Complexity analysis and chaos control of a R&D game with spillovers and endogenous demand in a triopoly

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Abstract: A R&D input game model with spillovers and endogenous demand in a triopoly market is considered in which all the players are bounded rational. Assume that the two players make up a cooperative team, they share the technology achievements completely and have same cost. Technology spillover exists between the cooperative team and the third player. R&D investment helps to expand market demand. On the basis of analyzing the stability of the only Nash equilibrium point, the local stable regions are obtained. Three-dimensional stable regions are investigated and the results show that the spillover rate of R&D has an obvious impact on the stable regions. Impact of technology spillover rate and endogenous demand on the profits is studied, and it is interesting that profits of different players could not lose stability synchronously. The complex dynamics, such as bifurcation scenarios, route to chaos and attractors are displayed by 2D bifurcation diagrams, the results show that if the adjustment speed of R&D input is high, the economic system tends to lose stability. The chaos is eliminated by using the feedback control method. Basins of attraction are investigated and we find the domain of attraction become smaller with increase of spillover rate and R&D input modification speed.

Key–Words: R&D game; spillovers; endogenous demand; teamwork ; basins of attraction;

1 Introduction

R&D has been one of the main driving force for the development of enterprises, which helps to reduce production cost, and improve production efficiency. The core competitiveness of enterprise has been gradually transformed into R&D competition. Since D'Aspremont and Jacqueminde [1] did an initiative research on R&D, numerous researches have been published. Gersbach and Schmutzler[2] considered a three-location duopoly game model and studied the impact of internal and external knowledge spillovers on the agglomeration of the players. Jiang et al. [3] and Vandekerckhove et al. [4] studied R&D cooperation model under the condition of innovation ability asymmetry and technology spillovers asymmetry, pointed out if there is ability asymmetry between the enterprises, an income transfer mechanism is needed to enhance the level of cooperation. Suetens [5] analyzed the relation between R&D and technological spillovers cooperation in a duopoly experiment and found that without technological spillovers, binding R&D contracts are needed for R&D decisions to deviate from the subgame perfect Nash R&D level towards the cooperative level. Goel and Haruna[6] discussed cooperative and noncooperative R&D with spillovers in the case of labor-managed players, and examined strategic interactions between labor-managed players in a duopoly. Breton [7] constructed a dynamic duopoly game model, by comparing Cournot and Bertrand equilibrium, he found Bertrand competition is more efficient if the R&D level is low or differences between companies are great in R&D. Cellini[8] analyzed dynamic R&D for process innovation in a Cournot duopoly. By comparing the profit and social welfare performances in steady state, they showed that the effect of private and social incentives towards R&D is consistent for all admissible levels of the technological spillovers.In the papers above, business decision - making is assumed to be completely rational, but the research on R&D competition among bounded rational manufacturers is not rich. Whitby et al.[9] studied the R&D duopoly competition with "agitation type" and "aggressive strategy type" character classes, numerical simulations showed that, chaos will appear in R&D competition if the two players were in different personality types. Sheng [10] investigated the global stability of R&D investment in the dynamic competition system. They found that under certain conditions, the system may have multiple attractors including chaotic attractor and periodic attractors. Li and Ma [11] set up a R&D games model in three oligarchs' market and took competition and cooperation into account, comparative analysis and complex dynamical behavior in the R&D were shown by numerical simulation.

While these literatures did not show influence of the R&D investment on market demand, in this paper, a R&D game model with spillovers and endogenous demand in a triopoly is set up in which the R&D investments contribute to the expansion of market demand. Assume all the oligarchs are bounded rational players and two oligarchs make up a cooperative R&D team. The main aim of this paper is to investigate the dynamic behaviors of three players in R&D input decisions. The paper is organized as follows. In Section 2, a R&D game model with spillovers, endogenous demand and teamwork is established. In Section 3, equilibrium points and local stability is discussed. Impact of spillovers and endogenous demand on the R&D investment and profits is studied in Section 4. In Section 5, dynamical behaviors of the game are investigated by 2D bifurcation diagrams[9]. In Section 6, the chaos is controlled by using the feedback control method. Basins of attraction[10] are investigated in Section 7. In Section 8, conclusions are drawn from our analysis.

2 The Triopoly game model

We consider a R&D game model with spillovers and endogenous demand in a triopoly. Assume player 1 and player 2 team up to carry out R&D and the research results are fully shared, so they have the same cost of production $c_i(t), i = 1, 2$. The price and the demand of player i' products is denoted by $p_i(t)$ and $q_i(t)$,$i = 1, 2, 3$. In the triopoly market, all the players select independently their output $q_i(t)$ and their own R&D investment level $x_i(t)$. The inverse demand functions are as follows:

$$\begin{aligned} p_1(t) = p_2(t) = a - b(q_1(t) + q_2(t) + q_3(t)) \\ +\delta(x_1(t) + x_2(t) + \beta x_3(t)) \\ p_3(t) = a - b(q_1(t) + q_2(t) + q_3(t)) \\ +\delta(\beta(x_1(t) + x_2(t)) + x_3(t)) \end{aligned} \tag{1}$$

where $a > 0, b > 0$ and endogenous demand parameter $\delta > 0$ denotes the effect of R&D activities on the market demand. Just as the arrival of Apple phones stimulated the enthusiasm of the consumers. $\beta \in [0, 1]$ measures technology spillover rate between player 3 and the cooperation team composed of player 1 and player 2. The technology spillover rate within the team is 1. $x_i(t)$ denotes the investment of player i in R&D. The following $c_i(t)$ denotes the unit cost of production of player i after R&D activities.

$$\begin{aligned} c_1(t) = c_2(t) = A - x_1(t) - x_2(t) - \beta x_3(t) \\ c_3(t) = A - x_3(t) - \beta(x_1(t) + x_2(t)) \end{aligned} \tag{2}$$

where A is the unit cost if the players do not invest on R&D. The profit of player i in period t is

$$\pi_i(t) = (p_i(t) - c_i(t))q_i(t) - \frac{\gamma x_i^2(t)}{2}, \quad i = 1, 2, 3. \tag{3}$$

where $\frac{\gamma x_i^2(t)}{2}$ is the cost function of investment in R&D in which $x_i(t)$ denotes the investment in R&D. The parameter γ is inversely related to player's cost effectiveness in R&D. The cost function of investment in R&D has the quadratic form because of the following reasons: technological innovation is only related to R&D, and has nothing to do with the economic scale, so the player may face diminishing returns to R&D scale and the unit input cost increases with increase of the scale of R&D. If there is no technology mutation, improvement of technology needs to invest more resources, which accords with practical. Use(1)-(2)in (3), and let

$$\frac{\partial \pi_i(t)}{\partial q_i(t)} = 0, \quad i = 1, 2, 3.$$

the players' output response functions can be obtained in (4), which show players' optimal outputs in a period t. The R&D investment $x_i(t)$ is the variable to functions (4).

$$\begin{cases} q_1(t) = \frac{a-A+(1+\delta)((2-\beta)(x_1(t)+x_2(t))+(2\beta-1)x_3(t))}{4b} \\ q_2(t) = \frac{a-A+(1+\delta)((2-\beta)(x_1(t)+x_2(t))+(2\beta-1)x_3(t))}{4b} \\ q_3(t) = \frac{a-A+(1+\delta)((3\beta-2)(x_1(t)+x_2(t))+(3-2\beta)x_3(t))}{4b} \end{cases} \tag{4}$$

In order to maximize profits, in period t, all the players make investment decisions on R&D according to marginal profits with respect to $x_i(t)$, combining(1)-(4), marginal profits functions on $x_1(t)$ are obtained as follows:

$$\begin{cases} \frac{\partial \pi_1(t)}{\partial x_1(t)} = \frac{1}{8b}((1+\delta)(2-\beta)(a-A+(1+\delta)((2-\beta) \\ (x_1(t)+x_2(t))+(2\beta-1)x_3(t)))-8b\gamma x_1(t)) \\ \frac{\partial \pi_2(t)}{\partial x_2(t)} = \frac{1}{8b}((1+\delta)(2-\beta)(a-A+(1+\delta)((2-\beta) \\ (x_1(t)+x_2(t))+(2\beta-1)x_3(t)))-8b\gamma x_2(t)) \\ \frac{\partial \pi_3(t)}{\partial x_3(t)} = \frac{1}{8b}((1+\delta)(3-2\beta)(a-A+(1+\delta)((3\beta-2) \\ (x_1(t)+x_2(t))+(3-2\beta)x_3))-8b\gamma x_3(t)) \end{cases} \tag{5}$$

While in practice, the players can not have complete information about the market and the other competitors, they may do not know other players investment decisions in the next-period in advance, so they cannot figure out the investment on the basis of the marginal profits functions (5). Suppose all the players are bounded rational players, that is, if the marginal profits are positive, they increase their investment on R&D in the next period; otherwise, they decrease the investment. Then the dynamical system can be described by the following nonlinear difference equations:

$$x_i(t+1) = x_i(t) + \alpha_i x_i(t)\frac{\partial \pi_i(t)}{\partial x_i(t)}, \quad i=1,2,3 \tag{6}$$

where $\alpha_i(0 < \alpha_i < 1)$ denotes the adjustment speed of R&D inputs of player i . Use (5) in (6), the following dynamic system can be obtained:

$$\begin{cases} x_1(t+1) = x_1(t) + \frac{\alpha_1 x_1(t)}{8b}((1+\delta)(2-\beta)(a-A \\ +(1+\delta)((2-\beta)(x_1(t)+x_2(t))+(2\beta-1)x_3(t))) \\ -8b\gamma x_1(t)) \\ x_2(t+1) = x_2(t) + \frac{\alpha_2 x_2(t)}{8b}((1+\delta)(2-\beta)(a-A \\ +(1+\delta)((2-\beta)(x_1(t)+x_2(t))+(2\beta-1)x_3(t))) \\ -8b\gamma x_2(t)) \\ x_3(t+1) = x_3(t) + \frac{\alpha_3 x_3(t)}{8b}((1+\delta)(3-2\beta)(a-A \\ +(1+\delta)((3\beta-2)(x_1(t)+x_2(t))+(3-2\beta)x_3)) \\ -8b\gamma x_3(t)) \end{cases} \tag{7}$$

3 Equilibrium points and local stability

3.1 Equilibrium points and local stability of bounded equilibrium points

According to system (7), let $x_i(t+1) = x_i(t)$, we can get

$$\begin{cases} x_1(t)((1+\delta)(2-\beta)(a-A+(1+\delta)((2-\beta) \\ (x_1(t)+x_2(t))+(2\beta-1)x_3(t)))-8b\gamma x_1(t))=0 \\ x_2(t)((1+\delta)(2-\beta)(a-A+(1+\delta)((2-\beta) \\ (x_1(t)+x_2(t))+(2\beta-1)x_3(t)))-8b\gamma x_2(t))=0 \\ x_3(t)((1+\delta)(3-2\beta)(a-A+(1+\delta)((3\beta-2) \\ (x_1(t)+x_2(t))+(3-2\beta)x_3(t)))-8b\gamma x_3(t))=0 \end{cases} \tag{8}$$

For convenience, numerical simulations are performed to show the equilibrium points and local stability of the game. We set the parameters as follows:

$$a=20, b=2, \gamma=1.2, A=2, \beta=0.6, \delta=0.6 \tag{9}$$

According to the parameters above, eight fixed points of system (7)can be obtained:

$$E_1 = (0,0,0)$$
$$E_2 = (0,0,4.75)$$
$$E_3 = (0,2.84,0)$$
$$E_4 = (2.84,0,0)$$
$$E_5 = (4.40,4.40,0)$$
$$E_6 = (0,3.07,4.49)$$
$$E_7 = (3.07,0,4.49)$$
$$E_8 = (4.71,4.71,3.96)$$

it is clear that E_8 is the only Nash equilibrium point $E_1 - E_7$ are boundary equilibria.

In order to analyze the stability of the equilibrium points , the Jacobian matrix for discrete dynamic system (7) is found as follows:

$$J(E) = \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{pmatrix} \tag{10}$$

where
$J_{11} = 1 + \frac{\alpha_1}{8b}[(1+\delta)(2-\beta)(a-A+(1+\delta)((2-\beta)(2x_1(t)+x_2(t))+(2\beta-1)x_3(t)))-16b\gamma x_1(t)]$;

$J_{12} = \frac{\alpha_1 x_1(t)}{8b}(1+\delta)^2(2-\beta)^2$;

$J_{13} = \frac{\alpha_1 x_1(t)}{8b}(1+\delta)^2(2-\beta)(2\beta-1)$;

$J_{21} = \frac{\alpha_2 x_2(t)}{8b}(1+\delta)^2(2-\beta)^2;$

$J_{22} = 1+\frac{\alpha_2}{8b}[(1+\delta)(2-\beta)(a-A+(1+\delta)((2-\beta)(2x_2(t)+x_1(t))+(2\beta-1)x_3(t)))-16b\gamma x_2(t)];$

$J_{23} = \frac{\alpha_2 x_2(t)}{8b}(1+\delta)^2(2-\beta)(2\beta-1);$

$J_{31} = \frac{\alpha_3 x_3(t)}{8b}(1+\delta)^2(3-2\beta)(3\beta-2);$

$J_{32} = \frac{\alpha_3 x_3(t)}{8b}(1+\delta)^2(3-2\beta)(3\beta-2);$

$J_{33} = 1+\frac{\alpha_3}{8b}[(1+\delta)(3-2\beta)(a-A+(1+\delta)((3\beta-2)(x_2(t)+x_1(t))+2(3-2\beta)x_3(t)))-16b\gamma x_3(t)]$

According to Routh-Hurwitz condition, the necessary and sufficient condition of asymptotic stability at E is that all the eigenvalues are inside the unit circle in complex plane.

Then put (9) into the parameters in (10), we get the eigenvalues of $E_1 - E_7$ by $J(E)$ respectively. According to the conditions for the stability of fixed points, the modulis of all characteristic roots are less than 1 , so $E_1 - E_7$ are unstable equilibrium points.

3.2 Three-dimensional stable regions of Nash equilibrium point E_8

As for $E_8 = (4.71, 4.71, 3.96)$, its Jacobian matrix is

$$J(E_8) = \begin{pmatrix} 1.0-4.17\alpha_1 & 1.48\alpha_1 & 0.21\alpha_1 \\ 1.48\alpha_2 & 1.0-4.17\alpha_2 & 0.21\alpha_2 \\ -0.29\alpha_3 & -0.29\alpha_3 & 1.0-2.70\alpha_3 \end{pmatrix} \quad (11)$$

The characteristic equation of the Jacobian matrix of E_8 is $f(\lambda) = \lambda^3 + A_1\lambda^2 + B_1\lambda + C_1$,
In which

$A_1 = (4.17\alpha_1 + 4.17\alpha_2 + 2.70\alpha 3 - 3.0);$

$B_1 = (0.048\alpha_1\alpha_3 - 2.18\alpha_1\alpha_2 + 0.048\alpha_2\alpha_3 + (4.17\alpha_1-1.0)(4.17\alpha_2-1.0)+(2.7\alpha_3-1.0)(4.17\alpha_1+4.17\alpha_2-2.0));$

$C_1 = 0.23\alpha_3(0.31\alpha_1\alpha_2 - 0.22\alpha_1(4.17\alpha_1 - 1)) + 0.23\alpha_3(0.31\alpha_1\alpha_2 - 0.21\alpha_2(4.17\alpha_2 - 1)) - (2.18\alpha_1\alpha_2 - 1.0(4.17\alpha_1 - 1.0)(4.17\alpha_2 - 1.0))(2.70\alpha_3 - 1) + (0.048\alpha_1\alpha_3 + 0.048\alpha_2\alpha_3)(4.17\alpha_1 + 4.17\alpha_2 - 2.0);$

According to the Jury's stability criterion, the necessary and sufficient condition of asymptotic stabilization at E_8 calls for the following conditions:

$$\begin{cases} A_1 + B_1 + C_1 + 1 > 0 \\ 1 - A_1 + B_1 - C_1 > 0 \\ 1 - C_1^2 > 0 \\ (1-C_1^2)^2 - (B_1 + A_1C_1)^2 > 0 \end{cases} \quad (12)$$

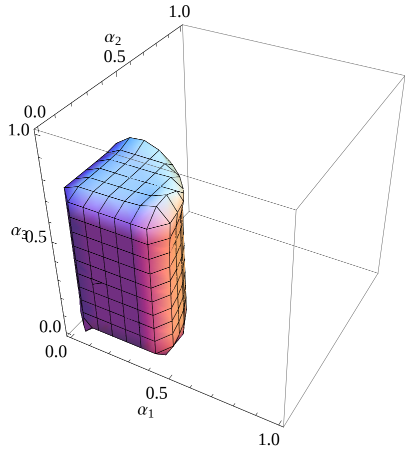

Figure 1: The stable region of Nash equilibrium point $E_8, \beta = 0.6$

A stable region of Nash equilibrium point E_8 can be determined by the above inequalities (12) in the space of $(\alpha_1, \alpha_2, \alpha_3)$ which can be shown in Figure 1. In the stable region, ranges of α_1 and α_2 are almost the same. The R&D inputs of three players will keep stable at E_8 after a limited number of games no matter what initial R&D inputs are chosen.The Nash equilibrium point E_8 may be unstable when $\alpha_1, \alpha_2, \alpha_3$ increase. If their adjustment speed of R&D inputs runs out of the stable region, the stable state of system (7) at point E_8 will be broken and the bifurcations, even chaos phenomena, will appear. From an economic point of view, the unordered competitions will happen in the market.

3.3 The Effects of technology spillover rate on stable region

In this paper, spillover rate is denoted by β, in order to analyze the effects of parameter β on stable region, make $\beta = 0.5, 0.7$ respectively and keep other parameters constant as (9), then the corresponding stable regions are shown in Figure 2 and 3. When $\beta = 0.5$, Nash equilibrium input $E_8^{'}$ is $(5.63, 5.63, 3.21)$ and when $\beta = 0.7$, Nash equilibrium input $E_8^{''}$ is $(4.04, 4.04, 3.91)$. By comparing Figure 1-Figure 3 and the 3 Nash equilibrium inputs, we find that with increase of technology spillover rate β ,

1) In the stable region, the range of α_3 narrows, while the range of α_1 and $\alpha_2$both expand.
2) The gap of the Nash equilibrium inputs among three players narrows.

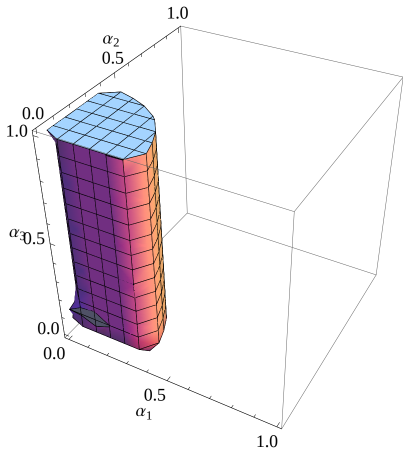

Figure 2: The stable region of Nash equilibrium point $E_8^{'}$,$\beta = 0.5$

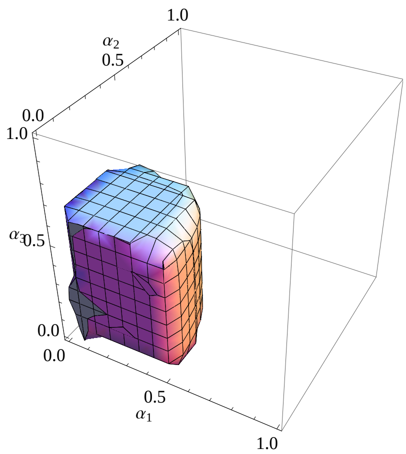

Figure 3: The stable region of Nash equilibrium point $E_8^{''}$,$\beta = 0.7$

That means if the technology spillover rate is relatively lower($\beta = 0.5$), the adjustment speed in R&D input of player who is outside the cooperation, i.e. the adjustment speed of player 3, has little effect on the system, while with increase of the technology spillover rate, the adjustment speed of player 3 must be kept in a certain range. If the technology spillover rate is relatively higher($\beta = 0.7$), the adjustment speed of player who is in the cooperation, i.e. player 1 and player 2, will have greater range of adjustment. If the spillover rate is relatively higher, the gap of the inputs among three players narrows.

4 The Effects of parameters β and δ on profits and investment

In this section, the effects of β on three players' profits and investment will be investigated firstly. Numerical simulations are performed to show the effects of β. For convenient, we set the parameters
$a = 20, b = 2, \gamma = 1.2, A = 2, \delta = 0.6, \alpha_1 = \alpha_2 = \alpha_3 = 0.3$
The investment diagram of (7)with respect to β is shown by Figure 4.

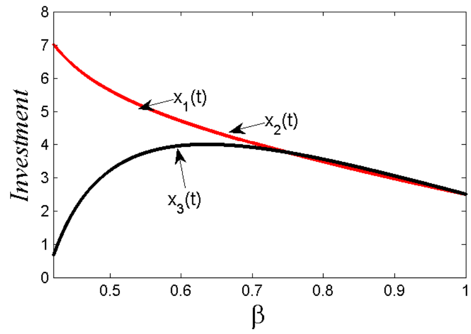

Figure 4: The investment diagram with respect to β when $\delta = 0.6$

Substituting (1),(2),(4)into(3), the profit function of all the players can be obtained as follows,

$$\pi_i(t) = bq_i^2(t) - \frac{\gamma x_i(t)^2}{2}, \quad i = 1, 2, 3$$

So the profit bifurcation diagram of (7) with respect to β is shown by Figure 5. The average profit with respect to β is shown by Figure 6.

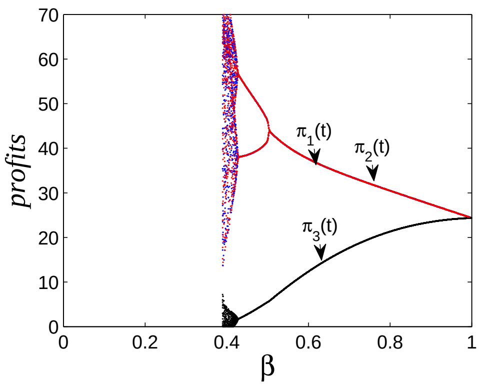

Figure 5: The profit bifurcation diagram with respect to β when $\delta = 0.6$

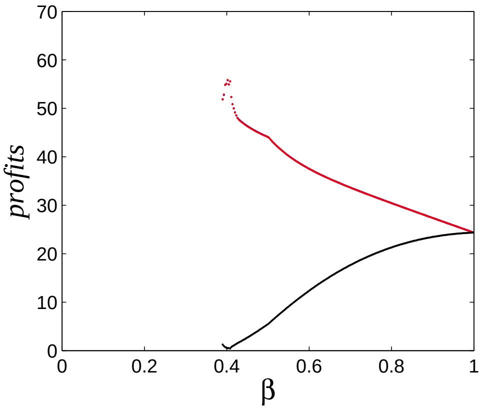

Figure 6: The average profits with respect to β when $\delta = 0.6$

In Figures 4, red set of points denotes $x_1(t)$ and $x_2(t)$, black set of points denotes $x_3(t)$. In Figures 5, blue set of points denotes $\pi_1(t)$ and red set of points denotes $\pi_2(t)$, black set of points denotes $\pi_3(t)$. Three features can be found as follows:

1) With increase of spillover rate β, investment of player 1 and player 2 decreases , while investment of player 3 increases before $\beta = 0.62$, decreases after $\beta = 0.62$. $x_1(t), x_1(t), x_3(t)$ become closer and after $\beta = 0.7$, they are almost the same.

2) When spillover rate $\beta = 1$, all the players have the same profits, with decrease of spillover rate β , all the players' profits will lose stability at last. But a very interesting phenomenon is that when profits of player 1 and player 2 experience 2-period doubling bifurcation at $\beta = 0.49$, profits of player 3 still remain stable.

3) With decrease of spillover rate β, average profits of player 1 and player 2 increase before 2-period doubling bifurcation, then begin to flutter. While profits of player 3 decrease before the bifurcation point, then increase slightly.

From an economic perspective, in order to obtain more profits stably, players in the cooperative team(player 1 and player 2) should keep spillover rate in a certain range, neither too high nor too low to avoid facing the shock of profits and make as much as possible .

The effects of δ on investment, profits and average profits are investigated secondly which is shown in Figure 7 8 and 9. The parameters are $a = 20, b = 2, \gamma = 1.2, A = 2, \beta = 0.6, \alpha_1 = \alpha_2 = \alpha_3 = 0.3$, with increase of δ, two features can be seen from Figure 7 and 8,

1)Investment of all the players increases, but investment of player 1 and player 2 increases faster than player 3, when $\delta > 0.43$, investment of player 1 and player 2 is more than player 3.

2)Profits of player 1 and player 2 increase, however average profits of player 3 almost do not change. when $\beta > 0.69$, profits of player 1 and player 2 enter into 2 cycles state. When $\beta > 0.75$ all the players' profits will fall into disorder.

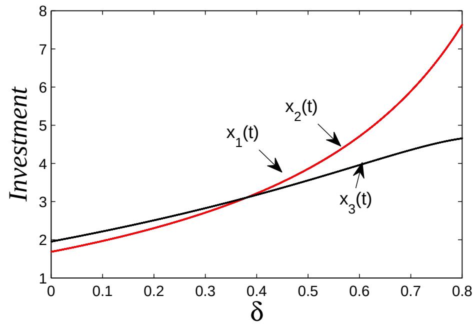

Figure 7: The investment diagram with respect to δ when $\beta = 0.6$

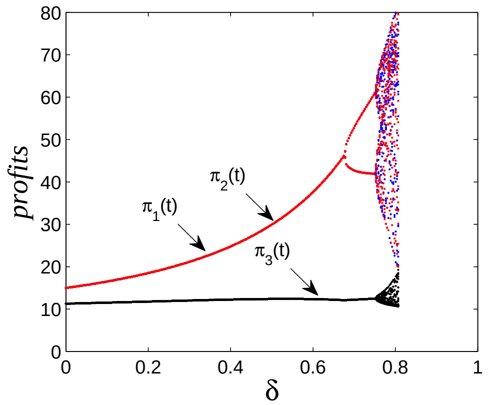

Figure 8: The profits bifurcation diagram with respect to δ when $\beta = 0.6$

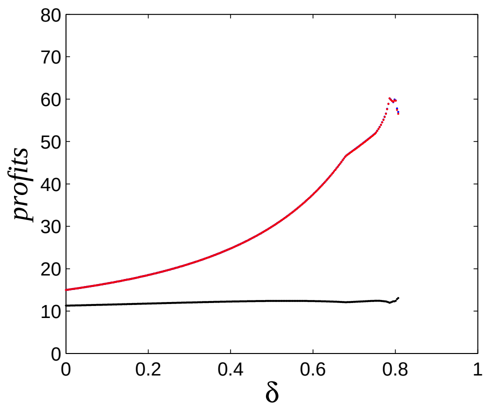

Figure 9: The average profits with respect to δ when $\beta = 0.6$

From an economic perspective, as we know, in the industry in which the products update and upgrade m-more quickly, for instance, consumer electronics, δ is relatively larger, investment of all the players is higher and the profits gap between team members and player 3 is wider; while in the industry in which the products update and upgrade slowly, such as mechanical engineering, δ is relatively smaller, and the profits gap is narrower.

5 2D bifurcation diagrams and interactive relationships among $\alpha_1, \alpha_2, \alpha_3$

2D bifurcation diagram is a more powerful tool in the numerical analysis of nonlinear dynamics than the 1D bifurcation diagrams. In the 2D bifurcation diagrams, bifurcation scenarios and route to chaos can be displayed clearly. In this section, the 2D bifurcation diagrams will be used to analyze the effects of players' adjustment speed on system stability. We set the same parameters as (9), then $(\alpha_1, \alpha_2)(\alpha_1, \alpha_3), (\alpha_2, \alpha_3)$ 2D bifurcation diagrams are shown in Figure 10, 11 and 12. In the 2D bifurcation diagrams, the system exhibits a sequence of flip bifurcations to quasi-periodic state or chaos, then to divergence(which means one of the players will be out of the market).
Different colors are assigned to each region to show its particular behavior. That is, brown, stable states; yellow, period-2 stable cycles; orange, period-3; blue, period-4; light green, period-5; dark gray, period-6; purple, period-8; grass green, period-9; red, period-10; light gray, quasi-periodic state or chaos; army green, escape.

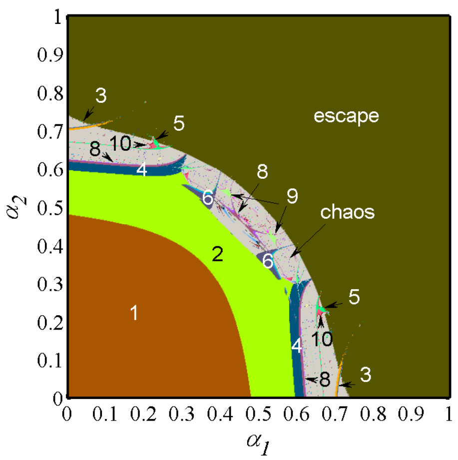

Figure 10: $(\alpha_1, \alpha_2) - 2D$ bifurcation diagram,$\alpha_3 = 0.3$

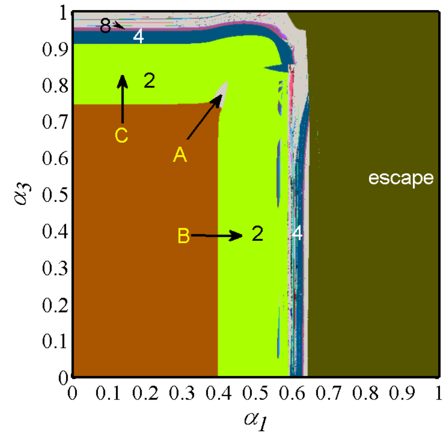

Figure 11: $(\alpha_1, \alpha_3) - 2D$ bifurcation diagram,$\alpha_2 = 0.3$

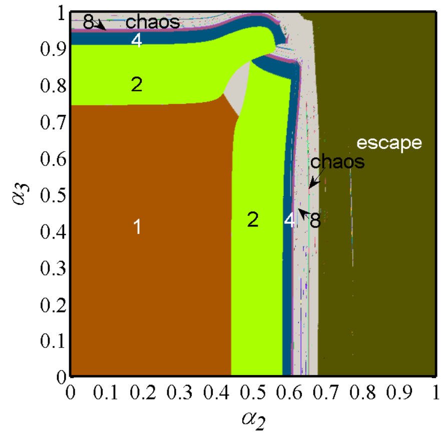

Figure 12: $(\alpha_2, \alpha_3) - 2D$ bifurcation diagram,$\alpha_1 = 0.3$

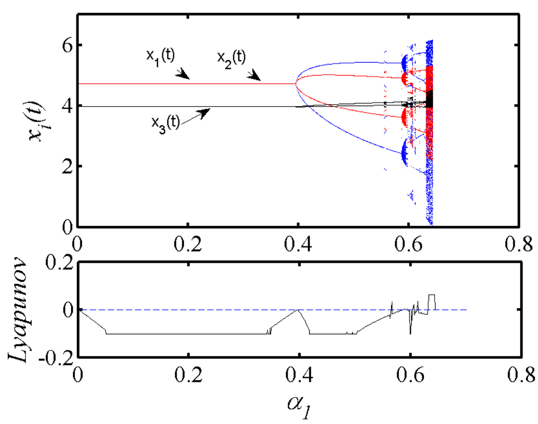

Figure 13: bifurcation and maximal Lyapunov exponent with respect to α_1 when $\alpha_2 = \alpha_3 = 0.3$

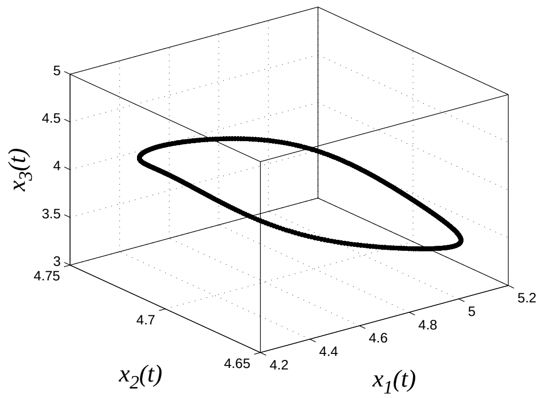

Figure 15: The quasi-periodic attractor of system when$\alpha_1 = 0.41, \alpha_2 = 0.3, \alpha_3 = 0.74$

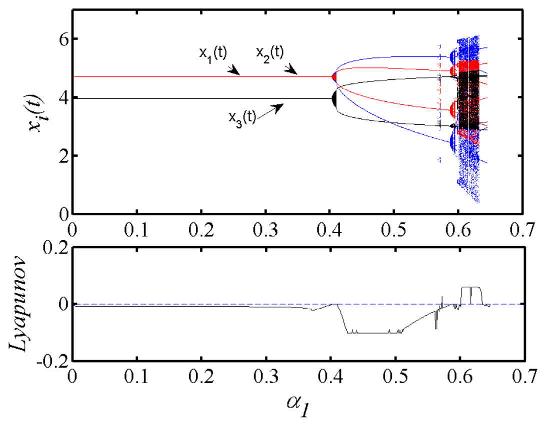

Figure 14: bifurcation and maximal Lyapunov exponent with respect to α_1 when $\alpha_2 = \alpha_3 = 0.3$

In Figure 10,$\alpha_1 \in [0, 1]$, $\alpha_2 \in [0, 1]$and $\alpha_3 = 0.3$, the system exhibits a sequence of flip bifurcations to chaos with increase of (α_1, α_2).
An approximate symmetrical structure is shown because player 1 and player 2 share research results completely. In addition, in the chaotic region, the system contains cycle parameters islands which has the self-similar structure such as the region period-5, period-10. As seen from Figure 10, if the team members' adjustment speed is relatively lower (in the brown area), the system (7) will be in a steady state. Along with increase of the adjustment speed , the economic system will experience cyclical shocks, chaos, even disappearance which means one of the players stop investing. Obviously, relatively larger adjustment speed is detrimental to the economic system.

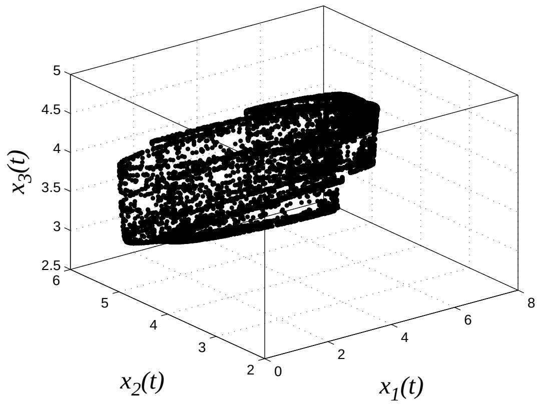

Figure 16: bifurcation and maximal Lyapunov exponent with respect to α_1 when $\alpha_2 = \alpha_3 = 0.3$

In Figure 11,$\alpha_1 \in [0, 1]$, $\alpha_3 \in [0, 1]$and $\alpha_2 = 0.3$, respectively,as can be seen, with increase of the adjustment speed α_1 and α_3, the economic system will experience cyclical shocks, quasi-periodic state ,chaos, even disappearance. When the parameters (α_1, α_3) pass through the borders as the black arrows B and C, system (7) loses its stability through flip bifurcation as shown in Figures 13(Figures 13 shows arrow B, C is similar to B). While it is interesting that when the parameters cross the borders as the arrow A , the dynamic behavior of systems is more complicated, the system enters into quasi-periodic state through Neimark-Sacker bifurcation, then enters period 2, and then evolves into chaos through flip bifurcation separately, as shown in Figure 14,15,16.
In Figure 12, $\alpha_2 \in [0, 1]$, $\alpha_3 \in [0, 1]$and $\alpha_1 = 0.3$ respectively, the system also exhibits a sequence of flip bifurcations with adjustment speed α_2 and α_3 , which is similar to Figure 11.

6 Chaos control

According to the numerical simulation in section 5, if the players' input adjustment speed is beyond the stable region, the market will experience cyclical shocks, and even fall into chaos. The appearance of chaos in the economic systems is harmful to all the players. In order to avert the risk, it is expedient for triopoly to maintain at Nash equilibrium input. Many methods for the chaos control have been proposed, such as OGY method[12], modified straight-line stabilization method[13], time-delayed feedback method[14], pole placement method [15]and so on. In this section, we use a feedback control method proposed by Elabbasy[16] et al. Consider the fluctuation of $x_1(t)$ and $\pi_1(t)$ shown above is higher, the controlling factor is applied on player 1. So the controlled system is given by:

$$\begin{cases} x_1(t+1) = x_1(t) + \frac{\alpha_1 x_1(t)}{8b}((1+\delta)(2-\beta)(a-A \\ +(1+\delta)((2-\beta)(x_1(t)+x_2(t)) + (2\beta-1)x_3(t)) \\ -8b\gamma x_1(t)) - k(x_1(t+1) - x_1(t)) \\ x_2(t+1) = x_2(t) + \frac{\alpha_2 x_2(t)}{8b}((1+\delta)(2-\beta)(a-A \\ +(1+\delta)((2-\beta)(x_1(t)+x_2(t)) + (2\beta-1)x_3(t)) \\ -\gamma x_2(t)) \\ x_3(t+1) = x_3(t) + \frac{\alpha_3 x_3(t)}{8b}((1+\delta)(3-2\beta)(a-A \\ +(1+\delta)((3\beta-2)(x_1(t)+x_2(t)) + (3-2\beta)x_3(t))) \\ -\gamma x_3(t)) \end{cases} \tag{13}$$

where k is the controlling factor. The equivalent system of (13) is

$$\begin{cases} x_1(t+1) = x_1(t) + \frac{\alpha_1 x_1(t)}{8b(k+1)}((1+\delta)(2-\beta)(a-A \\ +(1+\delta)((2-\beta)(x_1(t)+x_2(t)) + (2\beta-1)x_3(t)) \\ -8b\gamma x_1(t)) \\ x_2(t+1) = x_2(t) + \frac{\alpha_2 x_2(t)}{8b}((1+\delta)(2-\beta)(a-A \\ +(1+\delta)((2-\beta)(x_1(t)+x_2(t)) + (2\beta-1)x_3(t))) \\ -\gamma x_2(t)) \\ x_3(t+1) = x_3(t) + \frac{\alpha_3 x_3(t)}{8b}((1+\delta)(3-2\beta)(a-A \\ +(1+\delta)((3\beta-2)(x_1(t)+x_2(t)) + (3-2\beta)x_3(t))) \\ -\gamma x_3(t)) \end{cases} \tag{14}$$

In a real market, we can consider k as the adaptability or learning ability of the firm 1. For example, the firm 1 analyzed the information in the past, and adjusted the speed of investment.The parameters are chosen as Figure 13,that is, $a = 20, b = 2, \gamma = 1.2, A = 2, \beta = 0.6, \alpha_1 = \alpha_2 = \alpha_3 = 0.3$. As can be seen from Figure 17, when control factor $k = 0.5$, compared with Figure 13, the appearance of chaos is delayed. With the increasing of α_1 , the system falls into a chaotic state when $\alpha_1 = 0.95$.

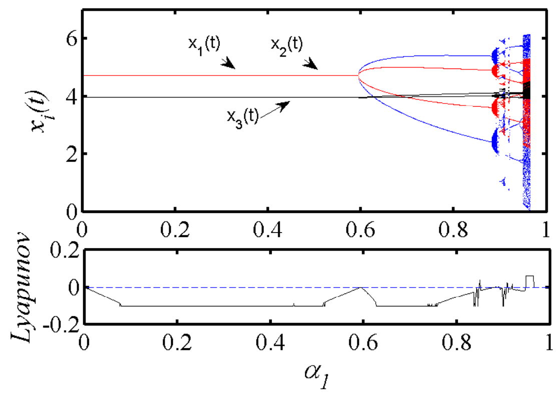

Figure 17: Bifurcation diagram maximal Lyapunov exponent with α_1 when $k = 0.5, \alpha_2 = \alpha_3 = 0.3$

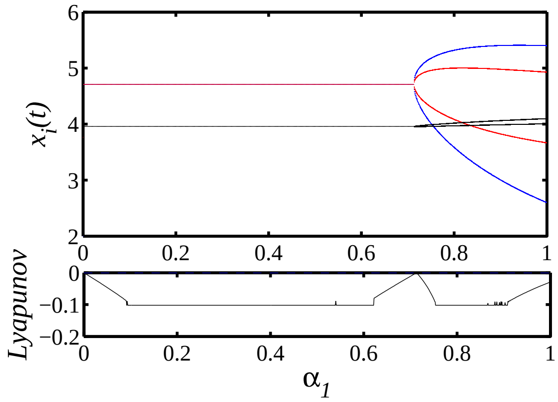

Figure 18: Bifurcation and maximal Lyapunov exponent with α_1 when $k = 0.8, \alpha_2 = \alpha_3 = 0.3$

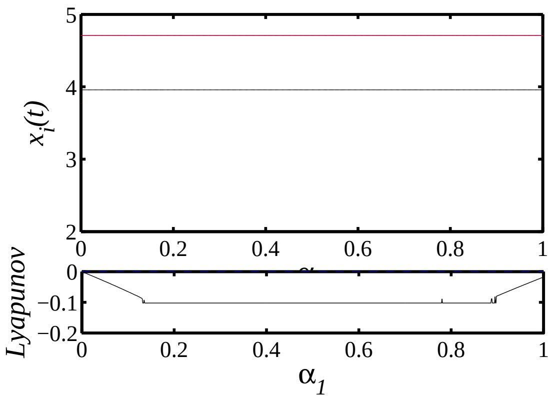

Figure 19: Bifurcation and maximal Lyapunov exponent with α_1 when $k = 1.6, \alpha_2 = \alpha_3 = 0.3$

As can be seen from Figure 18 , when control factor $k = 0.8$, chaos is eliminated. In Figure 19, $k = 1.6$, system 14 is in a stable state. From an economic point of view, if the first bounded rational player adopts this adjustment method, the R&D input game can be controlled to Nash equilibrium state with increase of k at last.

7 Basins of attraction of the cooperative team

In order to investigate the impact of technology spillover rate β and input adjustment speed α_i and on the domain of attraction, we introduce basins of attraction. The domain of attraction in the basins of attraction is the set of initial R&D input which can converge to the same attractor. If the domain of attraction in which the points converge to one equilibrium point, from an economic point of view, it is a safe region. That means if the initial R&D input of two sides is in the safe region, the system will remain stable after a number of games. If the initial input is in the escape area, the system will fall into divergence at last. We fix the initial R&D input of player 3 at 0.5 in (7), then (15) is obtained and the basins of attraction of player 1 and player 2 are investigated.

$$\begin{cases} x_1(t+1) = x_1(t) + \frac{\alpha_1 x_1(t)}{8b}((1+\delta)(2-\beta)[a-A \\ +(1+\delta)((2-\beta)(x_1(t)+x_2(t)) + (2\beta-1)x_3(t))] \\ -8b\gamma x_1(t)) \\ x_2(t+1) = x_2(t) + \frac{\alpha_2 x_2(t)}{8b}((1+\delta)(2-\beta)[a-A \\ +(1+\delta)((2-\beta)(x_1(t)+x_2(t)) + (2\beta-1)x_3(t))] \\ -8b\gamma x_2(t)) \\ x_3(t) = 0.5 \end{cases} \tag{15}$$

7.1 Impact of technology spillover rate on Basins of attraction of the cooperative team

Let $a = 20, b = 2, \gamma = 1.2, A = 2, \delta = 0.6, \alpha_1 = 0.5, \alpha_2 = 0.4$, and make $\beta = 0.55, 0.7, 0.8$, respectively, three basins of attraction about $(x_1(t), x_2(t))$ of the system are shown in Figure 20,21 and 22 in which the green set of points denotes domain of attraction, the red set of points denotes attractor, and the blue set of points denotes escape area.

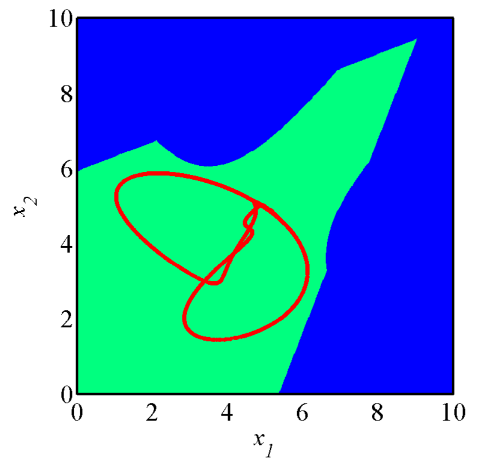

Figure 20: Basin of attraction with $\beta = 0.55$

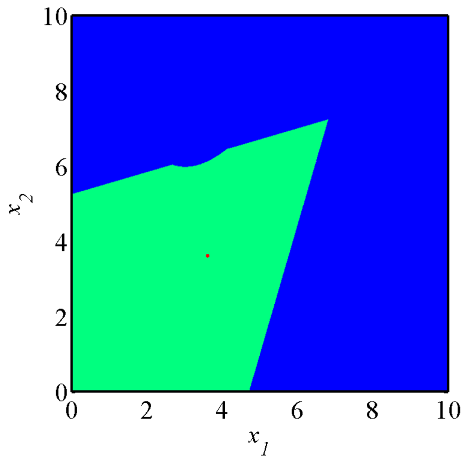

Figure 21: Basin of attraction with $\beta = 0.7$

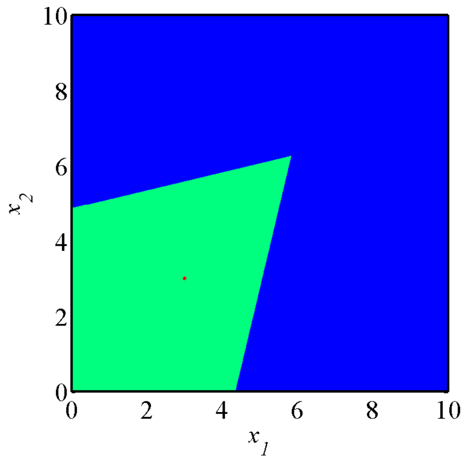

Figure 22: Basin of attraction with $\beta = 0.8$

In Figure 20, $\beta = 0.55$, the system(15) is in chaotic state,The input will converge to a chaotic attractor if the initial input is in the domain of attraction in Figure 20.

In Figure 21, $\beta = 0.7$, the system(15) is in stable state. We can see domain of attraction is an quadrilateral and it is a safe region. The attractor is the Nash equilibrium input that means if the initial R&D input of player 1 and player 2 is in this domain of attraction, the input will remain stable after iteration.

In Figure 22, $\beta = 0.8$, the system(15) is also in stable state. The domain of attraction also is a quadrilateral ,but it is smaller than the domain of attraction in Figure 21 ,it is a safe region,too

From the comparison of Figure 20,21 and 22, we find that the domain of attraction reduces with increase of technology spillover rate β under the conditions that the initial input of the third player is fixed. but if β is too low , the system (15) will be easy to fall into chaos.

From an economic perspective,the two players initial R&D input should be in the stable region in order to maintain the market stable after iteration, technology spillover rate should be kept within reasonable limits.

7.2 Impact of input adjustment speed on Basins of attraction of the cooperative team

Let $a = 20, b = 2, \gamma = 1.2, A = 2, \beta = 0.6, \delta = 0.6, \alpha_2 = 0.4$, and make $\alpha_1 = 0.5, 0.55, 0.64$, respectively, three basins of attraction about $(x_1(t), x_2(t))$ of the system are shown in Figure 23, 24, 25 .

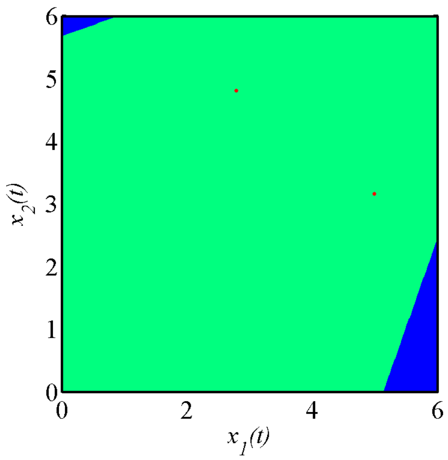

Figure 23: Basin of attraction with $\alpha_1 = 0.5$

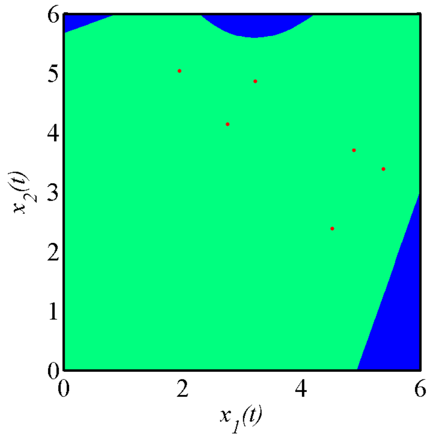

Figure 24: Basin of attraction with $\alpha_2 = 0.55$

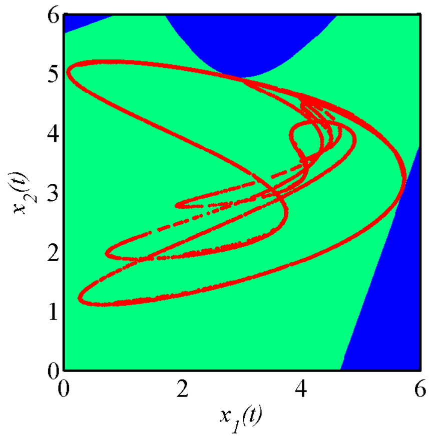

Figure 25: Basin of attraction with $\alpha_3 = 0.64$

In Figure 23,$\alpha_1 = 0.5$, the system(15) is in period-2 state.The attractor is a two period cycle that means if the initial R&D input of player 1 and player 2 is in this domain of attraction, the input will oscillate between the two points at last.

In Figure 24, $\alpha_1 = 0.55$, the system(15) is in period-6 state. The domain of attraction narrows and the attractor is a six period cycle.

In Figure 25, $\alpha_1 = 0.64$, the system(15) is in chaotic state. The input will converge to a chaotic attractor if the initial input is in the domain of attraction in Figure 25.

From the comparison of Figure 23, 24, 25, we find that the domain of attraction reduces with increase of R&D input modification speed under the conditions that the initial input of the third player is fixed. From an economic perspective, with increase of R&D input modification speed of player 1 and 2, the two players initial input should be lower in order to maintain the market stable.

8 Conclusion

In this paper, we consider a R&D input game model with spillover and endogenous demand in a triopoly market. Suppose two players in the triopoly make up a cooperative team to compete with the third one. The Nash equilibrium R&D input is unstable if the adjustment speed in R&D input is too high. 2D bifurcation diagram is introduced and we find that with increase of input modification speed, the system will lose stability via period-doubling bifurcations or Neimark-Sacker bifurcations. Effects of β and δ on profits are studied and we can see that that players' profits may not lose stability synchronously. A feedback control method is used to control the system to equilibrium state. Basins of attraction are investigated and show that, the domain of attraction becomes smaller with increase of spillover rate and input modification speed, and in order to maintain market stable, two team players' initial input must be kept within a certain range.

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