An Extended Solution to the Equations Describing a 3-Conductor Transmission Line

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Abstract: - The full derivation of the generalized and extended solution to the equations describing three-conductor transmission line is given in this paper; the brief results are presented in a previous paper. The considerations proceed from the C. Paul formulation of lossless transmission lines terminated by linear loads. In contrast to C. Paul, the conjoint interaction between the two lines is considered here and the influence of the receptor line is not neglected, that is the weak-coupling approximation is not applied. In result, an extended and generalized mathematical model compared the original model of C. Paul is obtained. In particular, a mixed problem for the hyperbolic system describing the three-conductor transmission line is formulated. It is shown that the formulated mixed problem is equivalent to an initial value problem for a functional system on the boundary of hyperbolic system’s domain with voltages and currents as the unknown functions in this system are the lines’. The system of functional equations can be resolved by a fixed-point method that enables us to find an approximated but explicit solution. The method elaborated in this paper might be applied also for linear as well as nonlinear boundary conditions.

Key-Words: three-conductor transmission line, electromagnetic compatibility, fixed point method, linear hyperbolic system, initial-boundary problem, mixed problem for hyperbolic system


1 Introduction

VLSI systems and their electromagnetic compatibility (EMC) aspects have attracted a lot of research interest (cf. [1]-[8]). In this paper an EMC model of a 3-conductor transmission line is considered taking into account the results of Clayton R. Paul [9]. In contrast to the considerations in [9], a generalized approach for solving the above problem is proposed. It is also proved that the weak coupling assumptions introduced in [9] are a particular case of the more general handling.

We obtain a general solution of the system by modeling pairwise the interacting 3-conductor transmission line introduced in [9] by keeping to the methods in [11]-[13] that were also used in other solutions such as [14], [15]. Our starting circuit is given in Fig. 1 (cf. [9]). The reference conductor for the line voltages is denoted by the ground symbol. Although it could represent an infinite ground plane, a wire, an overall shield, in our setup it is a land of a printed circuit board. The rest conductors are also lands of a printed circuit board, nevertheless they could be other objects as well. We presume the line to be an uniform and lossless line (cf. [7], [8]).

The top circuit is the generator circuit. It is terminated by a resistive load $R_L$ and it is driven by a voltage source with open-circuit voltage $U_S(t)$ and source resistance $R_S$. The bottom circuit is the receptor circuit. It is terminated by a resistive load $R_{NE}$ at the near end and by a resistive load $R_{FE}$ at the far end. At the terminals of the receptor circuit, the electric and magnetic fields originating by the voltage and current of the generator circuit, interact with the receptor circuit producing crosstalk voltages.
We aim at finding a solution for the crosstalk voltages based on a system that is more than the one in [9]. That is, we proceed from the hyperbolic system (1) obtained in accordance to the TEM mode of propagation (cf. [1]-[8]). The voltages with respect to the reference conductor \( u_k(x,t) \) \((k = 1; 2)\) and the currents of each circuit \( i_k(x,t) \) \((k = 1, 2)\) are functions of position \( x \) and time \( t \).

\[
\begin{align*}
\frac{\partial u_G(x,t)}{\partial x} + L_G \frac{\partial i_G(x,t)}{\partial t} &= -L_m \frac{\partial i_R(x,t)}{\partial t} \\
\frac{\partial i_G(x,t)}{\partial x} + (C_G + C_m) \frac{\partial u_G(x,t)}{\partial t} &= C_m \frac{\partial u_R(x,t)}{\partial t} \\
\frac{\partial u_R(x,t)}{\partial x} + L_R \frac{\partial i_R(x,t)}{\partial t} &= -L_m \frac{\partial i_G(x,t)}{\partial t} \\
\frac{\partial i_R(x,t)}{\partial x} + (C_R + C_m) \frac{\partial u_R(x,t)}{\partial t} &= C_m \frac{\partial u_G(x,t)}{\partial t}
\end{align*}
\]

(1)

with the following boundary

\[
\begin{align*}
u_G(0,t) = U_S(t) - R_S i_G(0,t), & \quad U_{NE} = u_R(0,t) = -R_{NE} i_R(0,t) \\
u_G(\Lambda,t) = R_A i_G(\Lambda,t), & \quad U_{FE} = u_R(\Lambda,t) = R_{FE} i_R(\Lambda,t)
\end{align*}
\]

(2)

and initial conditions:

\[
\begin{align*}
u_G(x,0) &= u_G(0), & \quad u_R(x,0) &= u_R(0) \\
i_G(x,0) &= i_G(0), & \quad i_R(x,0) &= i_R(0), \quad x \in [0,\Lambda]
\end{align*}
\]

(3)

Before going further, we would stress upon the fact that in our considerations, we do not apply the weak coupling assumption as opposite to [9] where this assumption is applied. This means that we do not neglect the right-hand side of (1). Therefore, our method is a more general case of (1).

Rewrite the above system (1) in the form

\[
\begin{align*}
(C_G + C_m) \frac{\partial u_G(x,t)}{\partial t} - C_m \frac{\partial u_R(x,t)}{\partial t} + \frac{\partial i_G(x,t)}{\partial x} &= 0 \\
-C_m \frac{\partial u_G(x,t)}{\partial t} + (C_R + C_m) \frac{\partial u_R(x,t)}{\partial t} + \frac{\partial i_R(x,t)}{\partial x} &= 0 \\
L_G \frac{\partial i_G(x,t)}{\partial t} + L_m \frac{\partial i_R(x,t)}{\partial t} + \frac{\partial u_G(x,t)}{\partial x} &= 0 \\
L_m \frac{\partial i_G(x,t)}{\partial t} + \frac{\partial u_R(x,t)}{\partial t} + \frac{\partial i_R(x,t)}{\partial x} &= 0
\end{align*}
\]

and introduce denotations

\[
\begin{align*}
u_1(x,t) &= u_G(x,t), & \quad \nu_2(x,t) &= u_R(x,t) \\
i_1(x,t) &= i_G(x,t), & \quad i_2(x,t) &= i_R(x,t) \\
C_{11} &= C_G + C_m, & \quad C_{12} &= C_{21} = -C_m, & \quad C_{22} &= C_R + C_m \\
L_1 &= L_G, & \quad L_2 &= L_{21} = L_m, & \quad L_{22} &= L_R \\
C &= \begin{pmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{pmatrix}, & \quad L = \begin{pmatrix} L_1 & L_{12} \\ L_{12} & L_{22} \end{pmatrix}
\end{align*}
\]

Then we reach the following mixed problem in the new denotations

\[
\begin{align*}
C_{11} \frac{\partial \nu_1(x,t)}{\partial t} + C_{12} \frac{\partial \nu_2(x,t)}{\partial t} + \frac{\partial i_1(x,t)}{\partial x} &= 0, \\
C_{12} \frac{\partial \nu_1(x,t)}{\partial t} + C_{22} \frac{\partial \nu_2(x,t)}{\partial t} + \frac{\partial i_2(x,t)}{\partial x} &= 0, \\
L_{11} \frac{\partial i_1(x,t)}{\partial t} + L_{12} \frac{\partial i_2(x,t)}{\partial t} + \frac{\partial \nu_1(x,t)}{\partial x} &= 0, \\
L_{12} \frac{\partial i_1(x,t)}{\partial t} + L_{22} \frac{\partial i_2(x,t)}{\partial t} + \frac{\partial \nu_2(x,t)}{\partial x} &= 0
\end{align*}
\]

(4)

\[
\begin{align*}
u_1(0,t) &= U_S(t) - R_S i_1(0,t), & \quad U_{NE} &= \nu_2(0,t) = -R_{NE} i_2(0,t), \\
u_1(\Lambda,t) &= R_A i_1(\Lambda,t), & \quad U_{FE} &= \nu_2(\Lambda,t) = R_{FE} i_2(\Lambda,t) \\
i_1(x,0) &= i_{10}(x), & \quad i_2(x,0) &= i_{20}(x), \quad x \in [0,\Lambda] \\
i_1(x,0) &= i_{10}(x), & \quad i_2(x,0) &= i_{20}(x), \quad x \in [0,\Lambda]
\end{align*}
\]

2 Hyperbolic system transformation

In a matrix form the above system (4) is
Since
\[ \Delta_c = C_1 C_2 - C_2^2 = (C_0 + C_m)(C_R + C_m) - C_m^2 = C_0 C_R + C_0 C_m + C_R C_m > 0 \]
we have to assume

**Assumption (L):** \( \Delta_L = L_0 L_R - L_m^2 = L_1 L_2 - L_{12}^2 \neq 0 \).

This implies
\[
[A] = \begin{bmatrix}
C_{11} & C_{12} & 0 & 0 \\
C_{12} & C_{22} & 0 & 0 \\
0 & 0 & L_{11} & L_{12} \\
0 & 0 & L_{12} & L_{22}
\end{bmatrix} = \Delta_c \Delta_L \neq 0
\]
and therefore \( A^{-1} \) does exist:

\[
A^{-1} = \frac{1}{\Delta_c \Delta_L} \begin{bmatrix}
C_{12} \Delta_L & -C_{12} \Delta_L & 0 & 0 \\
-C_{12} \Delta_L & C_{11} \Delta_L & 0 & 0 \\
0 & 0 & L_{22} \Delta_c & -L_{12} \Delta_c \\
0 & 0 & -L_{12} \Delta_c & L_{11} \Delta_c
\end{bmatrix}
\]

Multiplying (5) from the left by \( A^{-1} \) and in view of
\[
B = \begin{bmatrix}
\frac{C_{22}}{\Delta_c} & -\frac{C_{12}}{\Delta_c} & 0 & 0 \\
-\frac{C_{12}}{\Delta_c} & \frac{C_{11}}{\Delta_c} & 0 & 0 \\
0 & 0 & \frac{L_{22}}{\Delta_L} & -\frac{L_{12}}{\Delta_L} \\
0 & 0 & -\frac{L_{12}}{\Delta_L} & \frac{L_{11}}{\Delta_L}
\end{bmatrix}
\]
we obtain
\[
\begin{bmatrix}
\frac{\partial u_1}{\partial t} \\
\frac{\partial u_2}{\partial t} \\
\frac{\partial i_1}{\partial t} \\
\frac{\partial i_2}{\partial t}
\end{bmatrix} = \begin{bmatrix}
0 & 0 & \frac{C_{22}}{\Delta_c} & -\frac{C_{12}}{\Delta_c} \\
0 & 0 & -\frac{C_{12}}{\Delta_c} & \frac{C_{11}}{\Delta_c} \\
\frac{L_{22}}{\Delta_L} & -\frac{L_{12}}{\Delta_L} & 0 & 0 \\
-\frac{L_{12}}{\Delta_L} & \frac{L_{11}}{\Delta_L} & 0 & 0
\end{bmatrix} \begin{bmatrix}
\frac{\partial u_1}{\partial x} \\
\frac{\partial u_2}{\partial x} \\
\frac{\partial i_1}{\partial x} \\
\frac{\partial i_2}{\partial x}
\end{bmatrix} = 0 \quad (6)
\]

Rewrite (6) in a matrix form
\[
\frac{\partial U(x,t)}{\partial t} + B \frac{\partial U(x,t)}{\partial x} = 0 \quad (7)
\]

Substitute \( U(x,t) = H Z(x,t) \) in (7):
\[
H \frac{\partial Z(x,t)}{\partial t} + BH \frac{\partial Z(x,t)}{\partial x} = 0
\]

and multiplying by \( H^{-1} \) we obtain
\[
\frac{\partial Z(x,t)}{\partial t} + H^{-1}BH \frac{\partial Z(x,t)}{\partial x} = 0
\]

We have to find \( H \) such that \( H^{-1}BH = B^{can} \), where
\[
B^{can} = \begin{bmatrix}
\lambda_1 & 0 & 0 & 0 \\
0 & \lambda_2 & 0 & 0 \\
0 & 0 & \lambda_3 & 0 \\
0 & 0 & 0 & \lambda_4
\end{bmatrix}
\]

and \( \lambda_k (k = 1,2,3,4) \) are the eigen values of \( B \), i.e. the roots of the equation
\[
|B - \lambda I| = 0
\]
\[
\begin{align*}
\lambda^4 - L_{11} C_{11} + 2L_{12} C_{12} + L_{22} C_{22} \lambda^2 + \frac{1}{\Delta c \Delta L} &= 0 \\
\Delta c \Delta L \lambda^4 - (L_{11} C_{11} + 2L_{12} C_{12} + L_{22} C_{22}) \lambda^2 + 1 &= 0
\end{align*}
\]

We suppose

**Assumption (D):**

\[
D = (L_{11} C_{11} + 2L_{12} C_{12} + L_{22} C_{22})^2 - 4(L_{11} C_{11} - C_{12}^2)(L_{11} L_{22} - L_{12}^2) > 0
\]

For the characteristic roots we obtain

\[
\begin{align*}
\lambda_1 &= \sqrt{\frac{L_{11} C_{11} + 2L_{12} C_{12} + L_{22} C_{22}}{2(L_{11} C_{11} - C_{12}^2)}}, \\
\lambda_2 &= -\sqrt{\frac{L_{11} C_{11} + 2L_{12} C_{12} + L_{22} C_{22}}{2(L_{11} C_{11} - C_{12}^2)}}, \\
\lambda_3 &= -\sqrt{\frac{L_{11} C_{11} + 2L_{12} C_{12} + L_{22} C_{22}}{2(L_{11} C_{11} - C_{12}^2)}}, \\
\lambda_4 &= \sqrt{\frac{L_{11} C_{11} + 2L_{12} C_{12} + L_{22} C_{22}}{2(L_{11} C_{11} - C_{12}^2)}}.
\end{align*}
\]

**Remark.** For the sake of simplicity, we could find the eigenvectors of

\[
(B^{-1} - \mu_k I) H^{(k)} = 0;
\]

\[
\mu_k = 1/\lambda_k, \quad H^{(k)} = (\xi_{1k}, \xi_{2k}, \xi_{3k}, \xi_{4k})^T
\]

(instead of \((B - \lambda_k I) H^{(k)} = 0\)) because

\[
B^{-1} = \begin{pmatrix}
0 & 0 & L_{11} & L_{12} \\
0 & 0 & L_{12} & L_{22} \\
C_{11} & C_{12} & 0 & 0 \\
C_{12} & C_{22} & 0 & 0
\end{pmatrix}
\]

has a simpler form than \(B\).

Now we have to solve the following systems in order to obtain 4 eigenvectors \(H^{(k)}(k = 1, 2, 3, 4)\)

\[
\begin{align*}
(B - \lambda_1 I) H^{(1)} &= 0, \\
(B - \lambda_2 I) H^{(2)} &= 0, \\
(B - \lambda_3 I) H^{(3)} &= 0, \\
(B - \lambda_4 I) H^{(4)} &= 0
\end{align*}
\]

corresponding to eigenvector

\[
H^{(k)} = (\xi_{1k}, \xi_{2k}, \xi_{3k}, \xi_{4k})^T; \quad (k = 1, 2, 3, 4).
\]

To solve (9) we have to assume:

\[
L_{12} C_{11} + L_{22} C_{12} = L_m (C_G + C_m) - L_R C_m \neq 0
\]

and

\[
L_{12} C_{22} + L_{11} C_{12} = L_m (C_R + C_m) - L_G C_m \neq 0
\]

Therefore

\[
\begin{align*}
\xi_{1k} &= \frac{\lambda_k^2 \Delta_L C_{12} + L_{12}}{\lambda_k (L_{12} C_{11} + L_{22} C_{12})}, \\
\xi_{2k} &= \frac{L_{22} - \lambda_k^2 \Delta_L C_{11}}{\lambda_k (L_{12} C_{11} + L_{22} C_{12})}, \\
\xi_{3k} &= 1, \\
\xi_{4k} &= \frac{L_{12} C_{12} + L_{22} C_{22} - \lambda_k^2 \Delta_L \Delta_C}{(L_{12} C_{11} + L_{22} C_{12})}.
\end{align*}
\]

Introduce denotations:

\[
\gamma_k = \begin{pmatrix} 1/\lambda_k^2 \end{pmatrix} - \begin{pmatrix} L_{11} C_{11} + L_{12} C_{12} \end{pmatrix} = \frac{1 - \lambda_k^2 (L_{11} C_{11} + L_{12} C_{12})}{\lambda_k (L_{12} C_{11} + L_{22} C_{12})}.
\]

Note that

\[
\lambda_1 > \lambda_2 > 0; \quad \lambda_3 = -\lambda_1; \quad \lambda_4 = -\lambda_2;
\]

\[
\sqrt{D} \Rightarrow \gamma_2 - \gamma_1 = \frac{\sqrt{D}}{L_{12} C_{11} + L_{22} C_{12}}.
\]

Then we obtain the following eigenvectors:

\[
H^{(1)} = (p_1, q_1, 1, \gamma_1)^T, \quad H^{(2)} = (p_2, q_2, 1, \gamma_2)^T,
\]

\[
H^{(3)} = (-p_1, -q_1, 1, \gamma_1)^T, \quad H^{(4)} = (-p_2, -q_2, 1, \gamma_2)^T
\]

where

\[
\begin{align*}
p_k &= \frac{L_{12} + \lambda_k^2 \Delta_L C_{12}}{\lambda_k (L_{12} C_{11} + L_{22} C_{12})} = \frac{L_{11} + L_{12} \gamma_k}{\lambda_k (L_{12} C_{11} + L_{22} C_{12})}, \\
q_k &= \frac{L_{22} - \lambda_k^2 \Delta_L C_{11}}{\lambda_k (L_{12} C_{11} + L_{22} C_{12})} = \frac{L_{12} + L_{22} \gamma_k}{\lambda_k (L_{12} C_{11} + L_{22} C_{12})}, \quad (k = 1, 2)
\end{align*}
\]

Thus transformation matrix becomes
\[ H = \begin{bmatrix} p_1 & p_2 & -p_1 & -p_2 \\ q_1 & q_2 & -q_1 & -q_2 \\ 1 & 1 & 1 & 1 \end{bmatrix}. \]

Since \( |H| = 4 \sqrt{\Delta_L/\Delta_c} (\gamma_2 - \gamma_1)^2 \neq 0 \)

for the inverse matrix we obtain
\[
H^{-1} = \frac{1}{2(\gamma_2 - \gamma_1)} \begin{bmatrix} q_2 \sqrt{\Delta_L/\Delta_c} & -p_2 \sqrt{\Delta_L/\Delta_c} & \gamma_2 & -1 \\ -q_1 \sqrt{\Delta_L/\Delta_c} & p_1 \sqrt{\Delta_L/\Delta_c} & -\gamma_1 & 1 \\ -q_2 \sqrt{\Delta_L/\Delta_c} & p_2 \sqrt{\Delta_L/\Delta_c} & \gamma_2 & -1 \\ q_1 \sqrt{\Delta_L/\Delta_c} & -p_1 \sqrt{\Delta_L/\Delta_c} & -\gamma_1 & 1 \end{bmatrix}.
\]

because
\[
[p_1, p_2, q_1, q_2] = \lambda_1 \lambda_2 \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \Delta_L/\Delta_c (\gamma_2 - \gamma_1).
\]

### 3 Boundary conditions derivation with respect to the new variables

Introduce new variables \( U = HZ \) and \( Z = H^{-1} U \), where \( U = (u_1, u_2, i_1, i_2)^T \), \( Z = (I_1, I_2, I_3, I_4)^T \).

Then
\[
\begin{align*}
u_1(x,t) &= p_1 I_1(x,t) + p_2 I_2(x,t) - p_1 I_3(x,t) - p_2 I_4(x,t) \\
u_2(x,t) &= q_1 I_1(x,t) + q_2 I_2(x,t) - q_1 I_3(x,t) - q_2 I_4(x,t) \\
i_1(x,t) &= I_1(x,t) + I_2(x,t) + I_3(x,t) + I_4(x,t) \\
i_2(x,t) &= \gamma_1 I_1(x,t) + \gamma_2 I_2(x,t) + \gamma_3 I_3(x,t) + \gamma_4 I_4(x,t)
\end{align*}
\]

and
\[
\begin{align*}
I_1(x,t) &= \frac{1}{2(\gamma_2 - \gamma_1)} \begin{bmatrix} q_2 \sqrt{\Delta_L/\Delta_c} u_1(x,t) - p_2 \sqrt{\Delta_L/\Delta_c} u_2(x,t) + \gamma_2 i_1(x,t) - i_2(x,t) \\
-q_1 \sqrt{\Delta_L/\Delta_c} u_3(x,t) + p_1 \sqrt{\Delta_L/\Delta_c} u_4(x,t) - \gamma_1 i_3(x,t) + i_2(x,t) \end{bmatrix} \\
I_2(x,t) &= \frac{1}{2(\gamma_2 - \gamma_1)} \begin{bmatrix} q_1 \sqrt{\Delta_L/\Delta_c} u_1(x,t) + p_1 \sqrt{\Delta_L/\Delta_c} u_2(x,t) - \gamma_1 i_1(x,t) + i_2(x,t) \\
-q_2 \sqrt{\Delta_L/\Delta_c} u_3(x,t) - p_2 \sqrt{\Delta_L/\Delta_c} u_4(x,t) + \gamma_2 i_3(x,t) - i_2(x,t) \end{bmatrix}
\end{align*}
\]

Then the mixed problem (1)-(3) becomes as follows: to find a solution of the system
\[
\begin{align*}
\frac{\partial I_1(x,t)}{\partial t} + \lambda_1 \frac{\partial I_1(x,t)}{\partial x} &= 0, \\
\frac{\partial I_2(x,t)}{\partial t} + \lambda_2 \frac{\partial I_2(x,t)}{\partial x} &= 0, \\
\frac{\partial I_3(x,t)}{\partial t} - \lambda_1 \frac{\partial I_3(x,t)}{\partial x} &= 0, \\
\frac{\partial I_4(x,t)}{\partial t} - \lambda_2 \frac{\partial I_4(x,t)}{\partial x} &= 0
\end{align*}
\]

with initial conditions and boundary conditions in the new variables:

\[
\begin{align*}
I_1(x,0) &= \frac{1}{2(\gamma_2 - \gamma_1)} \begin{bmatrix} q_2 \sqrt{\Delta_L/\Delta_c} u_{i_0}(x) - p_2 \sqrt{\Delta_L/\Delta_c} u_{i_3}(x) + \gamma_2 i_{i_0}(x) - i_{i_3}(x) \\
-q_1 \sqrt{\Delta_L/\Delta_c} u_{i_1}(x) + p_1 \sqrt{\Delta_L/\Delta_c} u_{i_2}(x) - \gamma_1 i_{i_1}(x) + i_{i_2}(x) \end{bmatrix} = I_{i_0}(x) \\
I_2(x,0) &= \frac{1}{2(\gamma_2 - \gamma_1)} \begin{bmatrix} q_1 \sqrt{\Delta_L/\Delta_c} u_{i_0}(x) + p_1 \sqrt{\Delta_L/\Delta_c} u_{i_3}(x) - \gamma_1 i_{i_0}(x) + i_{i_3}(x) \\
-q_2 \sqrt{\Delta_L/\Delta_c} u_{i_1}(x) - p_2 \sqrt{\Delta_L/\Delta_c} u_{i_2}(x) + \gamma_2 i_{i_1}(x) - i_{i_2}(x) \end{bmatrix} = I_{i_3}(x) \\
I_3(x,0) &= \frac{1}{2(\gamma_2 - \gamma_1)} \begin{bmatrix} q_1 \sqrt{\Delta_L/\Delta_c} u_{i_0}(x) + p_1 \sqrt{\Delta_L/\Delta_c} u_{i_3}(x) + \gamma_1 i_{i_0}(x) + i_{i_3}(x) \\
-q_2 \sqrt{\Delta_L/\Delta_c} u_{i_1}(x) - p_2 \sqrt{\Delta_L/\Delta_c} u_{i_2}(x) - \gamma_2 i_{i_1}(x) - i_{i_2}(x) \end{bmatrix} = I_{i_1}(x) \\
I_4(x,0) &= \frac{1}{2(\gamma_2 - \gamma_1)} \begin{bmatrix} q_2 \sqrt{\Delta_L/\Delta_c} u_{i_0}(x) - p_2 \sqrt{\Delta_L/\Delta_c} u_{i_3}(x) - \gamma_2 i_{i_0}(x) + i_{i_3}(x) \\
-q_1 \sqrt{\Delta_L/\Delta_c} u_{i_1}(x) + p_1 \sqrt{\Delta_L/\Delta_c} u_{i_2}(x) + \gamma_1 i_{i_1}(x) - i_{i_2}(x) \end{bmatrix} = I_{i_2}(x)
\end{align*}
\]

To obtain the boundary conditions
\[
\begin{align*}
u_1(0,t) &= U_s(0,t) - R_s i_1(0,t), \\
u_1(\Lambda, t) &= R_s i_1(\Lambda, t), \\
u_{NE} &= u_2(0,t) - R_{NE} i_2(0,t), \\
u_{SE} &= u_2(\Lambda, t) = R_{SE} i_2(\Lambda, t)
\end{align*}
\]

with respect to the new variables we take into account.
Proceeding as in [11], we find the characteristics of curves

\[ x - \lambda_p T = \text{const} \] (with \( \lambda_3 = -\lambda_1 \), \( \lambda_4 = -\lambda_2 \)) and the straight line through a point \((A, \hat{t})\) is

\[ x + \lambda_p \hat{t} = \Lambda \implies \lambda_p \hat{t} = \Lambda - \lambda_p \hat{t} \Rightarrow T_p = \Lambda / \lambda_p. \]

Similarly, a characteristic \( C_p \) \((p = 3,4)\) is

Integration of \((17)\) from \((0,t)\) to \((\Lambda, t)\) where the integration is a line integral along \( C_k \), we obtain

\[ I_k(\Lambda, t) = I_k(0,t) \] \((t \geq 0)\).

In the same way, by integrating in \((17)\) from \((0,t)\) to \((\Lambda, t)\), we get

\[ I_k(\Lambda, t) = I_k(0,t + T_k) \] \((t \geq 0)\).

4 Obtaining functional equations equivalent to the mixed problem

Present (12) in the form:

\[ \begin{align*}
\Delta_2 - \lambda_k \hat{t} + T_k = -\lambda_k \hat{t} & \Rightarrow T_k = \Lambda / \lambda_k. \\
\end{align*} \]

and

\[ u_A(\Lambda, t) = p_1 I_1(\Lambda, t) + p_2 I_2(\Lambda, t) - p_1 I_3(\Lambda, t) - p_2 I_4(\Lambda, t) \]

Consequently

\[ p_1 I_1(0,t) + p_2 I_2(0,t) - p_1 I_3(0,t) - p_2 I_4(0,t) = = U_1(t) - R_1[I_p(0,t) + I_3(0,t)] = = R_1[I_p(0,t) + I_3(0,t)] \]

Through each point \((x,t) \in \Pi = \{x,t\} \in [0,\Lambda] \times [0,T]\) there are 4 characteristics: \(C_1, C_2\) with positive slopes and \(C_3, C_4\) with negative slopes. A characteristic \(C_k\) \((k = 1,2)\) through a point \((0,\hat{t})\) intersects the boundary \(x = \) at some point \((\Lambda, \hat{t})\) where \(T_k\) can be found by integration of \(dx/dt = \lambda_k\). Since the characteristic \(C_k\) is \(x - \lambda_k T = \text{const}\), then the straight line through \((0,\hat{t})\) is

\[ x - \lambda_k \hat{t} = -\lambda_k \hat{t} \Rightarrow t = \frac{x}{\lambda_k} + \hat{t}. \]

Setting \( x = \Lambda \) and \( t = \hat{t} + T_k \) we obtain

\[ \Delta_2 - \lambda_k (\hat{t} + T_k) = -\lambda_k \hat{t} \Rightarrow T_k = \Lambda / \lambda_k. \]
Hence, we can solve first two equations with respect to $I_1(0, t)$, $I_2(0, t)$ which leads to

$$I_1(0, t) = A_{01}(t) + A_{11}I_3(0, t) + A_{12}I_4(0, t),$$

where

$$A_{01}(t) = \frac{(q_2 + R_{SE} \gamma_2)U_3(t)}{\Delta_{12}},$$

$$A_{11} = \frac{2(p_2 \gamma_1 R_{FE} - q_2 R_S)}{\Delta_{12}},$$

$$A_{12} = \frac{p_1 q_2 - p_2 q_1 - (q_2 + q_1) R_S}{\Delta_{12}} + \frac{(p_1 \gamma_2 + p_2 \gamma_1) R_{SE} + (\gamma_1 - \gamma_2) R_S R_{SE}}{\Delta_{12}};$$

$$I_2(0, t) = A_{20}(t) + A_{21}I_3(0, t) + A_{22}I_4(0, t),$$

where

$$A_{20}(t) = \frac{- (q_1 + R_{SE} \gamma_1)U_3(t)}{\Delta_{12}},$$

$$A_{21} = \frac{p_1 q_2 - p_2 q_1 + (q_2 + q_1) R_S}{\Delta_{12}} - \frac{(p_1 \gamma_2 + p_2 \gamma_1) R_{SE} + (\gamma_1 - \gamma_2) R_S R_{SE}}{\Delta_{12}},$$

$$A_{22} = \frac{-2 p_2 \gamma_1 + 2q_2 R_S}{\Delta_{12}}.$$

Similarly

$$\Delta_{34} = \left| \begin{array}{cc} p_2 + R_L & p_1 + R_L \\ q_2 + p_2 \gamma_1 R_{FE} & q_1 + p_1 \gamma_1 R_{FE} \end{array} \right| = (p_2 + R_L)(q_1 + \gamma_1 R_{FE}) - (q_2 + \gamma_2 R_{FE})(p_1 + R_L) =$$

$$= \frac{- \sqrt{D} \left( \sqrt{\Delta_L / \Delta_C + R_S R_{SE}} \right) + (\lambda_1 - \lambda_2)}{L_{12} C_{11} + L_{22} C_{12}} + \frac{(\lambda_1 - \lambda_2)}{L_{12} C_{11} + L_{22} C_{12}} \times \left( C_{22} \Delta_L + L_{11}(\Delta_L \Delta_C - R_{FE}^2) \right) R_{FE} + \left( \frac{L_{22} + \lambda_1 \lambda_2 \Delta_C}{\lambda_1 \lambda_2} \right) R_L \right) \neq 0.$$

Consequently

$$I_3(\Lambda, t) = B_{11}I_1(\Lambda, t) + B_{12}I_2(\Lambda, t),$$

where

$$B_{11} = \frac{2p_2 \gamma_1 R_{FE} - 2q_2 R_S}{\Delta_{34}};$$

$$B_{12} = \frac{(p_2 q_1 - p_1 q_2) - (q_1 + q_2) R_S}{\Delta_{34}} + \frac{(p_2 \gamma_1 + p_1 \gamma_2) R_{FE} + (\gamma_2 - \gamma_1) R_S R_{FE}}{\Delta_{34}}.$$

So, we have obtained a system of functional equations

$$I_1(0, t) = A_{00}(t) + A_{10}I_2(0, t) + A_{11}I_3(0, t) + A_{12}I_4(0, t),$$

$$I_2(0, t) = A_{20}(t) + A_{21}I_3(0, t) + A_{22}I_4(0, t),$$

$$I_3(\Lambda, t) = B_{21}I_1(\Lambda, t) + B_{22}I_2(\Lambda, t),$$

$$I_4(\Lambda, t) = B_{31}I_1(\Lambda, t) + B_{32}I_2(\Lambda, t).$$

Taking into account $I_k(\Lambda, t - T_k) = I_k(0, t)$, $(k = 3, 4)$ we can rewrite the first two equations in the following way:

$$I_1(0, t) = A_{00}(t) + A_{10}I_2(0, t) + A_{11}I_3(0, t) + A_{12}I_4(0, t),$$

$$I_2(0, t) = A_{20}(t) + A_{21}I_3(0, t) + A_{22}I_4(0, t).$$

Similarly in view of $I_k(\Lambda, t - T_k) = I_k(0, t)$ $(k = 1, 2)$ we obtain

$$I_3(\Lambda, t) = B_{31}I_1(0, t - T_1) + B_{32}I_2(0, t - T_2).$$
Denoting the unknown functions by
\[ I_1(t, \tau) = I_1(\tau), \quad I_2(0, t) = I_2(t), \]
\[ I_3(t) = I_3(A, t), \quad I_4(t) = I_4(A, t) \]
and taking into account \( T_1 = T_3, \) \( T_2 = T_4 \) we obtain the following system:
\[
\begin{align*}
I_1(t) &= A_0(t) + A_1 I_3(t - T_1) + A_2 I_4(t - T_2), \\
I_2(t) &= A_{30}(t) + A_1 I_3(t - T_1) + A_{32} I_4(t - T_2), \\
I_3(t) &= B_1 I_1(t - T_1) + B_1 I_2(t - T_2), \\
I_4(t) &= B_{21} I_1(t - T_1) + B_{22} I_2(t - T_2).
\end{align*}
\]

To obtain initial conditions on the intervals \([-T_1, 0], [-T_2, 0]\) one can shifted the initial functions \(I_{10}(x), I_{20}(x), i_{10}(x), i_{20}(x)\) from the interval \([0, \infty)\) along the characteristics to the intervals \([-\tau_1, 0], [-\tau_2, 0]\) (cf. [12]).

The obtained functions after the above transformation on the boundary we denote by
\[ I_{10}(t), \quad I_{20}(t), \quad I_{30}(t), \quad I_{40}(t). \]

If \(u_{10}(x), u_{20}(x), i_{10}(x), i_{20}(x)\) are periodic functions then \(I_{10}(t), I_{20}(t), I_{30}(t), I_{40}(t)\) are periodic functions too.

5 Operator presentation of the periodic problem

Introduce the sets
\[ M_1 = \left\{ I_1(t) \in C_{\tau_1}(0, \infty) : \left| I_1(t) \right| \leq I_{10} e^{\mu_0 (t-T_{\tau_1})}, \quad t \in [kT_{\tau_1}, (k+1)T_{\tau_1}], \quad I_1(t) = I_{10}(t), t \in [-T_1, 0) \right\}, \]
\[ M_2 = \left\{ I_2(t) \in C_{\tau_1}(0, \infty) : \left| I_2(t) \right| \leq I_{20} e^{\mu_0 (t-T_{\tau_2})}, \quad t \in [kT_{\tau_2}, (k+1)T_{\tau_2}], \quad I_2(t) = I_{20}(t), t \in [-T_2, 0) \right\}, \]
\[ M_3 = \left\{ I_3(t) \in C_{\tau_1}(0, \infty) : \left| I_3(t) \right| \leq I_{30} e^{\mu_0 (t-T_{\tau_3})}, \quad t \in [kT_{\tau_3}, (k+1)T_{\tau_3}], \quad I_3(t) = I_{30}(t), t \in [-T_3, 0) \right\}, \]
\[ M_4 = \left\{ I_4(t) \in C_{\tau_1}(0, \infty) : \left| I_4(t) \right| \leq I_{40} e^{\mu_0 (t-T_{\tau_4})}, \quad t \in [kT_{\tau_4}, (k+1)T_{\tau_4}], \quad I_4(t) = I_{40}(t), t \in [-T_4, 0) \right\}. \]

\[ (p_1, p_2, p_3, p_4) \in N_0 \times N_0 \times N_0 \times N_0; \quad N_0 = \{0, 1, 2, \ldots\} \]
corresponding to the initial points of the intervals
\[ [p_1 T_1, (p_1 + 1) T_1] \times [p_2 T_2, (p_2 + 1) T_2] \times [p_3 T_3, (p_3 + 1) T_3] \times [p_4 T_4, (p_4 + 1) T_4]. \]

Introduce maps \(j_n(k) : N_0 \rightarrow N_0\) \((n = 1, 2)\) in the following way:
\[ j_1 : N_0 \rightarrow N_0, [kT_1, (k+1)T_1] \rightarrow [kT_1, T_2, k+1, T_1 - T_2] \]
\[ j_2 : N_0 \rightarrow N_0, [kT_2, (k+1)T_2] \rightarrow [kT_2, T_3, k+1, T_2 - T_3]. \]

We suppose that \( T_p = m_p T_{\tau_p}, \) \((p = 1, 2)\). Therefore
\[ [kT_p - T_p, (k+1)T_p - T_p] = [kT_{\tau_p} - T_{\tau_p}, k+1, T_{\tau_p} - T_{\tau_p}] \]
and then \( j_p(k) : k \rightarrow k - m_p \) provided \( k - m_p \geq 0; \)
\[ j_p^n(k) = j_p(j_p^{n-1}(k)), j_p^0(k) = k. \]

The definition of \( j_p \) implies that \( j_p^n(k) \in N_0 \) only for finite \( m. \)

Define
\[ j(p_1, p_2, p_3, p_4) = \left( j_1(p_1), j_2(p_2), j_3(p_3), j_4(p_4) \right), \]
\[ j^2(p_1, p_2, p_3, p_4) = \left( j_1^2(p_1), j_2^2(p_2), j_3^2(p_3), j_4^2(p_4) \right). \]

In particular,
\[ j(p, p, p, p) = \left( j_1(p), j_2(p), j_3(p), j_4(p) \right). \]

The set \( M \) turns out into a complete uniform space with a saturated family of pseudometrics (cf. [13])
\[
\rho_{(p_1, p_2, p_3, p_4)}(I_1, I_2, I_3, I_4, (T_1, T_2, T_3, T_4)) = 
\rho^{(p_1)}(I_1, T_1) + \rho^{(p_2)}(I_2, T_2) + \rho^{(p_3)}(I_3, T_3) + \rho^{(p_4)}(I_4, T_4) 
\]

Now we formulate the main problem: to find a \(T_0\)-periodic solution of:

\[
\begin{align*}
I_1(t) &= A_{10}(t) + A_{11}I_1(t-T_1) + A_{12}I_2(t-T_2), \\
I_2(t) &= A_{20}(t) + A_{21}I_1(t-T_1) + A_{22}I_2(t-T_2), \\
I_3(t) &= B_{10}I_1(t-T_1) + B_{11}I_2(t-T_2), \\
I_4(t) &= B_{20}I_1(t-T_1) + B_{21}I_2(t-T_2),
\end{align*}
\]

where

\[
\begin{align*}
I_1(t) &= I_{10}(t), \quad t \in [-T_1, 0], \\
I_2(t) &= I_{20}(t), \quad t \in [-T_2, 0], \\
I_3(t) &= I_{30}(t), \quad t \in [-T_1, 0], \\
I_4(t) &= I_{40}(t), \quad t \in [-T_2, 0]
\end{align*}
\]

We define an operator with components \(B = (B_1, B_2, B_3, B_4)\) by the formulas:

\[
\begin{align*}
B_1(I_1, I_2, I_3, I_4)(t) := \\
&\begin{cases} \\
A_{10}(t) + A_{11}I_1(t-T_1) + A_{12}I_2(t-T_2) & t \in [0, \infty) \\
I_1(t) = I_{10}(t), & t \in [-T_1, 0] \\
I_4(t) = I_{40}(t), & t \in [-T_2, 0]
\end{cases}
\end{align*}
\]

\[
\begin{align*}
B_2(I_1, I_2, I_3, I_4)(t) := \\
&\begin{cases} \\
A_{20}(t) + A_{21}I_1(t-T_1) + A_{22}I_2(t-T_2) & t \in [0, \infty) \\
I_1(t) = I_{10}(t), & t \in [-T_1, 0] \\
I_4(t) = I_{40}(t), & t \in [-T_2, 0]
\end{cases}
\end{align*}
\]

\[
\begin{align*}
B_3(I_1, I_2, I_3, I_4)(t) := \\
&\begin{cases} \\
B_{10}I_1(t-T_1) + B_{11}I_2(t-T_2) & t \in [0, \infty) \\
I_1(t) = I_{10}(t), & t \in [-T_1, 0] \\
I_2(t) = I_{20}(t), & t \in [-T_2, 0]
\end{cases}
\end{align*}
\]

\[
\begin{align*}
B_4(I_1, I_2, I_3, I_4)(t) := \\
&\begin{cases} \\
B_{20}I_1(t-T_1) + B_{22}I_2(t-T_2) & t \in [0, \infty) \\
I_1(t) = I_{10}(t), & t \in [-T_1, 0] \\
I_2(t) = I_{20}(t), & t \in [-T_2, 0]
\end{cases}
\end{align*}
\]

\[
I_{10}(\cdot), I_{30}(\cdot) \in C_{\mathcal{R}_1}[-T_1, 0], \quad I_{20}(\cdot), I_{40}(\cdot) \in C_{\mathcal{R}_2}[-T_2, 0]
\]

\[
U_S(\cdot) \in C_{\mathcal{R}_3}[0, \infty), \quad U_S = \max \{U_S(t); t \in [0, T_0]\}
\]

Assumptions (D) and (L) are valid:

\[
T_1 = m_1T_0, T_2 = m_2T_0 \quad \text{for positive integers } m_1, m_2
\]

\[
\left[\frac{q_2 + R_{NE}T_2}{|\Delta_{12}|}\right] U_S = \frac{1}{|\Delta_{12}|} \left[\frac{\lambda_2 L_{12} + (L_{22} \lambda_2 + R_{NE}) \times }{\lambda_2^2 (L_{12} C_{11} + L_{12} C_{12})} \right] < \min \{I_{10}, I_{20}\}
\]

Then there exists a unique \(T_0\)-periodic solution of (19).

**Proof:** The set \(M_1 \times M_2 \times M_3 \times M_4\) is a uniform space with the above saturated family of pseudometrics. In view of \(\lambda_1 > \lambda_2 \Rightarrow \lambda / \lambda_1 > \lambda / \lambda_2 \Rightarrow T_2 > T_1\) we show that \(B\) maps \(M_1 \times M_2 \times M_3 \times M_4\) into itself. It is easy to verify that all components of the operator \(B\) are periodic functions.

For \(t \in [kT_0, (k+1)T_0]\) and sufficiently large \(\mu > 0\), having in mind (12), (13) we obtain:

\[
\left[\frac{B^{(p)}(I_1, I_2, I_3, I_4)(t)}{\mu}\right] \leq \left|A_{10}(t)\right| + \left|A_{11}\right|I_1(t-T_1) + \left|A_{12}\right|I_2(t-T_2) \leq \left|A_{10}(t)\right| + \left|A_{11}\right| I_1 e^{\mu(t-T_1)} + \left|A_{12}\right| I_2 e^{\mu(t-T_2)} \leq e^{\mu t} \left[\frac{q_1 + R_{NE}T_1}{|\Delta_{12}|}\right] U_S + \left|A_{11}\right| I_1 e^{\mu t} + \left|A_{12}\right| I_2 e^{\mu t} \leq e^{\mu t} \left[\frac{q_1 + R_{NE}T_1}{|\Delta_{12}|}\right] U_S + \left|A_{11}\right| I_1 e^{\mu t} + \left|A_{12}\right| I_2 e^{\mu t}
\]

6 Existence–uniqueness of periodic solution

The main result is:

**Theorem 1.** Let the following conditions be fulfilled:
\[ |B_{14}^{(1)}(I_1, I_2, I_3, I_4)(t) | \leq \\
\leq [A_{32}(t)] + |A_{42}| |I_2(t - T_2)| + |A_{42}| |I_1(t - T_1)| \\
\leq \left[ |B_{11}| + R_{\alpha_{\infty}}^{(1)} |\mathcal{U}_S| + |A_{12}| I_{30} e^{\mu(t - T_1 - T_2)} + |A_{22}| I_{40} e^{\mu(t - T_1 - T_2)} \right] \\
\leq e^{\mu(t - T_1 - T_2)} \left[ |B_{11}| + R_{\alpha_{\infty}}^{(1)} |\mathcal{U}_S| + |A_{12}| I_{30} e^{\mu T_1} + |A_{22}| I_{40} e^{-\mu T_1} \right] \\
\leq e^{\mu(t - T_1 - T_2)} I_{20}
\]

\[ |B_{12}^{(1)}(I_1, I_2, I_3, I_4)(t) | \leq \\
\leq |B_{11}| |I_1(t - T_1)| + |B_{22}| |I_2(t - T_2)| \\
\leq |B_{11}| I_{10} e^{\mu T_1} + |B_{22}| I_{20} e^{-\mu T_1} \\
\leq e^{\mu(t - T_1 - T_2)} I_{20}
\]

\[ |B_{13}^{(1)}(I_1, I_2, I_3, I_4)(t) | \leq \\
\leq |B_{11}| |I_1(t - T_1)| + |B_{22}| |I_2(t - T_2)| \\
\leq |B_{11}| I_{10} e^{\mu T_1} + |B_{22}| I_{20} e^{-\mu T_1} \\
\leq e^{\mu(t - T_1 - T_2)} I_{20}
\]

It remains to show that \( B \) is contractive operator. We notice that \( T_2 > T_1 \Rightarrow -\mu T_2 < -\mu T_1 \Rightarrow e^{-\mu T_1} < e^{-\mu T_2} \).

Then

\[
\max \left\{ |A_{11}| e^{-\mu T_1} + |A_{12}| e^{-\mu T_1} | A_{21}, e^{-\mu T_1} + |A_{22}| e^{-\mu T_1} \right\} \leq e^{-\mu T_1} \max \left\{ |A_{11}| + |A_{12}|, |A_{21}| + |A_{22}| \right\} = K(\mu)
\]

and for \( t \in [kT_0, (k+1)T_0] \) we have

\[
|B_{14}^{(1)}(I_3, I_4)(t) - B_{14}^{(1)}(I_3, I_4)(t) | \leq \\
\leq |A_{11}| |I_3(t - T_1)| + |A_{21}| |I_2(t - T_2)| \\
\leq |A_{11}| |I_3(t - T_1)| + |A_{21}| |I_2(t - T_2)| e^{\mu(t - T_1 - T_2)} e^{\mu(t - T_1 - T_2)} \\
\leq e^{\mu(t - T_1 - T_2)} |A_{11}| |I_3(t - T_1)| + |A_{21}| |I_2(t - T_2)| e^{\mu(t - T_1 - T_2)}
\]

that implies

\[
\rho^{(1)}(B_{14}^{(1)}(I_3, I_4), B_{14}^{(1)}(I_3, I_4) | \leq \\
\leq |A_{11}| e^{\mu T_1} \rho^{(1)}(I_3, I_3) + |A_{21}| e^{\mu T_1} \rho^{(1)}(I_4, I_4) \\
\leq |A_{11}| e^{\mu T_1} (I_3, I_3) + |A_{21}| e^{\mu T_1} (I_4, I_4) + \rho^{(1)}(I_3, I_3) + \rho^{(1)}(I_4, I_4)
\]

which implies

\[
\rho^{(1)}(B_{12}^{(1)}(I_1, I_4)(t) - B_{12}^{(1)}(I_1, I_4)(t) | \leq \\
\leq |A_{11}| |I_1(t - T_1)| - |I_1(t - T_1)| + |A_{22}| |I_2(t - T_2)| - |I_2(t - T_2)| \\
\leq |A_{11}| |I_1(t - T_1)| - |I_1(t - T_1)| e^{\mu(t - T_1 - T_2)} e^{\mu(t - T_1 - T_2)} \\
\leq e^{\mu(t - T_1 - T_2)} |A_{11}| |I_1(t - T_1)| + |A_{22}| |I_2(t - T_2)| e^{\mu(t - T_1 - T_2)}
\]

Further on we have

\[
\rho^{(1)}(B_{13}^{(1)}(I_3, I_4)(t) - B_{13}^{(1)}(I_3, I_4)(t) | \leq \\
\leq |A_{11}| |I_3(t - T_1)| - |I_3(t - T_1)| + |A_{22}| |I_2(t - T_2)| - |I_2(t - T_2)| \\
\leq |A_{11}| |I_3(t - T_1)| - |I_3(t - T_1)| e^{\mu(t - T_1 - T_2)} e^{\mu(t - T_1 - T_2)} \\
\leq e^{\mu(t - T_1 - T_2)} |A_{11}| |I_3(t - T_1)| + |A_{22}| |I_2(t - T_2)| e^{\mu(t - T_1 - T_2)}
\]

that implies
\[ \rho^{(k)} \left( B^{(k)}(I_1, I_2), B^{(k)}(\bar{T}, \bar{T}) \right) \leq \]
\[ \leq [B_2] \left| e^{-\mu T} \rho^{(k)} (I_1, \bar{T}) + [B_{2}] e^{-\mu T} \rho^{(k)} (I_2, \bar{T}) \right| \]
\[ \leq [B_2] \left| e^{-\mu T} \rho^{(k)} (I_1, \bar{T}) + [B_{2}] e^{-\mu T} \rho^{(k)} (I_2, \bar{T}) \right| \]
\[ \leq K(\mu) \left( \rho^{(k)} (I_1, \bar{T}) + \rho^{(k)} (I_2, \bar{T}) + \rho^{(k)} (I_4, \bar{T}) \right) \]

Finally, we have
\[ \left[ B^{(k)}(I_1, I_2), B^{(k)}(\bar{T}, \bar{T}) \right] \leq \]
\[ \leq [B_2] \left| e^{-\mu T} \rho^{(k)} (I_1, \bar{T}) + [B_{2}] e^{-\mu T} \rho^{(k)} (I_2, \bar{T}) \right| \]
\[ \leq [B_2] \left| e^{-\mu T} \rho^{(k)} (I_1, \bar{T}) + [B_{2}] e^{-\mu T} \rho^{(k)} (I_2, \bar{T}) \right| \]
\[ \leq [B_2] \left| e^{-\mu T} \rho^{(k)} (I_1, \bar{T}) + [B_{2}] e^{-\mu T} \rho^{(k)} (I_2, \bar{T}) \right| \]
\[ \leq K(\mu) \left( \rho^{(k)} (I_1, \bar{T}) + \rho^{(k)} (I_2, \bar{T}) + \rho^{(k)} (I_4, \bar{T}) \right) \]

which implies
\[ \rho^{(k)} \left( B^{(k)}(I_1, I_2), B^{(k)}(\bar{T}, \bar{T}) \right) \leq \]
\[ \leq [B_2] \left| e^{-\mu T} \rho^{(k)} (I_1, \bar{T}) + [B_{2}] e^{-\mu T} \rho^{(k)} (I_2, \bar{T}) \right| \]
\[ \leq [B_2] \left| e^{-\mu T} \rho^{(k)} (I_1, \bar{T}) + [B_{2}] e^{-\mu T} \rho^{(k)} (I_2, \bar{T}) \right| \]
\[ \leq [B_2] \left| e^{-\mu T} \rho^{(k)} (I_1, \bar{T}) + [B_{2}] e^{-\mu T} \rho^{(k)} (I_2, \bar{T}) \right| \]
\[ \leq K(\mu) \left( \rho^{(k)} (I_1, \bar{T}) + \rho^{(k)} (I_2, \bar{T}) + \rho^{(k)} (I_4, \bar{T}) \right) \]

Therefore
\[ \rho^{(k)} \left( B^{(k)}(I_1, I_2), B^{(k)}(\bar{T}, \bar{T}) \right) \leq \]
\[ \leq [B_2] \left| e^{-\mu T} \rho^{(k)} (I_1, \bar{T}) + [B_{2}] e^{-\mu T} \rho^{(k)} (I_2, \bar{T}) \right| \]
\[ \leq [B_2] \left| e^{-\mu T} \rho^{(k)} (I_1, \bar{T}) + [B_{2}] e^{-\mu T} \rho^{(k)} (I_2, \bar{T}) \right| \]
\[ \leq K(\mu) \left( \rho^{(k)} (I_1, \bar{T}) + \rho^{(k)} (I_2, \bar{T}) + \rho^{(k)} (I_4, \bar{T}) \right) \]
\[ \leq K(\mu) \left( \rho^{(k)} (I_1, \bar{T}) + \rho^{(k)} (I_2, \bar{T}) + \rho^{(k)} (I_4, \bar{T}) \right) \]

It is easy to see that \( j^*(k, k, k, k) < Q(k) < \infty \) \( (n=1, 2, \ldots) \) is uniformly bounded by \( n \); \( Q \) is a positive constant not depending on \( n \). Indeed, every interval goes to the left from the initial point after a finite number \( n \) of iteration of \( j \). This means that the operator \( B \) is contractive one in the sense of definition from [13] and has a unique fixed point \( (I_1(t), I_2(t), I_3(t), I_4(t)) \), which is a solution of (19).

Finally, we note that the solution can be approximated by a sequence of successive approximations with advanced prescribed accuracy.

Theorem 1 is thus proved.

7 Results validation

Since our goal is to find \( U_{NE} = u_z(0, t); U_{FE} = u_z(\Lambda, t) \) we have (cf. (12), (13)):
\[ u_z(0, t) = q_1 I_1(0, t) + q_2 J_2(0, t) - q_3 J_3(0, t) - q_4 J_4(0, t) = \]
\[ = q_1 I_1(t) + q_2 J_2(t) - q_3 J_3(t) - q_4 J_4(t) \]
\[ u_z(\Lambda, t) = q_1 I_1(\Lambda, t) + q_2 J_2(\Lambda, t) - q_3 J_3(\Lambda, t) - q_4 J_4(0, t) = \]
\[ = q_1 I_1(t - \bar{T}) + q_2 J_2(t - \bar{T}) - q_3 J_3(t - \bar{T}) - q_4 J_4(t) \]

where \( (I_1(t), I_2(t), I_3(t), I_4(t)) \) is the solution obtained in the above theorem.

We have to check the conditions of our Theorem 1 referring to the data from [9]:
\[ L_{11} = L_G = L_R = L_22 = 0.8529 \ \mu H/m; \]
\[ L_m = 0.3725 \ \mu H/m; \]
\[ L_{12} = L_{21} = L_m \; ; \]
\[ C_{11} = C_G + C_m = C_R + C_m = C_{22} = 46.762 \ \text{pF/m}; \]
\[ C_{12} = C_{21} = -C_m = -18.036 \ \text{pF/m}; \]
\[ L_{12} C_{12} + L_{22} C_{22} = L_m (C_G + C_m) - L_R C_m = 0.3725 \times \]
\[ \times 46.762 - 0.8529 \times 18.036 = 2.036 \neq 0; \]
\[ L_{22} C_{22} + L_{11} C_{11} = L_m (C_R + C_m) - L_R C_m = 0.3725 \times \]
\[ \times 46.762 - 0.8529 \times 18.036 = 2.036 \neq 0; \]
\[ \Delta_C = C_{11} C_{22} - C_{12}^2 = (C_G + C_m) (C_R + C_m) - C_m^2 = \]
\[ = 46.762^2 - (18.036)^2 \approx 1861.3874 > 0; \]
\[ \Delta_L = L_{11} L_{22} - L_m^2 = L_{11} L_{22} - L_{12}^2 = \]
\[ = 0.8529^2 - 0.3725^2 \approx 0.5887 > 0; \]
\[ \lambda = \frac{L_{11} C_{11} + 2 L_{12} C_{12} + L_{22} C_{22}}{2 \Delta_C \Delta_L} + \]
\[ + \sqrt{\left( L_{11} C_{11} + 2 L_{12} C_{12} + L_{22} C_{22} \right)^2 - 4 \Delta_C \Delta_L} \approx \]
\[ \approx 0.0321157 \approx 0.1792 \]
\[
\lambda_2 = \frac{L_1 C_{11} + 2 L_2 C_{12} + L_2 C_{22}}{2 \Delta_c \Delta_t} - 4 \Delta_c \Delta_t \approx \left(\sqrt{(L_1 C_{11} + 2 L_2 C_{12})^2 - 4 \Delta_c \Delta_t} \right) \approx \sqrt{0.0284} \approx 0.1686
\]

\[
L_{22} + \lambda_1 \lambda_2 \Delta_c C_{11} = 0.8529 + 0.1792 \times 0.1686 \times 0.5887 \times 16.762 \approx 1.6846
\]

\[
C_{22} \Delta_c + L_1 \sqrt{\Delta_c \Delta_t} = 46.762 \times 0.5887 + 0.8529 \sqrt{0.5887 \times 1861.3874} \approx 55.7546
\]

\[
\Delta_{12} = \sqrt{D} \left(\frac{\lambda_1}{\Delta_c + R_s R_{xe}} + \lambda_1 \Delta_c C_{11} \right) + \lambda_1 - \lambda_2 \times \frac{L_1 C_{11} + L_2 C_{12}}{L_1 C_{11} + L_2 C_{12}} \times \left(\frac{C_{22} \Delta_c + L_1 \sqrt{\Delta_c \Delta_t} R_{xe} + L_2 \lambda_2 \Delta_c C_{11} R_s}{\lambda_1 \lambda_2} \right)
\]

\[
\Delta_{14} = \sqrt{D} \left(\frac{\lambda_1}{\Delta_c + R_s R_{xe}} + \lambda_1 \Delta_c C_{11} \right) - \lambda_1 - \lambda_2 \times \frac{L_1 C_{11} + L_2 C_{12}}{L_1 C_{11} + L_2 C_{12}} \times \left(\frac{C_{22} \Delta_c + L_1 \sqrt{\Delta_c \Delta_t} R_{xe} + L_2 \lambda_2 \Delta_c C_{11} R_s}{\lambda_1 \lambda_2} \right)
\]

\[
\gamma_1 = 1 - \lambda_1 \Delta_c C_{11} + \lambda_1 \Delta_c C_{12} = \frac{1 - \lambda_1 \Delta_c C_{11} + \lambda_1 \Delta_c C_{12}}{\lambda_1 \Delta_c C_{11} + \lambda_1 \Delta_c C_{12}} \times 1 - 0.0332157(0.8529 \times 46.762 - 0.3725 \times 18.036) \approx 0.0321157(0.3725 \times 46.762 - 0.8529 \times 18.036) \approx 1 - 1.0646 \approx -0.987979;
\]

\[
\gamma_2 = 1 - \lambda_2 \Delta_c C_{11} + \lambda_2 \Delta_c C_{12} = \frac{1 - \lambda_2 \Delta_c C_{11} + \lambda_2 \Delta_c C_{12}}{\lambda_2 \Delta_c C_{11} + \lambda_2 \Delta_c C_{12}} \times 1 - 0.0284(0.8529 \times 46.762 - 0.3725 \times 18.036) \approx 0.0284(0.3725 \times 46.762 - 0.8529 \times 18.036) \approx 1 - 0.94 \approx 1.038
\]

The inequalities from the main theorem are:

\[
|\lambda_1 \lambda_2 + (L_{22} \lambda_2 + R_{xe}) \gamma_1| \leq \frac{|\lambda_1 \lambda_2 + (L_{22} \lambda_2 + R_{xe}) \gamma_1|}{\left|\lambda_1 \lambda_2 + (L_{22} \lambda_2 + R_{xe}) \gamma_1\right|} \leq \min \{I_{12}, I_{20}\}
\]

In what follows we verify how the same data satisfy the conditions generated by the particular case under weak coupling assumptions. Indeed, the main system becomes

\[
(C_G + C_m) \frac{\partial u_G(x,t)}{\partial t} + \frac{\partial i_G(x,t)}{\partial x} = 0
\]

\[
(C_G + C_m) \frac{\partial u_G(x,t)}{\partial t} + \frac{\partial i_G(x,t)}{\partial x} = 0
\]

\[
L_G \frac{\partial i_G(x,t)}{\partial t} + \frac{\partial u_G(x,t)}{\partial x} = 0
\]

or in a matrix form

\[
\begin{bmatrix}
C_{11} & 0 & 0 & 0 \\
0 & C_{22} & 0 & 0 \\
0 & 0 & L_{11} & 0 \\
0 & 0 & L_{12} & L_{22}
\end{bmatrix}
\begin{bmatrix}
\partial u_G / \partial t \\
\partial i_G / \partial x \\
\partial i_G / \partial t \\
\partial i_G / \partial x
\end{bmatrix}
= 0.
\]

Since

\[
\Delta_c = C_{11} C_{22} = (C_G + C_m)(C_G + C_m) > 0
\]

\[
(L) \lambda_e = L_{11} L_{22} = L_G L_R > 0
\]

it follows \( |A| = \Delta_c \lambda_e \neq 0 \)

and therefore

\[
\frac{|\lambda_1 \lambda_2 + (L_{22} \lambda_2 + R_{xe}) \gamma_1|}{\left|\lambda_1 \lambda_2 + (L_{22} \lambda_2 + R_{xe}) \gamma_1\right|} \leq \min \{I_{12}, I_{20}\}
\]
In view of $L_{12}C_{11} + L_{22}C_{12} \neq 0$ and $L_{12}C_{22} + L_{11}C_{12} \neq 0$ we obtain the roots of

$$|B^{-1} - \mu I| = (L_{11}C_{11} - \mu^2)(L_{22}C_{22} - \mu^2) = 0$$

From the matrix

$$B = \begin{bmatrix}
    -\mu & 0 & L_{11} & 0 \\
    0 & -\mu & L_{12} & L_{22} \\
    C_{11} & 0 & -\mu & 0 \\
    C_{12} & C_{22} & 0 & -\mu
\end{bmatrix}$$

we obtain for $\mu_i = -\mu_k = \sqrt{L_{11}C_{11}}$ the eigenvectors

$$\xi_{12} = \frac{L_{12}C_{11} + L_{22}C_{12}}{L_{11}C_{12} + L_{12}C_{22}} \sqrt{C_{11}}, \quad \xi_{31} = \frac{L_{11}C_{11} - L_{22}C_{22}}{L_{11}C_{12} + L_{12}C_{22}} \sqrt{C_{11}}$$

$$\xi_{41} = 1, \quad \xi_{13} = \frac{L_{12}C_{11} - L_{22}C_{22}}{L_{11}C_{12} + L_{12}C_{22}} \sqrt{C_{11}}$$

Similarly for $\mu_2 = -\mu_4 = \sqrt{L_{22}C_{22}} \Rightarrow$

$$\begin{pmatrix}
    -\sqrt{L_{22}C_{22}} & 0 & L_{11} & 0 \\
    0 & -\sqrt{L_{22}C_{22}} & L_{12} & L_{22} \\
    0 & 0 & L_{11}C_{11} - L_{22}C_{22} & 0 \\
    0 & 0 & L_{11}C_{12} - L_{12}C_{22} & 0
\end{pmatrix}$$

we have

$$\xi_{12} = 0, \quad \xi_{22} = \sqrt{L_{22}/C_{22}}, \quad \xi_{32} = 0, \quad \xi_{42} = 1,$n

$$\xi_{14} = 0, \quad \xi_{24} = -\sqrt{L_{22}/C_{22}}, \quad \xi_{34} = 0, \quad \xi_{44} = 1.$$n

Consequently, the transformation matrix is:

$$A^{-1} = \begin{bmatrix}
    C_{22} & 0 & 0 & 0 \\
    \Delta_C & 0 & 0 & 0 \\
    -C_{12} / \Delta_C & C_{11} \Delta_C & 0 & 0 \\
    0 & 0 & L_{22} \Delta_C & 0 \\
    0 & 0 & -L_{12} \Delta_L & L_{11} \Delta_L
\end{bmatrix}$$

We find the eigenvectors

$$H^{(k)} = (\xi_{1k}, \xi_{2k}, \xi_{3k}, \xi_{4k})^T; (k = 1, 2, 3, 4)$$

of $(B^{-1} - \mu_k I)H^{(k)} = 0$; $\mu_k = 1/\Delta_k$, where

$$\begin{bmatrix}
    0 & 0 & L_{11} & 0 \\
    0 & 0 & L_{12} & L_{22} \\
    C_{11} & 0 & 0 & 0 \\
    C_{12} & C_{22} & 0 & 0
\end{bmatrix}$$

We have

$$\xi_{12} = 0, \quad \xi_{22} = \sqrt{L_{22}/C_{22}}, \quad \xi_{32} = 0, \quad \xi_{42} = 1,$n

$$\xi_{14} = 0, \quad \xi_{24} = -\sqrt{L_{22}/C_{22}}, \quad \xi_{34} = 0, \quad \xi_{44} = 1.$$
And then in view of (12) we have

\[
\begin{align*}
\dot{u}_1(x,t) &= \frac{L_{11}C_{11} - L_{22}C_{22}}{L_{11}C_{11} + L_{12}C_{22}} \sqrt{L_{11}} \cdot I_1(x,t) - \\
&- \frac{L_{11}C_{11} - L_{22}C_{22}}{L_{11}C_{11} + L_{12}C_{22}} \sqrt{L_{12}} \cdot I_2(x,t); \\
\dot{u}_2(x,t) &= \frac{L_{12}C_{11} + L_{22}C_{12}}{L_{11}C_{11} + L_{12}C_{22}} \sqrt{L_{22}} \cdot I_1(x,t) + \\
&\frac{L_{22}C_{22}}{L_{11}C_{11} + L_{12}C_{22}} \sqrt{L_{22}} \cdot I_2(x,t) - \\
&- \frac{L_{12}C_{11} + L_{22}C_{12}}{L_{11}C_{11} + L_{12}C_{22}} \sqrt{L_{12}} \cdot I_3(x,t); \\
i_i(x,t) &= \frac{L_{11}C_{11} - L_{22}C_{22}}{L_{11}C_{11} + L_{12}C_{22}} I_1(x,t) + \\
\frac{L_{22}C_{22}}{L_{11}C_{11} + L_{12}C_{22}} I_2(x,t); \\
\dot{u}_3(x,t) &= I_1(x,t) + I_2(x,t) + I_3(x,t) + I_4(x,t).
\end{align*}
\]

If we take the specific parameters again from [9]

\[
L_{11} = L_G = L_R = L_{22} = 0.8529 \ \mu \text{H/m;}
\]

\[
C_{11} = C_G + C_m = C_R + C_m = C_{22} = 46.762 \ \text{pF/m}
\]

it is obvious that \( L_{11} C_{11} - L_{22} C_{22} = 0 \). This implies \( u_i(x,t) = u_{ij}(x,t) = 0 \). The contradiction obtained shows the advantages of our method.

8 Conclusion

In this paper we presented the full derivation of the equations describing a 3-conductor transmission line terminated by linear loads. In such way, we extended the general method from [12] by shrinking the mixed problem for the hyperbolic system expressing TEM propagation lengthwise the lines to a functional system on the boundary. In result, by applying the fixed point method we can obtain in an explicit form the solution to the system of functional equations by successive approximations beginning with simple initial approximation. Our method is applicable to nonlinear boundary conditions too. Besides, in this paper we prove existence-uniqueness of a more general periodic solution and demonstrated the benefits of our method on the samples related to examinations of cross-talks. It should be noted that the method elaborated here can be applied to nonlinear boundary conditions.

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