-1})B(k,q^{-1})U(k)\\ &+H(k,q)W(k)+[L(k,q^{-1})\\ &+P(k,q^{-1})A^{-1}(k,q^{-1})G(k,q^{-1})]W(k)\end{aligned} \tag{19}$$

where the only term that depends on the future noise is $H(k,q)W(k)$. Because of independency of the noise it is impossible to know this term at current time k and our best estimate of this term is its mean value. Taking mathematic expectation on both sides of (19) conditioned on the data up to and including current time k we have the minimum variance prediction of the filtered output,

$$\hat{\Psi}(k) = P(k,q^{-1})A^{-1}(k,q^{-1})B(k,q^{-1})U(k) + [L(k,q^{-1}) + P(k,q^{-1})A^{-1}(k,q^{-1})G(k,q^{-1})]W(k) \quad (20)$$

3.1 GMVC Theorem

If the LTV AROs $A^{-1}(k,q^{-1})$, $C^{-1}(k,q^{-1})$ and $P^{-1}(k,q^{-1})$ are exponentially stable, the LTV MIMO GMVC for CARMA model (1) with multiple delays is given by

$$\hat{W}(k) = D(q^{-1})C^{-1}(k,q^{-1})A(k,q^{-1})D(q)Y(k) - D(q^{-1})C^{-1}(k,q^{-1})B(k,q^{-1})U(k) \quad (21)$$

$$U(k) = T^{-1}(k,q^{-1})\{-[A^{-1}(k,q^{-1})G(k,q^{-1}) + P^{-1}(k,q^{-1})L(k,q^{-1})]\hat{W}(k) + P^{-1}(k,q^{-1})Q(k,q^{-1})D(q)S(k)\} \quad (22)$$

where

$$T(k,q^{-1}) = A^{-1}(k,q^{-1})B(k,q^{-1}) + P^{-1}(k,q^{-1})\Lambda^{-1}(k)B_0^{-T}R_0^T(k)V(k)R(k,q^{-1}) \quad (23)$$

3.2 Proof

Subtracting (20) from (19) we have

$$\Psi(k+d) = \hat{\Psi}(k+d) + H(k,q^{-1})W(k+d) \quad (24)$$

Substituting (24) into GMVC cost functional (9) it follows that

$$J(k) = (\hat{\Psi}(k) - Z(k))^T\Lambda(k)(\hat{\Psi}(k) - Z(k)) + (R(k,q^{-1})U(k))^T V(k)R(k,q^{-1})U(k) + E[(H(k,q^{-1})W(k))^T\Lambda(k) H(k,q^{-1})W(k)/Data(k)] \quad (25)$$

Thus

$$\frac{\partial J(k)}{\partial U(k)} = 2B_0^T(k)\Lambda(k)[\hat{\Psi}(k) - Z(k)] + 2R_0^T(k)V(k)R(k,q^{-1})U(k) \quad (26)$$

and

$$\frac{\partial^2 J(k)}{\partial U^2(k)} = 2B_0^T(k)\Lambda(k)B_0(k) + 2R_0^T(k)V(k)R_0(k) \quad (27)$$

Because the second derivative in (27) is positive definite the cost functional (9) can be minimized by equating derivative matrix (26) to zero for solving control variable *U(k).* Substituting (20) into (26), setting it to zero and left dividing the result by $2B_0^T(k)\Lambda(k)$ we have

$$\Lambda^{-1}(k)B_0^{-T}R_0^T(k)V(k)R(k,q^{-1})U(k) = Z(k) - P(k,q^{-1})[A^{-1}(k,q^{-1})B(k,q^{-1})U(k) + A^{-1}(k,q^{-1})G(k,q^{-1})]W(k) - L(k,q^{-1})W(k) \quad (28)$$

Left dividing the above equation by $P(k,q^{-1})$ and solving for *U(k)* we have

$$[P^{-1}(k,q^{-1})\Lambda^{-1}(k)B_0^{-T}R_0^T(k)V(k)R(k,q^{-1}) + A^{-1}(k,q^{-1})B(k,q^{-1})]U(k) = P^{-1}(k,q^{-1})Z(k) - [A^{-1}(k,q^{-1})G(k,q^{-1}) + P^{-1}(k,q^{-1})L(k,q^{-1})]W(k) \quad (29)$$

The noise in the above equation can be determined using CARMA model (1) and *Data(k).* Left multiplying (1) using $D(q^{-1})C^{-1}(k,q^{-1})$ on both sides we have

$$W(k) = D(q^{-1})C^{-1}(k,q^{-1})B(k,q^{-1})U(k) - D(q^{-1})C^{-1}(k,q^{-1})A(k,q^{-1})D(q)Y(k) \quad (30)$$

Comparing (21) with (30) we have

$$C(k,q^{-1})D(q)\tilde{W}(k) = O \quad (31)$$

where O is a zero matrix with appropriate dimension and

$$\tilde{W}(k) = W(k) - \hat{W}(k) \quad (32)$$

is the estimation error. It will decay exponentially to zero because of exponential stability of LTV ARO $C^{-1}(k,q^{-1})$. Substituting (32) into (22) we have

$$\begin{aligned} U(k) = T^{-1}(k,q^{-1})\{ & -[A^{-1}(k,q^{-1})G(k,q^{-1}) \\ & + P^{-1}(k,q^{-1})L(k,q^{-1})][W(k) - \tilde{W}(k)] \\ & + P^{-1}(k,q^{-1})Q(k,q^{-1})D(q)S(k)\} \end{aligned} \quad (33)$$

Noting (1), (32) and (33) we have the closed-loop system for the LTV GMVC,

$$\begin{bmatrix} C(k,q^{-1})D(q) & o & o \\ -E(k,q^{-1}) & T(k,q^{-1}) & o \\ o & -B(k,q^{-1}) & A(k,q^{-1})D(q) \end{bmatrix} \begin{bmatrix} \tilde{W}(k) \\ U(k) \\ Y(k) \end{bmatrix}$$

$$= \begin{bmatrix} o & o \\ -E(k,q^{-1}) & P^{-1}(k,q^{-1})Q(k,q^{-1})D(q) \\ C(k,q^{-1})D(q) & o \end{bmatrix} \begin{bmatrix} W(k) \\ S(k) \end{bmatrix} \quad (34)$$

where

$$\begin{aligned} E(k,q^{-1}) = & A^{-1}(k,q^{-1})G(k,q^{-1}) \\ & + P^{-1}(k,q^{-1})L(k,q^{-1}) \end{aligned} \quad (35)$$

The inverse of the left most matrix in close loop (34) is the closed-loop LTV ARO that determines closed-loop stability. It is determined by the three diagonal LTV MAO's of the matrix because of its triangular form. In (34) the diagonal delay matrix *D(q)* does not affect stability because it is simply a delay operator. Therefore the closed-loop stability is determined by the LTV ARO's $A^{-1}(k,q^{-1})$, $C^{-1}(k,q^{-1})$ and $T^{-1}(k,q^{-1})$. Because of exponential stability of the LTV ARO's $A^{-1}(k,q^{-1})$ and $C^{-1}(k,q^{-1})$ the closed-loop control system is exponentially stable if and only if $T^{-1}(k,q^{-1})$ is exponentially stable. As shown by (23) stability of $T^{-1}(k,q^{-1})$ can be chosen by us because the choice of the weighting matrices and the filtering LTV MAO's is in our hands.

4 Simulation

We consider a first order 2I2O LTV CARMA model. The delay matrix is

$$D(q) = diag(q, q^2) \quad (36)$$

The three LTV MAO's of the CARMA model have the forms,

$$\begin{aligned} A(k,q^{-1}) &= I + A(k)q^{-1} \\ B(k,q^{-1}) &= I + B(k)q^{-1} \\ C(k,q^{-1}) &= I + C(k)q^{-1} \end{aligned} \quad (37)$$

where

$$A(k) = \begin{bmatrix} 0.7\cos(0.3k) & 0.3\cos(0.2k) \\ 0.5sign(\cos(0.2k)) & 0 \end{bmatrix}$$

$$B(k) = \begin{bmatrix} 1 & 0.5\sin(0.5k)*\cos(0.05k) \\ 0 & 0.5sign(\cos(0.5k)) \end{bmatrix} \quad (38)$$

$$C(k) = \begin{bmatrix} 0.9\dfrac{k+8}{k+5} & 0.9(1+e^{-0.5(k+1)})\sin(0.5k) \\ 0 & 0.8\cos(0.3(k+1)) \end{bmatrix}$$

LTV MAO's $A(k,q^{-1})$, $B(k,q^{-1})$ and $C(k,q^{-1})$ are triangular matrices. Stability of their ARO's is determined by the diagonal matrix *A(k)*, *B(k)* and *C(k)*. Both LTV ARO's $A^{-1}(k,q^{-1})$, and $C^{-1}(k,q^{-1})$ are exponentially stable because the absolute values of the diagonal elements of *A(k)* and *C(k)* are uniformly less than unit. However, $B^{-1}(k,q^{-1})$ is not exponentially stable because *B(k)* is a triangular matrix and the absolute value of its first diagonal element is one. The weighting matrices are chosen as

$$\Lambda(k) = \begin{bmatrix} 6 & 0 \\ 0 & 5 \end{bmatrix}, \ V(k) = \begin{bmatrix} 1 & 0 \\ 0 & 0.2 \end{bmatrix} \quad (39)$$

The input filter is

$$R(k,q^{-1}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} q^{-1} \quad (40)$$

The output and reference filters are chosen as

$$P(k,q^{-1}) = I + \begin{bmatrix} 0.8 & 0 \\ 0 & 0.9 \end{bmatrix} q^{-1} = Q(k,q^{-1}) \quad (41)$$

Here we use the same output and reference filter in order to make the actual output to follow the reference. Fig.1 and Fig.2 show that both outputs follow their references well. The control variable is shown in Fig.3.

5 Conclusion

A flexible GMVC has been developed for MIMO LTV systems that have multiple delays. It extends our previous LTV GMVC from single delay MIMO plants for multiple delay systems. The LTV GMVC uses LTV MAO filters for flexibility and robustness of closed-loop control systems. It is able to ensure optimal control for a large class of LTV systems even when there are multiple delays between the plant outputs and inputs. This GMVC can be extended for a generalized predictive controller (GPC) extending the LTV GPC [11] from LTV SISO systems for LTV MIMO systems with multiple delays.

References:

[1] S. Donoghue, J. W. Finch, D. Giaouris and A. Jones, Generalized Minimum Variance Controller as a Velocity Loop Controller of a Casting Drum Drive in a Polyester Manufacturing Line, *Proceedings of the World Congress on Engineering 2008*, Vol III, WCE 2008.

[2] M. Ezzaraa, M. A. Lafkih, M. Ramzi, Dj.M. Maric, P.F. Meier and S.K. Estreicher, Adaptive Generalized Minimum Variance of DC Motor, *European Journal of Scientific Research*, Vol.67 No.1 2011, pp. 66-74.

[3] S. C. Mahato and J. P. Tiwari, Adaptive Generalized Minimum Variance Control of DC Motor, *International Journal of Engineering Research & Technology (IJERT)*, Vol. 1 No5, 2012.

[4] P. Zhao, J. Chen, Y. Song, X. Tao, T. Xu and T. Mei, Design of a Control System for an Autonomous Vehicle Based on Adaptive-PID, *International Journal of Advanced Robotic Systems*, Vol. 9, 44, 2012

[5] H. N. Koivo, A Multivariable Self-Tuning Controller, *Automatica*, Vol. 16, 1980, pp351-366.

[6] D. W. Clarke, and P. J. Gawthrop, Self-Tuning Controller, *Proc. IEE Part. D*, Vol.122, 1975, pp929-934.

[7] D. W. Clarke and P. J. Gawthrop, Self-Tuning Control, *Proc. IEE Part. D*, Vol.126, 1979, pp633-640.

[8] A.S. Silvera and A.A.R. Coelho, Generalized Minimum Variance Control State-Space Design, *IET Control Theory and Applications*, Vol.5, No.15. 2011, pp. 1709-1715.

[9] Z. Li and R. J. Evans, Generalised Minimum Variance Control of Linear Time-Varying Systems, *IEE Proc.-Control Theory Appl.*, Vol. 149, 2002, pp.111-116.

[10] Z. Li, A Flexible Generalized Minimum Variance Controller for MIMO Time-Varying Systems, *International Journal of Control Systems and Robotics*, Vol.1, 2016, pp123-127.

[11] Z. Li and G. L. Wang, Generalized predictive control of linear time-varying systems, Journal of Franklin Institute, 2017 DOI: 10.1016/j.jfranklin.2016.10.021.

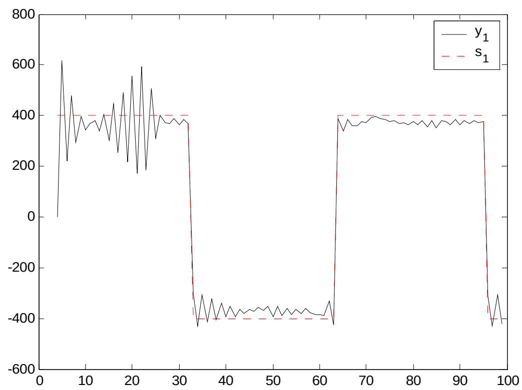

Fig.1 Simulation for the first output.

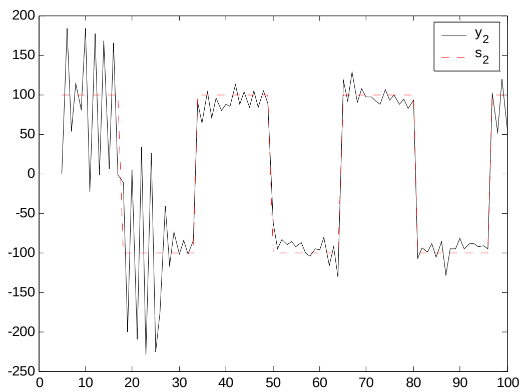

Fig.2 Simulation for the Second output.

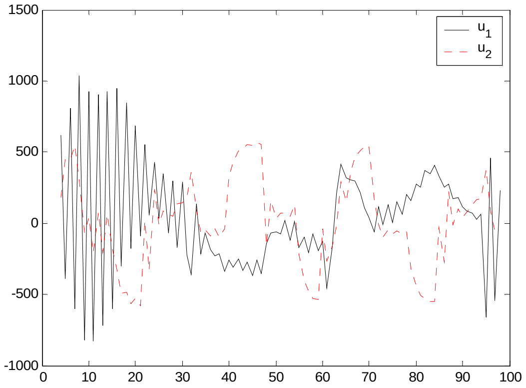

Fig.3 GMVC output.