LSSVR-based Solutions for Three-point BVPs of Second-order ODEs

YANMENG LI, HUAIJIANG SUN, WENZHU YAN
School of computer science and Engineering
Nanjing University of Science and Technology
No. 200 Xiaolingwei Street, Xuanwu District, Nanjing City, Jiangsu Province
Beijing. P.R. CHINA
yanmengli6@126.com; sunhjteach@163.com; ywznanj@163.com

Abstract: We know that most of the numerical methods for solving ordinary differential equations (ODEs) are based on iterative techniques or Taylor expansion techniques. In this paper, take three-point boundary value problems (BVPs) of linear second-order ODEs for example, we try to study the numerical solutions of ODEs from a new perspectiveł-machine learning. By means of the idea of least squares support vector regression (LSSVR), we propose a new numerical solving method for three-point BVPs of linear second-order ODEs. From the derivative process of the proposed method, we can see that it has generality and can be used for solving some other kinds of ODEs. In order to verify the effectiveness of the proposed method, we perform a series of comparative experiments with four specific linear second-order ODEs. Experimental results show that the proposed method is a effective menthod.

Key-Words: numerical solution; ordinary differential equation; least squares; support vector regression

1 Introduction

ODEs can be found in a wide aspect, for example in mathematical formulation and physical phenomena extraordinary in science and engineering. Depending on the orders of ODEs, they can be divided into three main categories: first-order, second-order and higher-order ODEs. Exact solutions for ODEs are not generally available and hence numerical methods must be considered. Multi-point boundary value problems for ODEs can arise in solving linear partial differential equations by using the steparation variable method [1]. Moreover, in the engineering problem to increase the stability of a rod, one also imposes a fixed interior point except for the ends of the rod [2]. Along this line, the solutions of multi point boundary value problems for ordinary differential equations have great significance in mathematical theory and practical applications [3-5]. It is seen that the three-point boundary value problems for nonlinear second-order ordinary differential equations have attractef much attention [6-10].

As shown in past works, the solution of solving differential equations are almost based on iterative methods or Taylor expansion methods, and now we focused on a new method which based on LS-SVR. However, from the viewpoint of practical applications, the explicit solutions or approximate solutions of three-point BVPs must be given, so that we can have a compare between them. In this paper, we hope to study a class of three-point BVPs of linear second-order ODEs from a new perspective.

$$
\begin{aligned}
& y''(x) + p(x)y'(x) + q(x)y(x) = g(x),\ x \in [a, b], \\
& y(a) = p_0,\ y(b) + \lambda y(\mu) = q_0,\ \mu \in (a, b), \\
& \text{or} \\
& y(b) = q_0,\ y(a) + \delta y(\mu) = p_0,\ \mu \in (a, b)
\end{aligned} \tag{1}
$$

where the known functions $p(x) \in C^1[a,b]$, $q(x), g(x) \in C[a,b]$, λ, μ are the constants. Here we just discuss the second condition.

Based on optimization modelings we have least square support vector regressions (LS-SVRs) and it's a powerful method for solving pattern recognition and function estimation problems which based on Vapnik and Chervonenkis structural risk minimization principle [11]. But, I think they show a better generalization ability compared to other machine methods on a wide variety of real-world problems, such as optimal control [12], time series prediction [13], image segmentation [14] and so on. In this method one maps data into a high dimensional feature space by means of a kernel function and then solves linear regression problems, which results in solving quadratic programming problems (QPPs). The main challenge in developing a useful regression modeling is to capture accurately the underlying functional relationship between the given inputs and their output values. It can be used as a tool for analysis, simulation and prediction, once the resulting model is obtained.

For a given set of data points and a kernel function, LS-SVR aims at determining a regressor where the input data are taken into a higher dimensional feature space. The basic idea of kernel LS-SVR with ε-insensitive loss function proposed by Vapnik is to find a linear function $y(x)$ in the higher dimensional feature space such that, on the one hand, more training samples locate in the ε-intensive tube between $y(x)-\varepsilon$ and $y(x)+\varepsilon$, and on the another hand, the function $y(x)$ is as flat as possible, which leads to introduce the regularization term. Thus, the structural risk minimization principle is implemented.

In the following, we mainly research how to use kernel LS-SVR method to obtain numerical solutions of the Eq.(1). For this end, we first briefly introduce kernel LS-SVR and some basic concepts in Section 2 and then present a new approximation method for the solution of Eq.(1) by means of kernel LS-SVR in Section 3. In order to verify the effectiveness of the presented method, we perform a series of comparative experiments with six specific linear second-order ODEs in Section 4 and give some concluding remarks in Section 5.

In this paper, we mainly consider the following three-point boundary value problems for linear second-order ordinary differential equations with variable coefficients:

$$\begin{aligned} & y''(t)+p(t)y'(t)+q(t)y(t)=g(t),\ t\in[a,b], \\ & y(a)=p_0, \\ & y(b)+\lambda y(\mu)=q_0,\ \mu\in(a,b), \end{aligned} \tag{2}$$

where the known functions $p(t)\in C^1[a,b]$, $q(t), g(t)\in C[a,b]$, λ, μ are the constants. Here we just discuss the second condition.

2 Kernel LS-SVR and Some Concepts

ODEs can In this section, we briefly recall kernel SVR, for details, see [11]. Let $\mathrm{T}=\{(x_i,y_i)\}_{i=1}^{l}$ be a set of sample data, where $x_i\in\mathrm{R}^n$ and $y_i\in\mathrm{R}$ are input and output of the i-th sample data, respectively. Let $k:\mathrm{R}^n\times\mathrm{R}^n\to R$ be a Gaussian RBF kernel function with the reproducing kernel Hilbert space (RKHS) H and the nonlinear feature mapping $\phi:\mathrm{R}^n\to\mathrm{H}$. The space H is a higher dimensional feature space and can be expressed as an expended space of the mapped inputs $\{\phi(x_i)\}_{i=1}^{l}$, that is, $H=span\{\phi(x_1),\cdots,\phi(x_l)\}$. It is known that $k(u,v)=\langle\phi(u,\phi(v))\rangle_H$ for all $u,v\in\mathrm{R}^n$, where $\langle\cdot,\cdot\rangle_H$ denotes the inner product in RKBS H. Let $\mathrm{K}=[k(x_i,y_i)]\in R^{l\times l}$ be the kernel matrix and K_i denote the i-th column of K. LS-SVR is based on the ε-insensitive loss function defined by

$$L_\varepsilon=\begin{cases} 0, \left|\mathrm{f(x)\text{ - }y}\right|\le\varepsilon, \\ \left|\mathrm{f(x)\text{ - }y}\right|\le\varepsilon, \text{otherwise}, \end{cases}$$

Where $f(x)$ and y are the predicted value and the true value of the input $x\in\mathrm{R}^n$, respectively. The regression function obtained by using kernel-based LS-SVR has the form $f(x)=w^T\phi(x)+b$, where unknown the normal vector w and bias b can be got by solving the following optimization problem:

$$\begin{aligned} & \min_{w,b}\ \frac{1}{2}\|w\|^2 \\ & s.t.\ \mathrm{w}^{\mathrm{T}}\phi(x_i)+b-y_i\le\varepsilon\ \ \mathrm{i}=1,\cdots,\mathrm{l}, \\ & \qquad y_i-\mathrm{w}^{\mathrm{T}}\phi(x_i)+b\le\varepsilon\ \ \mathrm{i}=1,\cdots,\mathrm{l}, \end{aligned} \tag{3}$$

where $C>0$ is a pre-specified value. Since $H=span\{\phi(x_1),\cdots,\phi(x_l)\}$, we can set $w=\sum_{i=1}^{l}\beta_i\phi(x_i)$. Put $\beta=(\beta_1,\cdots,\beta_l)^T$, $y=(y_1,\cdots,y_l)^T$, $e_l=(1,\cdots,1)^T\in R^l$, then $w^T\phi(x_i)=K_i^T\beta$ and the problem (2) can be expressed the matrix form

$$\begin{aligned} & \min_{\beta}\ \frac{1}{2}\beta^T K\beta \\ & s.t.\ K\beta+be_l-y\le\varepsilon e_l \\ & \qquad y-(K\beta+be_l)\le\varepsilon e_l \end{aligned} \tag{4}$$

By solving the Wolfe dual form of the problem (4)

$$\begin{aligned} & \min_{\alpha_1,\alpha_2}\ \frac{1}{2}(\alpha_1-\alpha_2)^T K(\alpha_1-\alpha_2)-y^T(\alpha_1-\alpha_2)+\varepsilon e_l^T(\alpha_1+\alpha_2) \\ & s.t.\ e_l^T(\alpha_1-\alpha_2)=0, \\ & \quad 0\le\alpha_1,\alpha_2\le Ce_l \end{aligned}$$

we can obtain the optimal Lagrange multipliers α_1^*,α_2^* and then $\beta^*=\alpha_1^*-\alpha_2^*$ and $b^*=y_i-K_i^T\beta^*$ for some $i:0<\alpha_{1i}^*<C$ or $b^*=y_k-K_k^T\beta^*-\varepsilon$ for some $k:0<\alpha_{2k}^*<C$. Therefore, the optimal regression function is $f(x)=[k(x_1,x),\cdots,k(x_l,x)]\beta^*+b^*$.

For the sake of convenience, let

$$\nabla_n^m k(u,v)=\frac{\partial^{n+m}k(u,v)}{\partial u^n\partial v^m},\forall u,v\in R, n,m=1,2,\cdots,$$

then

$\nabla_n^m k(u,v) = (\phi^{(n)}(u))^T \phi^{(m)}(v), \forall u,v \in R, n,m = 1,2,\cdots,$
and then

$$K_0^0 = [\nabla_0^0 k(x_i,x_j)] = [k(x_i,x_j)] = K \in R^{l\times l},$$

$$K_n^m = [\nabla_n^m k(x_i,x_j)] = [\nabla_n^m k(u,v)|_{u=x_i,v=x_j}] \in R^{l\times l}$$

If we denote the i-th column of the matrix K_n^m by $(K_m^n)_i$, then

$$\begin{aligned}
\hat{y}(t) &= w^T\phi(t) + b = \sum\nolimits_{j=1}^{l}\beta_j\phi(t_i)^T\phi(t) + b = \sum\nolimits_{j=1}^{l}\beta_j k(t_j,t) + b,\\
\hat{y}'(t) &= w^T\phi'(t) + b = \sum\nolimits_{j=1}^{l}\beta_j\phi(t_i)^T\phi'(t) + b = \sum\nolimits_{j=1}^{l}\beta_j\nabla_0^1 k(t_j,t),\\
\hat{y}''(t) &= w^T\phi''(t) + b = \sum\nolimits_{j=1}^{l}\beta_j\phi(t_i)^T\phi''(t) + b = \sum\nolimits_{j=1}^{l}\beta_j\nabla_0^2 k(t_j,t)
\end{aligned} \tag{5}$$

3 Approximate Solutions based on Kernel LS-SVR

In this section, we mainly study how to find numerical solutions of the Eq.(1) by using LS-SVR. Specifically, we assume that the numerical solution of the Eq.(1) has the form $\hat{f}(x) = w^T\phi(x) + b$ and wish that the unknown $w \in H$ and $b \in R$ can be learned by LS-SVR. For this purpose, we first discrete the domain $[a,b]$ of the Eq.(1) into a set of collocation points $a = x_1 < x_2 < \cdots < x_l = b$ with the same stepsize $h = \frac{b-a}{l-1}$ such that $\mu = t_n \in (t_1, t_l)$. Let

$$L(y(t)) = y''(t) + p(t)y'(t) + q(t)y(t)$$

If $\hat{f}(x)$ is the exact solution of the Eq.(1), then $\hat{y}(a) = p_0, \hat{y}(b) + \lambda\hat{y}(\mu) = q_0$ and $L(\hat{f}(x_i)) = g(x_i)$ for all $\mathrm{i} = 1,\cdots,\mathrm{l}$. But usually, $\hat{y}(a), \hat{y}(b) + \lambda\hat{y}(\mu)$ and $L(\hat{f}(x_i))$ are necessarily equal to p_0, q_0 and $g(x_i)$, respectively. So, we hope the smaller the better of the values $|\hat{y}(a) - p_0|, |\hat{y}(b) + \lambda\hat{y}(\mu) - q_0|$ and $|L(\hat{f}(x_i)) - g(x_i)|$ for $\mathrm{i} = 1,\cdots,\mathrm{l}$. For this end, we view $\{(\mathrm{x_j}, \mathrm{g}(\mathrm{x_j}))\}_{\mathrm{j}=1}^{\mathrm{l}}$ as the sample set and consider the following three optimization modelings. According to the discussion in Section 2, we can let $w = \sum\nolimits_{j=1}^{l}\beta_j\phi(t_j)$.

In order to insure the values $|L(\hat{f}(x_i)) - g(x_i)|, \mathrm{i} = 1,\cdots,\mathrm{l}$ being as small as possible and satisfy $\hat{y}(a) = p_0, \hat{y}(b) + \lambda\hat{y}(\mu) = q_0$, we consider the following optimization problem:

$$\begin{aligned}
\min_{w,b}\ & \frac{1}{2}\|w\|^2 + \frac{\mathrm{C}}{2}\xi^T\xi\\
s.t.\ & L(\hat{\mathrm{y}}(\mathrm{t_i})) - g(\mathrm{t_i}) = \xi_{\mathrm{i}}\quad \mathrm{i} = 1,\cdots,\mathrm{l},\\
& y(\mathrm{a}) = \mathrm{p}_0,\\
& \mathrm{y(b)} + \lambda\hat{\mathrm{y}}(\mu) = \mathrm{q}_0
\end{aligned} \tag{6}$$

where $C > 0$ is a pre-specified value and ξ_i are slack variables. According to (6), we have

$$\hat{y}(t_i) = \sum\nolimits_{\mathrm{j}=1}^{\mathrm{l}}\beta_j k(t_j,t_i) + b = K_i^T\beta + b,$$

$$\hat{y}(a) = \hat{y}(t_1) = K_1^T\beta + b,$$

$$\hat{y}(b) + \lambda\hat{y}(\mu) = \hat{y}(t_l) + \lambda\hat{y}(t_n) = (K_l + \lambda K_n)^T\beta + (1+\lambda)b,$$

$$\hat{y}'(t_i) = \sum\nolimits_{\mathrm{j}=1}^{\mathrm{l}}\beta_j\nabla_0^1 k(t_j,t_i) = (K_0^1)_i^T\beta,$$

$$\hat{y}''(t_i) = \sum\nolimits_{\mathrm{j}=1}^{\mathrm{l}}\beta_j\nabla_0^2 k(t_j,t_i) = (K_0^2)_i^T\beta$$

for all $\mathrm{i} = 1,\cdots,\mathrm{l}$ and then

$$\hat{y}(t_i) = \sum\nolimits_{\mathrm{j}=1}^{\mathrm{l}}\beta_j k(t_j,t_i) + b = K_i^T\beta + b,$$

$$L(\hat{y}(t_i)) = [(K_0^2)_i + p(t_i)(K_0^1)_i + q(t_i)K_i]^T\beta + q(t_i)b$$

Put

$$m_i = (K_0^2)_i + p(t_i)(K_0^1)_i + q(t_i)K_i \in R^l, \mathrm{i} = 1,\cdots,\mathrm{l},$$

$$\mathrm{M} = [\mathrm{m_1},\cdots,\mathrm{m_l}] \in R^{l\times l}, g = (g(t_1),\cdots,g(t_1))^T \in R^l,$$

$$q = (q(t_1),\cdots,q(t_1))^T \in R^l$$

then the problem (6) can be translated into the matrix form:

$$\begin{aligned}
\min_{\beta,b,\xi}\ & \frac{1}{2}\beta^T K\beta + \frac{\mathrm{C}}{2}\xi^T\xi\\
s.t.\ & M^{\mathrm{T}}\beta + qb - g = \xi,\\
& K_1^T\beta + b = p_0,\\
& (\mathrm{K_n} + \lambda\mathrm{K_l})^{\mathrm{T}}\beta + (1+\lambda)\mathrm{b} = \mathrm{q}_0
\end{aligned} \tag{7}$$

By means of the constraint $b = p_0 - K_1^T\beta$ and let $S = M^T - qK_1^T, h = qp_0 - g, t = K_n + \lambda K_l - (1+\lambda)K_1$ and $a = (1+\lambda)p_0 - q_0$, we can deduce that $\xi = S\beta + h$, $t^T\beta + a = 0$ and then the problem (7) can be rewritten as

$$\begin{aligned}
\min_{\beta,b,\xi}\ & \frac{1}{2}\beta^T K\beta + \frac{\mathrm{C}}{2}\|S\beta + h\|^2\\
s.t.\ & t^{\mathrm{T}}\beta + a = 0
\end{aligned} \tag{8}$$

Considering the Lagrange function of the problem (8)

$$\mathrm{L}(\beta,\alpha) = \frac{1}{2}\beta^T K\beta + \frac{\mathrm{C}}{2}\|S\beta + h\|^2 + \alpha(t^{\mathrm{T}}\beta + a), \forall\alpha \in R$$

and let $\frac{\partial L}{\partial\beta} = 0$,then we have

$\beta = -K^{-1}(CS^T S\beta + CS^T h + \alpha t)$. Without loss of generality, we can assume that K is a nonsingular matrix. Otherwise, since K is a symmetric nonnegative definite matrix, we can regularize K by replacing $K + \delta I_l$ into K, where $\delta > 0$ is a sufficiently small number and I_l denotes the l order unit matrix. Substituting β into the Lagrange function, we can get

$$\mathrm{L}(\alpha) = -\frac{1}{2}\beta^T K\beta - \frac{\mathrm{C}}{2}\beta^T S^T S\beta + \alpha^T a \quad (9)$$

and then the Wolfe dual form of the problem (9) is

$$\min_{\alpha}\ \mathrm{L}(\alpha) = -\frac{1}{2}\beta^T K\beta - \frac{\mathrm{C}}{2}\beta^T S^T S\beta + \alpha^T a \quad (10)$$

Let $\frac{\partial L}{\partial \alpha} = 0$ we can get

$$\alpha = \frac{1}{2}(t^T Dt)^{-1}(a - ct^T(K^T + cS^T S)^{-1} S^T h)$$

insertion of α into β and b then we can get

$$\beta = -(K + cS^T)^{-1}(t\alpha + cS^T h)$$

$$= -(K + cS^T)^{-1}(t(\frac{1}{2}(t^T Dt)^{-1}(a - ct^T(K^T + cS^T S)^{-1} S^T h)) + cS^T h)$$

and

$$b = p_0 - k_1^T\beta = p_0 - k_1^T(-(K + cS^T)^{-1}$$

$$(t(\frac{1}{2}(t^T Dt)^{-1}(a - ct^T(K^T + cS^T S)^{-1} S^T h)) + cS^T h))$$

So, if the RBF kernel function is chosen in experiment we can have the approximate solution

$$y(t) = \exp\{-\frac{\|x(i) - x(j)\|^2}{2\sigma^2}\}\beta + b.$$

4 Numerical Examples

In order to demonstrate the effectiveness of the proposed method, in this section, we perform a series of comparative experiments between numerical solutions and exact solutions in the following four elaborated three-point BVPs of linear second-order ODEs with variable coefficients. The RBF kernel function is chosen in experiment. In order to demonstrate the effectiveness of the proposed method, in this section, we consider four integral equations and their analytic solutions and listed them in following tables. There have exact solution e^{-x} in (1)-(3), e^{x} in (4).

Example 1. Consider a three-point BVP of linear second-order ODE with variable coefficients:

$$\begin{cases} \varphi''(x) + \varphi'(x) + x\varphi(x) = xe^{-x}, 0 \le \mathrm{x} \le 1, \\ \varphi(x) = 1, \varphi(1) + \varphi(1/2) = e^{-1} + e^{-1/2}. \end{cases}$$

Example 2. Consider a three-point BVP of linear second-order ODE with variable coefficients:

$$\begin{cases} \varphi''(x) - (1 + \sin x)\varphi(x) = -e^{-x}\sin x, 0 \le \mathrm{x} \le 1, \\ \varphi(0) = 1, \varphi(1) + \varphi(1/2) = e^{-1} + e^{-1/2}. \end{cases}$$

Example 3. Consider a three-point BVP of linear second-order ODE with variable coefficients:

$$\begin{cases} \varphi''(x) + x^2\varphi'(x) - x\varphi(x) = (1 - x - x^2)e^{-x}, 0 \le \mathrm{x} \le 1, \\ \varphi(0) = 1, \varphi(1) + \varphi(1/2) = e^{-1} + e^{-1/2}. \end{cases}$$

Example 4. Consider a three-point BVP of linear second-order ODE with variable coefficients:

$$\begin{cases} \varphi''(x) - (1 + \sin(x))\varphi(x) = -e^{-x}\sin(x), 0 \le \mathrm{x} \le 1, \\ \varphi(0) = 1, \varphi(1) + \varphi(1/2) = e^{-1} + e^{-1/2}. \end{cases}$$

From these results, it can be seen that the largest absolute errors are not beyond 10^{-4} magnitude order, which show that the proposed algorithm could reach a quite agreeable accuracy. There for we can conclude that the proposed method is feasible in solving linear second-order ODE with three-point boundary value problems for with LS-SVR.

Table 1: Comparison for Example 1

x	Exact solution	Numerical solution	Absolute error
0	1.0000	1.0000	0
0.1	0.9048	0.9141	0.0092
0.2	0.8187	0.8262	0.0075
0.3	0.7403	0.7394	0.0014
0.4	0.6703	0.6565	0.0139
0.5	0.6065	0.5806	0.0260
0.6	0.5488	0.5146	0.0343
0.7	0.4966	0.4611	0.0355
0.8	0.4493	0.4222	0.0271
0.9	0.4066	0.3995	0.0070
1	0.3679	0.3939	0.0260

Table 2: Comparison for Example 2

x	Exact solution	Numerical solution	Absolute error
0	1.0000	1.0000	0
0.1	0.9048	0.9088	0.0040
0.2	0.8187	0.8237	0.0050
0.3	0.7403	0.7448	0.0040

0.4	0.6703	0.6721	0.0018
0.5	0.6065	0.6057	0.0009
0.6	0.5488	0.5455	0.0033
0.7	0.4966	0.4917	0.0048
0.8	0.4493	0.4443	0.0050
0.9	0.4066	0.4033	0.0032
1	0.3679	0.3688	0.0009

Table 3: Comparison for Example 3

x	Exact solution	Numerical solution	Absolute error
0	1.0000	1.0000	0
0.1	0.9048	0.9054	0.0006
0.2	0.8187	0.8179	0.0009
0.3	0.7403	0.7375	0.0034
0.4	0.6703	0.6642	0.0061
0.5	0.6065	0.5981	0.0084
0.6	0.5488	0.5392	0.0096
0.7	0.4966	0.4876	0.0090
0.8	0.4493	0.4432	0.0061
0.9	0.4066	0.4061	0.0005
1	0.3679	0.3763	0.0084

Table 4: Comparison for Example 4

x	Exact solution	Numerical solution	Absolute error
0	1.0000	1.0000	0
0.1	1.1052	1.0961	0.0091
0.2	1.2214	1.2101	0.0113
0.3	1.3499	1.3418	0.0081
0.4	1.4918	1.4906	0.0012
0.5	1.6487	1.6560	0.0073
0.6	1.8221	1.8375	0.0154
0.7	2.0138	2.0345	0.0207
0.8	2.2255	2.2461	0.0206
0.9	2.4596	2.4719	0.0123
1	2.7183	2.7110	0.0073

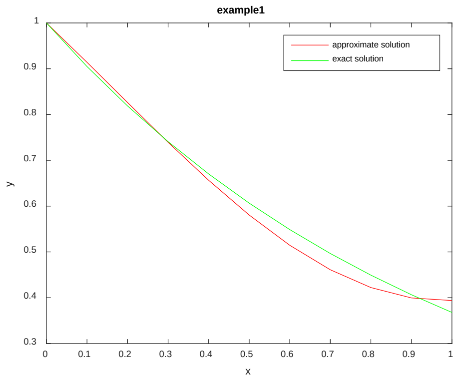

Fig.1 Comparison for Example 1

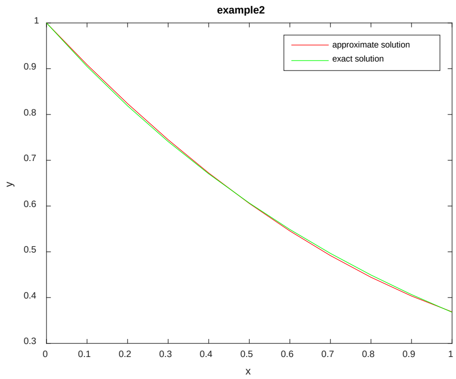

Fig.2 Comparison for Example 2

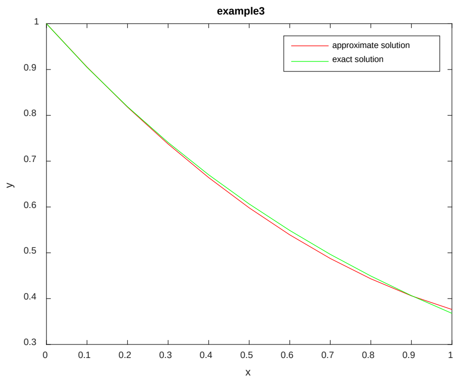

Fig.3 Comparison for Example 3

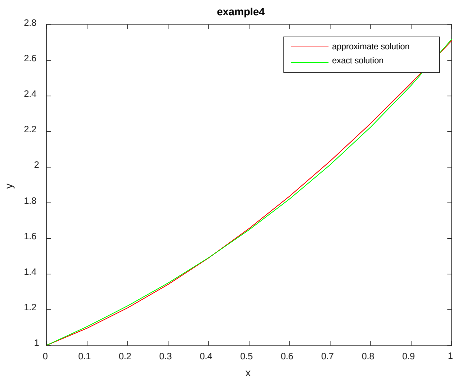

Fig.4 Comparison for Example 4

4 Conclusion

In this paper, we try to seek the numerical solutions for three-point BVPs of linear second-order ODEs with variable coefficients by using optimization technology. We know that kernel LS-SVR based on optimization modeling is a powerful methodology for solving function estimation problems and shows better generalization ability than other machine learning methods on a wide variety of real-world problems. But the main challenge in developing a useful regression model is to capture accurately the underlying functional relationship between the given inputs and their output values. In order to construct optimization modeling, we select the discrete points of the domain of equation as input values and values of the function $L(\hat{y}(x))$ at the discrete points as output values. By solving the Wolfe dual problem of the modeling, we propose a numerical method. From the experiment results, we can see that the proposed method has a good approximation property. From the derivation process, we can discover that the proposed method has a certain versatility and can be used to solve some other kinds of differential equations and integral equations.

References:

[1] Eric Weisstein at Wolfram Research, First-Order Ordinary Differential Equation, 13 (13) (2015) 572.

[2] M. A. Khams, Second Order Differential Equations Box 12395 - El Paso TX 79913 - USA 1412.

[3] Ismail Alia, Virginia Kiryakovab, Shyam L. Kalla, Solutions of fractional multi-order integral and differential equations using a Poisson-type transform, 269 (1) (2002) 172–199.

[4] Tohoku Math. J. Two-point boundary value problems for nonlinear second order differential equations in Hilbert spaces, 32 (2) (1980), 225–233.

[5] H. B. Thompson, C. Tisdell, Three-point boundary value problems for second-order, ordinary, differential equations, doi:10. 1016/S0895-7177(01)00063–2.

[6] Weibing Wanga, Jianhua Shenb, c, Zhiguo Luob, Multi-point boundary value problems for second-order functional differential equations, 56 (8) (2008) 2065–2072.

[7] C. Z. Bai, J. X. Fang, Existence of multiple positive solutions for nonlinear m-point boundary-value problems, Appl. Math. Comput. 140 (2003) 297–305.

[8] X. J. Liu, J. Q. Liu, Three positive solutions for second-order m-point boundary value problems, 156 (2004) 733–742.

[9] Z. Zhao, Solutions and Greens functions for some linear second-order three-point boundary value problems, Comput. Math. Appl. 56 (2008) 104–113.

[10] Xian-Ci Zhong, Qiong-Ao Huang, Approximate solution of three-point boundary value problems for second-order ordinary differential equations with variable coefficients, Applied Mathematics and Computation, 247 (2014) 18–29.

[11] Vapnik, V. N. The nature of statistical learning theory. New York: Springer. (2000), (2nd ed.).

[12] Suykens, J. A. K., Vandewalle, J. De Moor, B. Optimal control by least squares support vector machines. Neural Networks, 14 (1) (2001), 23–25.

[13] Lau, K. W. Wu, Q. H. Local prediction of non-linear time series using support vector regression. Pattern Recognition, 41 (5) (2008), 1539–1547.

[14] Chen, S. Wang, M. Seekingmulti-thresholdsdirectlyfromsupportvectors for image segmentation. Neurocomputing, 67 (1-4) (2005) 335–344.