New analysis on H_∞ control for exponential stability of neural network with mixed time-varying delays via hybrid feedback control

CHARUWAT CHANTAWAT
Khon Kaen University
Department of Mathematics
Khon Kaen 40002, THAILAND
charuwat_c@kkumail.com

THONGCHAI BOTMART*
Khon Kaen University
Department of Mathematics
Khon Kaen 40002, THAILAND
thongbo@kku.ac.th
*Corresponding author

WAJAREE WEERA
University of Phayao
Department of Mathematics
Phayao 56000, THAILAND
wajaree.we@up.ac.th

Abstract: This paper is concerned the problem of H_∞ control for neural networks with mixed time-varying delays which comprising different interval and distributed time-varying delays via hybrid feedback control. The interval and distributed time-varying delays are not necessary to be differentiable. The main purpose of this research is to estimate exponential stability of neural network with H_∞ performance attenuation level γ. The key features of the approach include the introduction of a new Lyapunov-Krasovskii functional with triple integral terms, the employment of a tighter bounding technique, some slack matrices and newly introduced convex combination condition in the calculation, improved delay-dependent suff cient conditions for the H_∞ control with exponential stability of the system are obtained in terms of linear matrix inequalities (LMIs). The results of this paper complement the previously known ones. Finally, a numerical example is presented to show the effectiveness of the proposed methods.

Key–Words: Neural networks, Exponential stability, H_∞ control, Hybrid feedback control.

1 Introduction

During the past decades, the problem of the reliable control has received much attention [1,3,11,12]. Neural networks have received considerable due to the effective use of many aspects such as signal processing, automatic control engineering, associative memories, parallel computation, fault diagnosis, combinatorial optimization and pattern recognition and so on [5, 13, 14]. It has been shown that the presence of time delay in a dynamical system is often a primary source of instability and performance degradation [6]. Many researchers have paid attentions to the problem of stability for systems with time delays [8, 9]. The H_∞ controller can be used to guarantee closed loop system not only a stability but also an adequate level of performance. In practical control systems, actuator faults, sensor faults or some component faults may happen, which often lead to unsatisfactory performance, even loss of stability. Therefore, research on reliable control is necessary.

Most of the works have been focused on the problem of designing a H_∞ controller that stabilizes linear systems with time-varying. Also, it is assumed that the perfect information is available for state feedback and the controlled output is disturbance free. The problem of H_∞ control design usually leads to solving an algebraic Lyaponov equation. It should be noted that some works have been dedicated to the problem of reliable control for nonlinear systems with time-varying delay [3, 11, 12]. However, to the best of the authors knowledge, so far the research on reliable H_∞ control is still an open problem, which is worth further investigation.

Motivated by above discussion, in this paper we have considered the problem of H_∞ control for neural networks with mixed time-varying delays which comprising different interval and distributed time-varying delays via hybrid feedback control. The interval and distributed time-varying delays are not necessary to be differentiable. The main purpose of this research is to estimate exponential stability of neural network with H_∞ performance attenuation level γ. The key features of the approach include the introduction of a new Lyapunov-Krasovskii functional with triple integral terms, the employment of a tighter bounding technique, some slack matrices and newly introduced convex combination condition in the calculation, improved delay-dependent suff cient conditions for the H_∞ control with exponential stability of the system are obtained in terms of linear matrix inequalities

(LMIs). The results of this paper complement the previously known ones. Finally, a numerical example is presented to show the effectiveness of the proposed methods.

The rest of this paper is organized as follows. In Section 2 , some notations, def nitions and some well-known technical lemmas are given. Section 3 presents the H_∞ control for exponential stability and the H_∞ control for exponential stability. The numerical examples and their computer simulations are provided in Section 4 to indicate the effectiveness of the proposed criteria. Finally, the paper is concluded in Section 5.

2 Definitions of Function Spaces and Notation

Notations

The following notation will be used in this paper: $\mathbb{R}^n$ denotes the $n-$dimensional space. A^T denotes the transpose of matrix A, A is symmetric if $A = A^T$, $\lambda(A)$ denotes all the eigenvalue of A, $\lambda_{\max}(A) = \max\{Re\ \lambda : \lambda \in \lambda(A)\}$, $\lambda_{\min}(A) = \min\{Re\ \lambda : \lambda \in \lambda(A)\}$, $A > 0$ or $A < 0$ denotes that the matrix A is a symmetric and positive def nite or negative def nite matrix. If A, B are symmetric matrices, $A > B$ means that $A - B$ is positive def nite matrix, I denotes the identity matrix with appropriate dimensions. The symmetric term in the matrix is denoted by $*$. The following norms will be used: $||\cdot||$ refer to the Euclidean vector norm; $||\phi||_c = \sup_{t\in[-\varrho,0]} ||\phi(t)||$ stands for the norm of a function $\phi(\cdot) \in \mathbb{C}[[-\varrho, 0], \mathbb{R}^n]$.

Consider the following neural network system with mixed time delays

$$\begin{aligned} \dot{x}(t) &= -Ax(t) + Bf(x(t)) + Cg(x(t-h(t))) \\ &\quad +D\int_{t-d(t)}^{t} h(x(s))ds + Ew(t) + \mathscr{U}(t), \\ z(t) &= A_1x(t) + B_4x(t-h(t)) + C_1u(t) \qquad (1) \\ &\quad +D_1\int_{t-d(t)}^{t} x(s)ds + E_1w(t), \\ x(t) &= \phi(t), \quad t \in [-\varrho, 0], \end{aligned}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $w(t) \in \mathbb{R}^n$ the deterministic disturbance input, $z(t) \in \mathbb{R}^n$ the system output, $f(x(t)), g(x(t)), h(x(t))$ the neuron activation function, $A = diag\{a_1, ..., a_n\} > 0$ a diagonal matrix, $B, C, D, E, A_1, B_4, C_1, D_1, E_1$ are the known real constant matrices with appropriate dimensions, $\phi(t) \in \mathbb{C}[[-\varrho, 0], \mathbb{R}^n]$ the initial function, The state hybrid feedback controller $\mathscr{U}(t)$ satisf es:

$$\begin{aligned} \mathscr{U}(t) &= B_1u(t) + B_2u(t-\tau(t)) \qquad (2) \\ &\quad +B_3\int_{t-d_1(t)}^{t} u(s)ds, \end{aligned}$$

where $u(t) = Kx(t)$ and K is a constant matrix control gain, B_1, B_2, B_3 are the known real constant matrices with appropriate dimensions. Then, substituting it into (1), it is easy to get the following:

$$\begin{aligned} \dot{x}(t) &= [-A + B_1K]x(t) + Bf(x(t)) + Ew(t) \\ &\quad +Cg(x(t-h(t))) + D\int_{t-d(t)}^{t} h(x(s))ds \\ &\quad +B_2Kx(t-\tau(t)) + B_3K\int_{t-d_1(t)}^{t} x(s)ds, \\ z(t) &= [A_1 + C_1K]x(t) + B_4x(t-h(t)) \qquad (3) \\ &\quad +D_1\int_{t-d(t)}^{t} x(s)ds + E_1w(t), \\ x(t) &= \phi(t), \quad t \in [-\varrho, 0], \end{aligned}$$

where the time-varying delays function $h(t)$, $\tau(t)$, $d(t)$ and $d_1(t)$ satisfy the condition

$$\begin{aligned} 0 \le h_1 \le h(t) \le h_2, &\qquad 0 \le d(t) \le d, \\ 0 \le \tau(t) \le \tau, &\qquad 0 \le d_1(t) \le d_1, \qquad (4) \end{aligned}$$

where h_1, h_2, τ, d, d_1, $\varrho = \max\{h_2, \tau, d, d_1\}$ are known real constant scalars and we denote $h_{12} = h_2 - h_1$.

In this paper, we consider activation functions $f(\cdot)$, $g(\cdot)$ and $h(\cdot)$ satisfy Lipschitzian with the Lipschitz constants $\hat{f}_i$, $\hat{g}_i$ and $\hat{h}_i > 0$:

$$\begin{aligned} |f_i(x_1) - f_i(x_2)| &\le \hat{f}_i|x_1 - x_2|, \\ |g_i(x_1) - g_i(x_2)| &\le \hat{g}_i|x_1 - x_2|, \qquad (5) \\ |h_i(x_1) - h_i(x_2)| &\le \hat{h}_i|x_1 - x_2|, \\ i = 1, 2, ..., n, &\quad \forall x_1, x_2 \in \mathbb{R}, \end{aligned}$$

and we denote that

$$\begin{aligned} F &= diag\{\hat{f}_i, i = 1, 2, ..., n\}, \\ G &= diag\{\hat{g}_i, i = 1, 2, ..., n\}, \qquad (6) \\ H &= diag\{\hat{h}_i, i = 1, 2, ..., n\}. \end{aligned}$$

Definition 1. *[10]. Given $\alpha > 0$. The zero solution of system (1), where $u(t) = 0$, $w(t) = 0$, is $\alpha-$stable if there is a positive number $N > 0$ such that every solution of the system satisfies*

$$||x(t, \phi)|| \le N||\phi||_c e^{-\alpha t}, \quad \forall t \le 0.$$

Definition 2. *[10]. Given $\alpha > 0$, $\gamma > 0$. The H_∞ control problem for system (1) has a solution if there exists a memoryless state feedback controller $u(t) = Kx(t)$ satisfying the following two requirements:*
(i) The zero solution of the closed-loop system, where $w(t) = 0$,

$$\begin{aligned}\dot{x}_i(t) &= -Ax(t) + Bf(x(t)) + Cg(x(t-h(t))) \\ &\quad + D\int_{t-d(t)}^{t} h(x(s))ds + \mathscr{U}(t),\end{aligned}$$

is $\alpha-$stable.
(ii) There is a number $c_0 > 0$ such that

$$\sup \frac{\int_0^\infty ||z(t)||^2 dt}{c_0||\phi||_c^2 + \int_0^\infty ||w(t)||^2 dt} \leq \gamma,$$

where the supremum is taken over all $\phi(t) \in \mathbb{C}[[-\varrho, 0], \mathbb{R}^n]$ and the non-zero uncertainty $w(t) \in L_2([0, \infty], \mathbb{R}^n)$.

Lemma 3. *[4]. (Cauchy inequality) For any symmetric positive definite matrix $N \in M^{n\times n}$ and $x, y \in \mathbb{R}^n$ we have*

$$\pm 2x^T y \leq x^T N x + y^T N^{-1} y.$$

Lemma 4. *[4]. Given a positive definite matrix $Z \in R^{n\times n}$, and two scalar $0 \leq r_1 < r_2$ and vector function $x : [r_1, r_2] \to \mathbb{R}^n$ such that the following integrations concerned are well defined, then we have*

$$\begin{aligned}&\Big(\int_{r_1}^{r_2} x(s)ds\Big)^T Z \Big(\int_{r_1}^{r_2} x(s)ds\Big) \\ \leq\ & (r_2 - r_1)\int_{r_1}^{r_2} x^T(s) Z x(s) ds.\end{aligned}$$

Lemma 5. *[7]. For any positive definite symmetric constant matrix P and scalar $\tau > 0$, such that the following integrations are well defined*

$$\begin{aligned}&-\int_{-\tau}^{0}\int_{t+\theta}^{t} x^T(s)Px(s)dsd\theta \leq -\frac{2}{\tau^2} \\ &\Big(\int_{-\tau}^{0}\int_{t+\theta}^{t} x(s)dsd\theta\Big)^T P \Big(\int_{-\tau}^{0}\int_{t+\theta}^{t} x(s)dsd\theta\Big).\end{aligned}$$

Lemma 6. *[4]. (Schur complement) Given constant matrices Z_1, Z_2, Z_3 where $Z_1 = Z_1^T$ and $Z_2 = Z_2^T > 0$. Then $Z_1 + Z_3^T Z_2^{-1} Z_3 < 0$ if and only if $\begin{bmatrix} Z_1 & Z_3^T \\ Z_3 & -Z_2 \end{bmatrix} < 0$ or $\begin{bmatrix} -Z_2 & Z_3 \\ Z_3^T & Z_1 \end{bmatrix} < 0$.*

3 Stability analysis

In this section, we will present stability criterion for system (3).
Consider a Lyapunov-Krasovskii functional candidate as

$$V(t, x_t) = \sum_{i=1}^{14} V_i(t, x_t), \tag{7}$$

where

$$\begin{aligned}
V_1(x_t) &= x^T(t)P_1x(t) + 2x^T(t)P_2\int_{t-h_2}^{t} x(s)ds \\
&\quad + \Big(\int_{t-h_2}^{t} x(s)ds\Big)^T P_3 \int_{t-h_2}^{t} x(s)ds \\
&\quad + 2x^T(t)P_4\int_{-h_2}^{0}\int_{t+s}^{t} x(\theta)d\theta ds \\
&\quad + 2\Big(\int_{t-h_2}^{t} x(s)ds\Big)^T P_5 \\
&\quad \times \int_{-h_2}^{0}\int_{t+s}^{t} x(\theta)d\theta ds \\
&\quad + \Big(\int_{-h_2}^{0}\int_{t+s}^{t} x(\theta)d\theta ds\Big)^T \\
&\quad \times P_6\int_{-h_2}^{0}\int_{t+s}^{t} x(\theta)d\theta ds, \\
V_2(x_t) &= \int_{t-h_1}^{t} e^{2\alpha(s-t)}x^T(s)R_1x(s)ds, \\
V_3(x_t) &= \int_{t-h_2}^{t} e^{2\alpha(s-t)}x^T(s)R_2x(s)ds, \\
V_4(x_t) &= h_1\int_{-h_1}^{0}\int_{t+s}^{t} e^{2\alpha(\theta-t)}\dot{x}^T(\theta)Q_1\dot{x}(\theta)d\theta ds, \\
V_5(x_t) &= h_2\int_{-h_2}^{0}\int_{t+s}^{t} e^{2\alpha(\theta-t)}\dot{x}^T(\theta)Q_2\dot{x}(\theta)d\theta ds, \\
V_6(x_t) &= h_{12}\int_{-h_2}^{-h_1}\int_{t+s}^{t} e^{2\alpha(\theta-t)}\dot{x}^T(\theta) \\
&\quad \times Z_2\dot{x}(\theta)d\theta ds, \\
V_7(x_t) &= \int_{-d}^{0}\int_{t+s}^{t} e^{2\alpha(\theta-t)}h^T(x(\theta)) \\
&\quad \times Uh(x(\theta))d\theta ds, \\
V_8(x_t) &= \int_{-d_1}^{0}\int_{t+s}^{t} e^{2\alpha(\theta-t)}u^T(\theta)S_2u(\theta)d\theta ds, \\
V_9(x_t) &= \tau\int_{-\tau}^{0}\int_{t+s}^{t} e^{-2\alpha(\theta-t)}\dot{u}^T(\theta)S_1\dot{u}(\theta)d\theta ds, \\
V_{10}(x_t) &= \int_{-h_2}^{-h_1}\int_{s}^{0}\int_{t+u}^{t} e^{2\alpha(\theta+u-t)}\dot{x}^T(\theta) \\
&\quad \times Z_1\dot{x}(\theta)d\theta du ds,
\end{aligned}$$

$$\begin{aligned}
V_{11}(x_t) &= \int_{-h_1}^{0}\int_{\tau}^{0}\int_{t+s}^{t} e^{2\alpha(\theta+s-t)}\dot{x}^T(\theta) \\
&\quad \times W_1\dot{x}(\theta)d\theta ds d\tau, \\
V_{12}(x_t) &= \int_{-h_2}^{0}\int_{\tau}^{0}\int_{t+s}^{t} e^{2\alpha(\theta+s-t)}\dot{x}^T(\theta) \\
&\quad \times W_2\dot{x}(\theta)d\theta ds d\tau, \\
V_{13}(x_t) &= \int_{-h_2}^{0}\int_{\tau}^{0}\int_{t+s}^{t} e^{2\alpha(\theta+s-t)}\dot{x}^T(\theta) \\
&\quad \times W_3\dot{x}(\theta)d\theta ds d\tau, \\
V_{14}(x_t) &= \int_{-d}^{0}\int_{t+s}^{t} e^{2\alpha(\theta-t)}x^T(\theta)Q_3x(\theta)d\theta ds.
\end{aligned}$$

Let us set

$$\begin{aligned}
\lambda_1 &= \lambda_{\min}(P_1), \\
\lambda_2 &= 3h_2^2\lambda_{\max}(\Theta) + h_1\lambda_{\max}(R_1) + h_1^2\lambda_{\max}(Q_1) \\
&\quad +(h_2-h_1)^2\lambda_{\max}(Z_2) + d^2\lambda_{\max}(H^TUH) \\
&\quad +d_1^2\lambda_{\max}(P_1^{-1}B_1^TS_2B_1P_1^{-1}) \\
&\quad +\tau^2\lambda_{\max}(P_1^{-1}B_1^TS_1B_1P_1^{-1}) \\
&\quad +(h_2-h_1)h_2^2\lambda_{\max}(Z_1) + h_1^2\lambda_{\max}(W_1) \\
&\quad +h_2^2\lambda_{\max}(W_2) + d^2\lambda_{\max}(Q_3).
\end{aligned}$$

Theorem 7. *Given $\alpha > 0$, The H_∞ control of system (3) has a solution if there exist symmetric positive definite matrices Q_1, Q_2, Q_3, R_1, R_2, S_1, S_2, S_3, W_1, W_2, W_3, Z_1, Z_2, Z_3, diagonal matrices $U > 0$, $U_2 > 0$, $U_3 > 0$, and matrices $P_1 = P_1^T$, $P_3 = P_3^T$, $P_6 = P_6^T$, P_2, P_4, P_5 such that the following LMI hold:*

$$\Xi_1 = \begin{bmatrix} \tilde{\Xi}_{1(1,1)} & \tilde{\Xi}_{1(1,2)} \\ * & \tilde{\Xi}_{1(2,2)} \end{bmatrix} < 0, \quad (8)$$

$$\Xi_2 = \begin{bmatrix} \tilde{\Xi}_{2(1,1)} & \tilde{\Xi}_{2(1,2)} \\ * & \tilde{\Xi}_{1(2,2)} \end{bmatrix} < 0, \quad (9)$$

$$\Xi_3 = \begin{bmatrix} -0.5e^{-2\alpha h_2}Q_2 + \mathscr{N} \end{bmatrix} < 0, \quad (10)$$

$$\Xi_4 = \begin{bmatrix} -0.1R_1 + \tau^2B_1^TS_1B_1 \end{bmatrix} < 0, \quad (11)$$

where

$$\tilde{\Xi}_{1(1,1)} = \begin{bmatrix} \Pi & F^TP_1 & P_1 & 2dP_1D \\ * & -U_2 & 0 & 0 \\ * & * & -U_3 & 0 \\ * & * & * & \Xi_{1(4,4)} \end{bmatrix},$$

$$\tilde{\Xi}_{1(1,2)} = \begin{bmatrix} 4P_1B_2 & 2d_1P_1B_3 & P_1E \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\tilde{\Xi}_{1(2,2)} = \begin{bmatrix} \Xi_{1(5,5)} & 0 & 0 \\ * & \Xi_{1(6,6)} & 0 \\ * & * & -0.5\gamma \end{bmatrix},$$

$$\tilde{\Xi}_{2(1,1)} = \begin{bmatrix} -0.4R_1 & R_1B & R_1C & 2dR_1D \\ * & -U_2 & 0 & 0 \\ * & * & -U_3 & 0 \\ * & * & * & \Xi_{2(4,4)} \end{bmatrix},$$

$$\tilde{\Xi}_{2(1,2)} = \begin{bmatrix} 4R_1B_2 & 2d_1R_1B_3 & R_1B_1 & R_1E \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\tilde{\Xi}_{2(2,2)} = \begin{bmatrix} \Xi_{2(5,5)} & 0 & 0 & 0 \\ * & \Xi_{2(6,6)} & 0 & 0 \\ * & * & -S_3 & 0 \\ * & * & * & -0.5\gamma \end{bmatrix},$$

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ * & \Pi_{22} \end{bmatrix} < 0,$$

$$\Pi_{11} = \begin{bmatrix} \Pi_{1,1} & \Pi_{1,2} & 0 & \Pi_{1,4} & \Pi_{1,5} \\ * & \Pi_{2,2} & \Pi_{2,3} & \Pi_{2,4} & 0 \\ * & * & \Pi_{3,3} & \Pi_{3,4} & 0 \\ * & * & * & \Pi_{4,4} & 0 \\ * & * & * & * & \Pi_{5,5} \end{bmatrix},$$

$$\Pi_{12} = \begin{bmatrix} \Pi_{1,6} & \Pi_{1,7} & 0 & \Pi_{1,9} & -A^TR_1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -P_3 & 0 & 0 & -P_5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Pi_{22} = \begin{bmatrix} \Pi_{6,6} & 0 & 0 & \Pi_{6,9} & P_2 \\ * & \Pi_{7,7} & 0 & 0 & 0 \\ * & * & \Pi_{8,8} & 0 & 0 \\ * & * & * & -2\alpha P_6 & P_4 \\ * & * & * & * & \Pi_{10,10} \end{bmatrix},$$

$$\begin{aligned}
\Pi_{1,1} &= -AP_1 - P_1A^T + B_1B_1 + B_1^TB_1^T + R_1 \\
&\quad +R_2 + B^TU_2B + P_2 + P_2^T + h_2P_4 \\
&\quad -e^{-2\alpha h_1}Q_1 - 0.5e^{-2\alpha h_2}Q_2 + dQ_3 \\
&\quad +dH^TUH - e^{-4\alpha h_1}(W_1 + W_1^T) \\
&\quad -e^{-4\alpha h_2}(Z_1 + Z_1^T + W_2 + W_2^T) \\
&\quad -e^{-4\alpha h_2}(W_3 + W_3^T) + h_2P_4^T \\
&\quad -2\alpha P_1 + F^TU_2F, \\
\Pi_{1,2} &= e^{-2\alpha h_1}Q_1, \Pi_{1,4} = -P_2 + e^{-2\alpha h_2}Q_2, \\
\Pi_{1,5} &= 2h_1^{-1}e^{-2\alpha h_1}W_1, \Pi_{1,6} = P_3 - P_4 \\
&\quad +h_2P_5^T + 2h_2^{-1}e^{-4\alpha h_2}(W_2 + W_3) \\
&\quad -2\alpha P_2, \Pi_{1,7} = 2h_{12}^{-1}e^{-4\alpha h_2}Z_1, \\
\Pi_{1,9} &= P_5 + h_2P_6 - 2\alpha P_4, \\
\Pi_{2,2} &= -e^{-2\alpha h_1}(R_1 + Q_1) - e^{-2\alpha h_2}Z_2, \\
\Pi_{2,3} &= e^{-2\alpha h_2}(Z_2 - Z_3),\ \Pi_{2,4} = e^{-2\alpha h_2}Z_3, \\
\Pi_{3,3} &= -e^{-2\alpha h_2}(Z_2 + Z_2^T) + e^{-2\alpha h_2}(Z_3 + Z_3^T) \\
&\quad +G^TC^TU_3CG + G^TU_3G, \\
\Pi_{3,4} &= e^{-2\alpha h_2}(Z_2 - Z_3), \\
\Pi_{4,4} &= -e^{-2\alpha h_2}(Z_2 + R_2 + Q_2), \\
\Pi_{5,5} &= -h_1^{-2}e^{-4\alpha h_1}(W_1 + W_1^T), \\
\Pi_{6,6} &= -P_5 - P_5^T - 2\alpha P_3 - h_2^{-2}e^{-4\alpha h_2}W_2, \\
&\quad -h_2^{-2}e^{-4\alpha h_2}(W_2^T + W_3 + W_3^T),
\end{aligned}$$

$$\begin{aligned}
\Pi_{7,7} &= -h_{12}^{-2}e^{-4\alpha h_2}(Z_1+Z_1^T),\\
\Pi_{8,8} &= -d^{-1}e^{-2\alpha d}Q_3,\\
\Pi_{10,10} &= -1.5R_1+h_1^2Q_1+h_2^2Q_2+h_{12}^2Z_2\\
&\quad +h_{12}h_2Z_1+h_1W_1+h_2W_2+h_2W_3\\
\Xi_{1(4,4)} &= -2de^{-2\alpha d}U,\ \Xi_{1(5,5)}=-4e^{-2\alpha\tau}S_1,\\
\Xi_{1(6,6)} &= -2d_1e^{-2\alpha d_1}S_2,\ \Xi_{2(4,4)}=-2de^{-2\alpha d}U,\\
\Xi_{2(5,5)} &= -4e^{-2\alpha\tau}S_1,\ \Xi_{2(6,6)}=-2d_1e^{-2\alpha d_1}S_2,\\
\mathscr{N} &= e^{-2\alpha\tau}B_1^TS_1B_1+dB_1^TS_2B_1+B_1^TS_3B_1.
\end{aligned}$$

Moreover, stabilizing feedback control is given by

$$u(t)=B_1P_1^{-1}x(t),\quad t\geq 0,$$

and the solution of the system satisfies

$$||x(t,\phi)|| \leq \sqrt{\frac{\lambda_2}{\lambda_1}}||\phi||_c e^{-\alpha t},\quad t\geq 0.$$

Proof: Choosing the Lyapunov-Krasovskii functional candidate as (7). It is easy to check that

$$\begin{aligned}
\lambda_1||x(t)||^2 &\leq V(t,x_t),\quad \forall t\geq 0, \quad (12)\\
\text{and}\quad V(0,x_0) &\leq \lambda_2||\phi||_c^2.
\end{aligned}$$

We take the time-derivative of $V_i(x_t)$ along the solutions of system (3)

$$\begin{aligned}
\dot V_1(x_t) &= -2x^T(t)AP_1x(t)+2x^T(t)B_1^TB_1x(t)\\
&\quad +2f^T(x(t))B^TP_1x(t)+2w^T(t)E^T\\
&\quad \times P_1x(t)+2u^T(t-\tau(t))B_2^TP_1x(t)\\
&\quad +2g^T(x(t-h(t)))C^TP_1x(t)\\
&\quad +2\left(\int_{t-d(t)}^t h(x(s))ds\right)^T D^TP_1x(t)\\
&\quad +2\left(\int_{t-d_1(t)}^t u(s)ds\right)^T B_3^TP_1x(t)\\
&\quad +2x^T(t)P_2[x(t)-x(t-h_2)]\\
&\quad +2\left(\int_{t-h_2}^t x(s)ds\right)^T P_2\dot x(t)\\
&\quad +2[x(t)-x(t-h_2)]^TP_3\int_{t-h_2}^t x(s)ds\\
&\quad +2x^T(t)P_4[h_2x(t)-\int_{t-h_2}^t x(s)ds]\\
&\quad +2\left(\int_{-h_2}^0\int_{t+s}^t x(\theta)d\theta ds\right)^T P_4\dot x(t)\\
&\quad +2\left(\int_{t-h_2}^t x(s)ds\right)^T P_5[h_2x(t)\\
&\quad -\int_{t-h_2}^t x(s)ds]+2[x(t)-x(t-h_2)]^T\\
&\quad \times P_5\int_{-h_2}^0\int_{t+s}^t x(\theta)d\theta ds+2[h_2x(t)\\
&\quad -x(t-h_2)]^TP_6\int_{-h_2}^0\int_{t+s}^t x(\theta)d\theta ds,\\
\dot V_2(x_t) &= x^T(t)R_1x(t)-e^{-2\alpha h_1}x^T(t-h_1)\\
&\quad \times R_1x(t-h_1)-2\alpha V_2,\\
\dot V_3(x_t) &= x^T(t)R_2x(t)-e^{-2\alpha h_2}x^T(t-h_2)\\
&\quad \times R_2x(t-h_2)-2\alpha V_3,\\
\dot V_4(x_t) &\leq h_1^2\dot x^T(t)Q_1\dot x(t)-h_1e^{-2\alpha h_1}\int_{t-h_1}^t\dot x^T(s)\\
&\quad \times Q_1\dot x(s)ds-2\alpha V_4,\\
\dot V_5(x_t) &\leq h_2^2\dot x^T(t)Q_2\dot x(t)-h_2e^{-2\alpha h_2}\int_{t-h_2}^t\dot x^T(s)\\
&\quad \times Q_2\dot x(s)ds-2\alpha V_5,\\
\dot V_6(x_t) &\leq h_{12}^2\dot x^T(t)Z_2\dot x(t)-h_{12}e^{-2\alpha h_2}\int_{t-h_2}^{t-h_1}\\
&\quad \dot x^T(s)Z_2\dot x(s)ds-2\alpha V_6,\\
\dot V_7(x_t) &\leq dh^T(x(t))Uh(x(t))-e^{-2\alpha d}\int_{t-d}^t\\
&\quad \times h^T(x(s))Uh(x(s))ds-2\alpha V_7,\\
\dot V_8(x_t) &\leq d_1u^T(t)S_2u(t)-e^{-2\alpha d_1}\int_{t-d_1}^t u^T(s)\\
&\quad \times S_2u(s)ds-2\alpha V_8, \quad (13)\\
\dot V_9(x_t) &\leq \tau^2\dot u^T(t)S_1\dot u(t)-\tau e^{-2\alpha\tau}\int_{t-\tau}^t\dot u^T(s)\\
&\quad \times S_2\dot u(s)ds-2\alpha V_9,\\
\dot V_{10}(x_t) &\leq h_{12}h_2\dot x^T(t)Z_1\dot x(t)-e^{-4\alpha h_2}\int_{-h_2}^{-h_1}\int_{t+\theta}^t\\
&\quad \dot x^T(u)Z_1\dot x(u)dud\theta-2\alpha V_{10},\\
\dot V_{11}(x_t) &\leq h_1\dot x^T(t)W_1\dot x(t)-e^{-4\alpha h_1}\int_{-h_1}^0\int_{t+\tau}^t\\
&\quad \dot x^T(s)W_1\dot x(s)dsd\tau-2\alpha V_{11},\\
\dot V_{12}(x_t) &\leq h_2\dot x^T(t)W_2\dot x(t)-e^{-4\alpha h_2}\int_{-h_2}^0\int_{t+\tau}^t\\
&\quad \dot x^T(s)W_2\dot x(s)dsd\tau-2\alpha V_{12},\\
\dot V_{13}(x_t) &\leq h_2\dot x^T(t)W_3\dot x(t)-e^{-4\alpha h_2}\int_{-h_2}^0\int_{t+\tau}^t\\
&\quad \dot x^T(s)W_3\dot x(s)dsd\tau-2\alpha V_{13},
\end{aligned}$$

$$\begin{aligned}\dot{V}_{14}(x_t) &\leq dx^T(t)Q_3x(t) - e^{-2\alpha d}\int_{t-d}^{t} x^T(s) \\ &\quad Q_3x(s)ds - 2\alpha V_{14}.\end{aligned}$$

By Lemma 3 and Lemma 4, we have

$$\begin{aligned}
&2f^T(x(t))B^TP_1x(t) \\
\leq\ & x^T(t)F^TP_1U_2^{-1}P_1Fx(t) \\
&+x^T(t)B^TU_2Bx(t), \\
&2g^T(x(t-h(t)))C^TP_1x(t) \\
\leq\ & x^T(t-h(t))G^TC^TU_3CGx(t-h(t)) \\
&+x^T(t)P_1U_3^{-1}P_1x(t), \\
&2w^T(t)E^TP_1x(t) \\
\leq\ & \frac{\gamma}{2}w^T(t)w(t) + \frac{2}{\gamma}x^T(t)P_1E^TEP_1x(t), \\
&2u^T(t-\tau(t))B_2^TP_1x(t) \\
\leq\ & \frac{e^{-2\alpha\tau}}{4}u^T(t-\tau(t))S_1u(t-\tau(t)) \qquad (14) \\
&+4e^{2\alpha\tau}x^T(t)P_1B_2S_1^{-1}B_2^TP_1x(t), \\
&2\left(\int_{t-d(t)}^{t} h(x(s))ds\right)^T D^TP_1x(t) \\
\leq\ & \frac{e^{-2\alpha d}}{2}\int_{t-d}^{t} h^T(x(s))Uh(x(s))ds \\
&+2de^{2\alpha d}x^T(t)P_1DU^{-1}D^TP_1x(t), \\
&2\left(\int_{t-d_1(t)}^{t} u(s)ds\right)^T B_3^TP_1x(t) \\
\leq\ & \frac{e^{-2\alpha d_1}}{2}\int_{t-d_1}^{t} u^T(s)S_2u(s)ds \\
&+2d_1e^{2\alpha d_1}x^T(t)P_1B_3S_2^{-1}B_3^TP_1x(t),
\end{aligned}$$

$$\begin{aligned}
dh^T(x(t))Uh(x(t)) &\leq dx^T(t)HUHx(t), \\
d_1u^T(t)S_2u(t) &= d_1x^T(t)P_1^{-1}B_1^T \\
&\quad \times S_2B_1P_1^{-1}x(t), \\
\tau^2\dot{u}^T(t)S_1\dot{u}(t) &= \tau^2\dot{x}^T(t)P_1^{-1}B_1^T \\
&\quad \times S_1B_1P_1^{-1}\dot{x}(t),
\end{aligned}$$

and the Leibniz-Newton formula gives

$$\begin{aligned}
&-\tau e^{-2\alpha\tau}\int_{t-\tau}^{t} \dot{u}^T(s)S_2\dot{u}(s)ds \\
\leq\ & -\tau(t)e^{-2\alpha\tau}\int_{t-\tau(t)}^{t} \dot{u}^T(s)S_2\dot{u}(s)ds \\
\leq\ & -e^{-2\alpha\tau}\left(\int_{t-\tau(t)}^{t} \dot{u}(s)ds\right)^T \\
&\times S_2\left(\int_{t-\tau(t)}^{t} \dot{u}(s)ds\right) \\
\leq\ & -e^{-2\alpha\tau}u^T(t)S_1u(t) + 2e^{-2\alpha\tau}u^T(t)S_1u(t) \\
&+\frac{e^{-2\alpha\tau}}{2}u^T(t-\tau(t))S_1S_1^{-1}S_1u(t-\tau(t)) \\
&-e^{-2\alpha\tau}u^T(t-\tau(t))S_1u(t-\tau(t)) \\
=\ & e^{-2\alpha\tau}x^T(t)P_1^{-1}B_1^TS_1B_1P_1^{-1}x(t) \\
&-\frac{e^{-2\alpha\tau}}{2}u^T(t-\tau(t))S_1u(t-\tau(t)). \qquad (15)
\end{aligned}$$

Denote

$$\begin{aligned}
\sigma_1(t) &= \int_{t-h_2}^{t-h(t)} \dot{x}(s)ds, \\
\sigma_2(t) &= \int_{t-h(t)}^{t-h_1} \dot{x}(s)ds. \qquad (16)
\end{aligned}$$

Next, when $0 < h_1 < h(t) < h_2$, we have

$$\begin{aligned}
&\int_{t-h_2}^{t-h_1} \dot{x}^T(s)Z_2\dot{x}(s)ds \\
=\ & \int_{t-h_2}^{t-h(t)} \dot{x}^T(s)Z_2\dot{x}(s)ds \\
&+\int_{t-h(t)}^{t-h_1} \dot{x}^T(s)Z_2\dot{x}(s)ds.
\end{aligned}$$

Using Lemma 4 gives

$$\begin{aligned}
&h_{12}\int_{t-h_2}^{t-h(t)} \dot{x}^T(s)Z_2\dot{x}(s)ds \\
\geq\ & \frac{h_{12}}{h_2-h(t)}\left(\int_{t-h_2}^{t-h(t)} \dot{x}(s)ds\right)^T \\
&\times Z_2\int_{t-h_2}^{t-h(t)} \dot{x}(s)ds \\
=\ & \frac{h_{12}}{h_2-h(t)}\sigma_1^T(t)Z_2\sigma_1(t).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&h_{12}\int_{t-h(t)}^{t-h_1} \dot{x}^T(s)Z_2\dot{x}(s)ds \\
\geq\ & \frac{h_{12}}{h(t)-h_1}\sigma_2^T(t)Z_2\sigma_2(t),
\end{aligned}$$

then

$$
\begin{aligned}
& h_{12}\int_{t-h_2}^{t-h_1}\dot{x}^T(s)Z_2\dot{x}(s)ds \\
\geq\ & \frac{h_{12}}{h_2-h(t)}\sigma_1^T(t)Z_2\sigma_1(t)+\frac{h_{12}}{h(t)-h_1}\sigma_2^T(t)Z_2\sigma_2(t) \\
=\ & \sigma_1^T(t)Z_2\sigma_1(t)+\frac{h(t)-h_1}{h_2-h(t)}\sigma_1^T(t)Z_2\sigma_1(t) \\
& +\sigma_2^T(t)Z_2\sigma_2(t)+\frac{h_2-h(t)}{h(t)-h_1}\sigma_2^T(t)Z_2\sigma_2(t). \qquad (17)
\end{aligned}
$$

By reciprocally convex with $a=\frac{h_2-h(t)}{h_{12}}$, $b=\frac{h(t)-h_1}{h_{12}}$, the following inequality holds:

$$
\begin{bmatrix} \sqrt{\frac{b}{a}}\sigma_1(t) \\ -\sqrt{\frac{a}{b}}\sigma_2(t)\end{bmatrix}^T \begin{bmatrix} Z_2 & Z_3 \\ Z_3^T & Z_2 \end{bmatrix}\begin{bmatrix} \sqrt{\frac{b}{a}}\sigma_1(t) \\ -\sqrt{\frac{a}{b}}\sigma_2(t)\end{bmatrix} \geq 0, (18)
$$

which implies

$$
\begin{aligned}
& \frac{h(t)-h_1}{h_2-h(t)}\sigma_1^T(t)Z_2\sigma_1(t)+\frac{h_2-h(t)}{h(t)-h_1}\sigma_2^T(t)Z_2\sigma_2(t) \\
\geq\ & \sigma_1^T(t)Z_3\sigma_2(t)+\sigma_2^T(t)Z_3^T\sigma_1(t). \qquad (19)
\end{aligned}
$$

Then, we can get from (16)-(19) that

$$
\begin{aligned}
& h_{12}\int_{t-h_2}^{t-h_1}\dot{x}^T(s)Z_2\dot{x}(s)ds \\
\geq\ & \sigma_1^T(t)Z_2\sigma_1(t)+\sigma_2^T(t)Z_2\sigma_2(t) \\
& +\sigma_1^T(t)Z_3\sigma_2(t)+\sigma_2^T(t)Z_3^T\sigma_1(t).
\end{aligned}
$$

Thus

$$
\begin{aligned}
- & \ e^{-2\alpha h_2}h_{12}\int_{t-h_2}^{t-h_1}\dot{x}^T(s)Z_2\dot{x}(s)ds \\
\leq & -e^{-2\alpha h_2}[x(t-h(t))-x(t-h_2)]^T \\
& \times Z_2[x(t-h(t))-x(t-h_2)] \\
& -e^{-2\alpha h_2}[x(t-h_1)-x(t-h(t))]^T \\
& \times Z_2[x(t-h_1)-x(t-h(t))] \\
& -e^{-2\alpha h_2}[x(t-h(t))-x(t-h_2)]^T \\
& \times Z_3[x(t-h_1)-x(t-h(t))] \\
& -e^{-2\alpha h_2}[x(t-h_1)-x(t-h(t))]^T \\
& \times Z_3^T[x(t-h(t))-x(t-h_2)]. \qquad (20)
\end{aligned}
$$

By using Lemma 4 and Lemma 5 and the following identity relation:

$$
\begin{aligned}
0 \ = & -2\dot{x}^TR_1\dot{x}(t)-2\dot{x}^TR_1Ax(t)+2\dot{x}^TR_1Bf(x(t)) \\
& +2\dot{x}^TR_1Cg(x(t-h(t)))+2\dot{x}^TR_1D \\
& \times\int_{t-d(t)}^{t}h(x(s))ds+2\dot{x}^TR_1Ew(t) \\
& +2\dot{x}^TR_1B_1u(t)+2\dot{x}^TR_1B_2u(t-\tau(t)) \\
& +2\dot{x}^TR_1B_3\int_{t-d_1(t)}^{t}u(s)ds. \qquad (21)
\end{aligned}
$$

and from (14)-(21), we obtain

$$
\begin{aligned}
& \dot{V}(t,x_t)+2\alpha V(t,x_t) \\
\leq\ & \gamma w^T(t)w(t)+\xi^T(t)\mathscr{M}_1\xi(t)+\dot{x}^T(t)\mathscr{M}_2\dot{x}(t) \\
& +x^T(t)\mathscr{M}_3x(t)+\dot{x}^T(t)\mathscr{M}_4\dot{x}(t) \\
& -x^T(t)\Big[A_1^TA_1+A_1^TC_1B_1P_1^{-1}+P_1^{-1}B_1^TC_1^TA_1 \\
& +P_1^{-1}B_1^TC_1^TB_1P_1^{-1}\Big]x(t) \\
& -2x^T(t)\Big[A_1^TB_4+P_1^{-1}B_1^TC_1^TB_4\Big]x(t-h(t)) \\
& -2x^T(t)\Big[A_1^TD_1+P_1^{-1}B_1^TC_1^TD_1\Big]\int_{t-d}^{t}x(s)ds \\
& -2x^T(t)\Big[A_1^TE_1+P_1^{-1}B_1^TC_1^TE_1\Big]w(t) \\
& -x^T(t-h(t))B_4^TB_4x(t-h(t)) \\
& -2x^T(t-h(t))B_4^TD_1\int_{t-d}^{t}x(s)ds \\
& -2x^T(t-h(t))B_4^TE_1w(t) \\
& -\Big(\int_{t-d}^{t}x(s)ds\Big)^TD_1^TD_1\int_{t-d}^{t}x(s)ds \\
& -2\Big(\int_{t-d}^{t}x(s)ds\Big)^TD_1^TE_1w(t) \\
& -w^T(t)E_1^TE_1w(t), \qquad (22)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathscr{M}_1 =\ & \Pi+F^TP_1U_2^{-1}P_1F+P_1U_3^{-1}P_1 \\
& +2de^{2\alpha d}P_1DU^{-1}D^TP_1 \\
& +4e^{2\alpha\tau}P_1B_2S_1^{-1}B_2^TP_1 \\
& +2d_1e^{2\alpha d_1}P_1B_3S_2^{-1}B_3^TP_1+\frac{2}{\gamma}P_1E^TEP_1, \\
\mathscr{M}_2 =\ & -0.4R_1+R_1BU_2^{-1}B^TR_1 \\
& +R_1CU_3^{-1}C^TR_1+2de^{2\alpha d}R_1DU^{-1}D^TR_1 \\
& +4e^{2\alpha\tau}R_1B_2S_1^{-1}B_2^TR_1+\frac{2}{\gamma}R_1E^TER_1 \\
& +2d_1e^{2\alpha d_1}R_1B_3S_2^{-1}B_3^TR_1+R_1B_1S_3^{-1}B_1^TR_1, \\
\mathscr{M}_3 =\ & -0.5e^{-2\alpha h_2}Q_2+dP_1^{-1}B_1^TS_2B_1P_1^{-1} \\
& +e^{-2\alpha\tau}P_1^{-1}B_1^TS_1B_1P_1^{-1}+P_1^{-1}B_1S_3B_1P_1^{-1}, \\
& \qquad (23)
\end{aligned}
$$

$$\begin{aligned}
\mathscr{M}_4 &= -0.1R_1 + \tau^2 P_1^{-1} B_1^T S_1 B_1 P_1^{-1}, \\
\xi^T(t) &= \Big[x^T(t)\ x^T(t-h_1)\ x^T(t-h(t))\ x^T(t-h_2) \\
&\quad \int_{t-h_1}^{t} x^T(s)ds\ \int_{t-h_2}^{t} x^T(s)ds \\
&\quad \int_{t-h_2}^{t-h_1} x^T(s)ds\ \int_{t-d}^{t} x^T(s)ds \\
&\quad \int_{-h_2}^{0}\int_{t+s}^{t} x^T(\theta)d\theta ds\ \dot{x}^T(t)\Big].
\end{aligned}$$

Using Schur complement lemma, pre-multiplying and post-multiplying $\mathscr{M}_1$, $\mathscr{M}_2$, $\mathscr{M}_3$ and $\mathscr{M}_4$ by P_1 and P_1 respectively, the inequality $\mathscr{M}_1$, $\mathscr{M}_2$, $\mathscr{M}_3$ and $\mathscr{M}_4$ are equivalent to $\Xi_1 < 0$, $\Xi_2 < 0$, $\Xi_3 < 0$ and $\Xi_4 < 0$ respectively, and from the inequality (22) it follow that

$$\begin{aligned}
&\dot{V}(t,x_t) + 2\alpha V(t,x_t) \\
\leq\ & \gamma w^T(t)w(t) - x^T(t)\Big[A_1^T A_1 + A_1^T C_1 B_1 P_1^{-1} \\
&+ P_1^{-1} B_1^T C_1^T A_1 + P_1^{-1} B_1^T C_1^T B_1 P_1^{-1}\Big]x(t) \\
&-2x^T(t)\Big[A_1^T B_4 + P_1^{-1} B_1^T C_1^T B_4\Big]x(t-h(t)) \\
&-2x^T(t)\Big[A_1^T D_1 + P_1^{-1} B_1^T C_1^T D_1\Big]\int_{t-d}^{t} x(s)ds \\
&-2x^T(t)\Big[A_1^T E_1 + P_1^{-1} B_1^T C_1^T E_1\Big]w(t) \\
&-x^T(t-h(t))B_4^T B_4 x(t-h(t)) \\
&-2x^T(t-h(t))B_4^T D_1 \int_{t-d}^{t} x(s)ds \\
&-2x^T(t-h(t))B_4^T E_1 w(t) \\
&-\Big(\int_{t-d}^{t} x(s)ds\Big)^T D_1^T D_1 \int_{t-d}^{t} x(s)ds \\
&-2\Big(\int_{t-d}^{t} x(s)ds\Big)^T D_1^T E_1 w(t) \\
&-w^T(t)E_1^T E_1 w(t). \qquad (24)
\end{aligned}$$

Letting $w(t) = 0$, we obtain from the inequality (24) that

$$\dot{V}(t,x_t) + 2\alpha V(t,x_t) \leq 0,$$

we have,

$$\dot{V}(t,x_t) \leq -2\alpha V(t,x_t), \quad \forall t \geq 0. \qquad (25)$$

Integrating both sides of (25) from 0 to t , we obtain

$$V(t,x_t) \leq V(0,x_0)e^{-2\alpha t}, \quad \forall t \geq 0.$$

Taking the condition (12) into account, we have

$$\begin{aligned}
&\lambda_1 ||x(t)||^2 \leq V(t,x_t) \leq V(0,x_0)e^{-2\alpha t} \\
&\leq \lambda_2 ||\phi||_c^2 e^{-2\alpha t}.
\end{aligned}$$

Then, the solution $||x(t,\phi)||$ of the system (3) satisfy

$$||x(t,\phi)|| \leq \sqrt{\frac{\lambda_2}{\lambda_1}}||\phi||_c e^{-\alpha t}, \quad \forall t \geq 0, \qquad (26)$$

which implies that the zero solution of the closed-loop system is $\alpha-$stable. To complete the proof of the theorem, it remains to show the $\gamma-$optimal level condition (ii). For this, we consider the following relation:

$$\begin{aligned}
&\int_0^t ||z(s)||^2 - \gamma ||w(s)||^2]ds \\
=\ & \int_0^t [||z(s)||^2 - \gamma ||w(s)||^2 + \dot{V}(s,x_s)]ds \\
&- \int_0^t \dot{V}(s,x_s)ds.
\end{aligned}$$

Since $V(t,x_t) \geq 0$, we obtain

$$-\int_0^t \dot{V}(s,x_s)ds = V(0,x_0) - V(t,x_t) \leq V(0,x_0).$$

Therefore, for all $t \leq 0$

$$\begin{aligned}
&\int_0^t ||z(s)||^2 - \gamma ||w(s)||^2]ds \qquad (27) \\
\leq\ & \int_0^t [||z(s)||^2 - \gamma ||w(s)||^2 + \dot{V}(s,x_s)]ds \\
&+ V(0,x_0).
\end{aligned}$$

From (24) and the value of $||z(t)||^2$ we obtain

$$\begin{aligned}
&\int_0^t ||z(s)||^2 - \gamma ||w(s)||^2]ds \qquad (28) \\
\leq\ & \int_0^t \Big[-2\alpha V(t,x_t)\Big]ds + V(0,x_0).
\end{aligned}$$

Hence, from (28) it follows that

$$\int_0^t ||z(s)||^2 - \gamma ||w(s)||^2]ds \leq V(0,x_0) \leq \lambda_2 ||\phi||_c^2,$$

equivalently,

$$\int_0^t ||z(s)||^2 dt \leq \int_0^t \gamma ||w(s)||^2]ds + \lambda_2 ||\phi||_c^2.$$

Letting $t \to \infty$, and setting $c_0 = \frac{\lambda_2}{\gamma}$, we obtain that

$$\frac{\int_0^\infty ||z(t)||^2 dt}{c_0 ||\phi||_c^2 + \int_0^\infty ||w(t)||^2 dt} \leq \gamma,$$

for all non-zero $w(t) \in L_2([0,\infty],\mathbb{R}^n)$, $\phi(t) \in \mathbb{C}[[-\varrho,0],\mathbb{R}^n]$. This completes the proof of the theorem. □

4 Numerical Examples

In this section, numerical example is given to present the effectiveness and applicability of our stability results.

Example 8. *Consider neural network (3) with parameters as follows:*

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}, \; B = \begin{bmatrix} -0.7 & 0.2 \\ 0.4 & -0.1 \end{bmatrix},$$
$$C = \begin{bmatrix} 0.7 & -0.8 \\ 0.5 & -0.9 \end{bmatrix}, \; D = \begin{bmatrix} 0.7 & -0.7 \\ -0.1 & -0.4 \end{bmatrix},$$
$$E = \begin{bmatrix} -0.1 & 0.2 \\ 0 & 0.4 \end{bmatrix}, \; F = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.3 \end{bmatrix},$$
$$G = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.4 \end{bmatrix}, \; H = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.2 \end{bmatrix},$$
$$A_1 = \begin{bmatrix} -0.2 & 0.3 \\ 0 & -0.4 \end{bmatrix}, \; B_1 = \begin{bmatrix} -0.4 & 0 \\ 0 & -0.4 \end{bmatrix},$$
$$B_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \; B_3 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},$$
$$B_4 = \begin{bmatrix} -0.4 & 0 \\ -0.1 & -0.5 \end{bmatrix}, \; C_1 = \begin{bmatrix} 0.2 & 0.1 \\ 0 & -0.5 \end{bmatrix},$$
$$D_1 = \begin{bmatrix} 0.2 & 0.1 \\ 0 & -0.4 \end{bmatrix}, \; E_1 = \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0.3 \end{bmatrix},$$
$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; h(\cdot) = tanh(\cdot)$$
$$f(\cdot) = g(\cdot) = 0.2 \begin{bmatrix} |x_1(t)+1| - |x_1(t)-1| \\ |x_2(t)+1| - |x_2(t)-1| \end{bmatrix},$$

From the conditions (8)-(11) of Theorem 7, we let $\alpha = 0.01$, $h_1 = 0.1$, $h_2 = 0.3$, $d = 0.3$, $d_1 = 0.5$, *and* $\tau = 0.4$. *By using the LMI Toolbox in MATLAB, we obtain* $\gamma = 1.7637$,

$$P_1 = \begin{bmatrix} 0.9248 & -0.1581 \\ -0.1581 & 0.7921 \end{bmatrix}, P_2 = \begin{bmatrix} 0.0948 & -0.0501 \\ -0.0521 & 0.1187 \end{bmatrix},$$
$$P_3 = \begin{bmatrix} -0.3225 & 0.0156 \\ 0.0156 & -0.3867 \end{bmatrix}, P_4 = \begin{bmatrix} 0.0011 & -0.0006 \\ -0.0006 & 0.0013 \end{bmatrix},$$
$$P_5 = \begin{bmatrix} -0.0020 & -0.0002 \\ -0.0001 & -0.0028 \end{bmatrix}, P_6 = \begin{bmatrix} 0.0146 & -0.0020 \\ -0.0020 & 0.0161 \end{bmatrix},$$
$$Q_1 = \begin{bmatrix} 0.5107 & -0.0462 \\ -0.0462 & 0.5004 \end{bmatrix}, Q_2 = \begin{bmatrix} 0.4714 & -0.0328 \\ -0.0328 & 0.5166 \end{bmatrix},$$
$$S_2 = \begin{bmatrix} 0.8199 & -0.0161 \\ -0.0161 & 0.8399 \end{bmatrix}, R_1 = \begin{bmatrix} 0.1392 & -0.0521 \\ -0.0521 & 0.1790 \end{bmatrix},$$
$$R_2 = \begin{bmatrix} 0.4409 & -0.0481 \\ -0.0481 & 0.4248 \end{bmatrix}, S_1 = \begin{bmatrix} 0.2493 & -0.0177 \\ -0.0177 & 0.2606 \end{bmatrix},$$

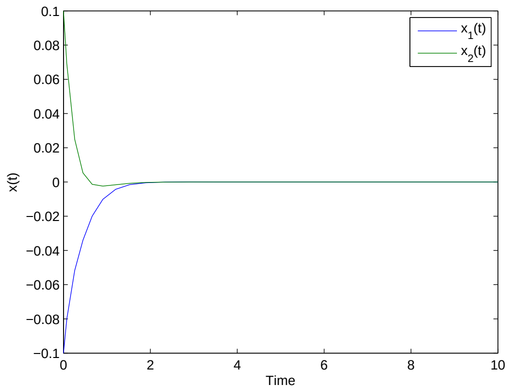

Figure 1: Response solution of the system (3) where $w(t) = 0$

$$S_3 = \begin{bmatrix} 0.5445 & -0.0353 \\ -0.0353 & 0.6032 \end{bmatrix}, Q_3 = \begin{bmatrix} 0.4332 & 0.0043 \\ 0.0043 & 0.4422 \end{bmatrix},$$
$$W_2 = \begin{bmatrix} 0.0518 & -0.0083 \\ -0.0083 & 0.0582 \end{bmatrix}, U = \begin{bmatrix} 1.5450 & 0 \\ 0 & 1.5450 \end{bmatrix},$$
$$W_3 = \begin{bmatrix} 0.0203 & -0.0007 \\ -0.0007 & 0.0207 \end{bmatrix}, U_2 = \begin{bmatrix} 1.0492 & 0 \\ 0 & 1.0492 \end{bmatrix},$$
$$Z_1 = \begin{bmatrix} 0.0349 & -0.0002 \\ -0.0002 & 0.0353 \end{bmatrix}, U_3 = \begin{bmatrix} 0.9085 & 0 \\ 0 & 0.9085 \end{bmatrix},$$
$$Z_2 = \begin{bmatrix} 0.8015 & -0.1874 \\ -0.1874 & 0.8118 \end{bmatrix}, Z_3 = \begin{bmatrix} 0.2777 & 0.0028 \\ 0.0028 & 0.3139 \end{bmatrix},$$
$$W_1 = 10^{-3} \begin{bmatrix} 7.4622 & -0.0421 \\ -0.0421 & 7.5146 \end{bmatrix},$$

The feedback control is given by

$$u(t) = B_1 P_1^{-1} x(t) = \begin{bmatrix} -0.4478 & -0.0894 \\ -0.0894 & -0.5228 \end{bmatrix}, \quad t \geq 0,$$

Moreover, the solution $x(t, \phi)$ *of the system satisfies*

$$||x(t,\phi)|| \leq 1.2300e^{-0.01t}||\phi||_c.$$

Figure 1 shows the response solution $x(t)$ of the neural network system (3) where $w(t) = 0$ and the initial condition $\phi(t) = [-0.1 \;\; 0.1]^T$.

Figure 2 shows the response solution $x(t)$ of the neural network system (3) where $w(t)$ is Gaussian noise with mean 0 and variance 1 and the initial condition $\phi(t) = [-0.1 \;\; 0.1]^T$.

5 Conclusions

In this paper, the problem of a H_∞ control for a neural network systems with interval and distributed time-varying delays was investigated. It is assumed that the interval and distributed time-varying delays are not necessary to be differentiable. Firstly, we considered an H_∞ control for exponential stability of neural

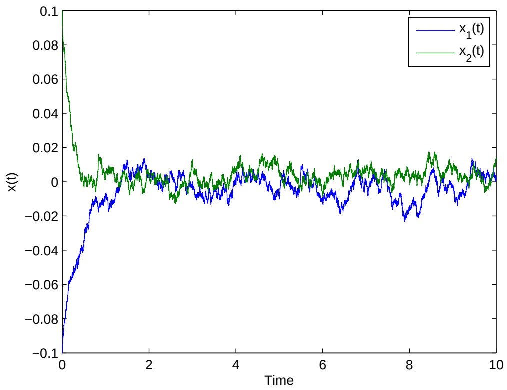

Figure 2: Response solution of the system (3)

network with interval and distributed time-varying delays via hybrid feedback control. Secondly, by using a novel LyapunovKarsovskii functional, the employment of a tighter bounding technique, some slack matrices and newly introduced convex combination condition in the calculation, improved delay-dependent suff cient conditions for the H_∞ control with exponential stability of the system are obtained. Finally, a numerical example has been given to illustrate the effectiveness of the proposed method. The results in this paper improve the corresponding results of the recent works.

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