Integrability of Differential Equations of Motion of an n-Dimensional Rigid Body in Nonconservative Fields for $n = 5$ and $n = 6$

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Abstract: In this review, we discuss new cases of integrable systems on the tangent bundles of finite-dimensiona spheres. Such systems appear in the dynamics of multidimensional rigid bodies in nonconservative fields These problems are described by systems with variable dissipation with zero mean. We found several new cases of integrability of equations of motion in terms of transcendental functions (in the sense of the classificatio of singularities) that can be expressed as finit combinations of elementary functions.

Key–Words: Dynamical system, Integrability, Transcendental firs integral

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We examine nonconservative systems that cannot be studied by ordinary methods of analysis of Hamiltonian systems. For such systems, we must directly integrate the fundamental equation of dynamics (see also [1, 2]). We propose a new, more universal presentation of complete integrable systems (both new and obtained earlier) in dimensions 5 and 6.

In the general case, it is quite difficul to construct a theory of integration of nonconservative systems (even in low-dimensional cases). However, in some cases where a system possesses certain additional symmetries, one can express firs integrals as finit combinations of elementary functions (see [3, 4, 5]).

We present general aspects of the dynamics of free, multi-dimensional rigid bodies: the notion of the tensor of angular velocity, joint dynamical equations of motion on the direct product $\mathbf{R}^n \times \mathrm{so}(n)$, the Euler and Rivals formulas in the multi-dimensional case, etc.

We discuss the tensor of inertia of f ve- and six-dimensional rigid bodies. In this activity, we consider only the cases where four of the f ve principal moments of inertia of a f ve-dimensional body coincide, i.e., $I_2 = I_3 = I_4 = I_5$ and f ve of the six principal moments of inertia of a six-dimensional body coincide, i.e., $I_2 = I_3 = I_4 = I_5 = I_6$.

The results presented in activity refer to the case where the interaction of a homogeneous fl w of a medium with a fi ed body is concentrated on a four-dimensional fla part (disk) of the surface of the f ve-dimensional body (and, respectively, on a f ve-dimensional fla part (disk) of the surface of the six-dimensional body), and the force acts perpendicularly to this disk. We systemize these results and present them in the invariant form. We also introduce an additional dependence of the moment of the nonconservative force acting in the system on the angular velocity. This dependence can be also considered in higher-dimensional cases.

1 General Preliminaries

1.1 Dynamical symmetries of five- and six-dimensional bodies

Assume that a f ve-dimensional (respectively, six-dimensional) rigid body Θ of mass m with a smooth four-dimensional (respectively, f ve-dimensional) boundary $\partial\Theta$ is under the influenc of a nonconservative force field Note that this can be treated as motion of the body in a resistive medium that fill up a f ve-dimensional (respectively, six-dimensional) domain of Eucludean space $\mathbf{E}^5$ (respectively, $\mathbf{E}^6$). Assume that the body is dynamically symmetric; in this case, there are several representations of its tensor of inertia: in the f ve-dimensional case, in some coordinate system $Dx_1x_2x_3x_4x_5$ attached to the body, the operator of inertia has either the form

$$\mathrm{diag}\{I_1, I_2, I_2, I_2, I_2\}, \qquad (1)$$

or the form $\text{diag}\{I_1, I_1, I_3, I_3, I_3\}$; respectively; in the six-dimensional case, in some coordinate system $Dx_1x_2x_3x_4x_5x_6$ attached to the body, the operator of inertia has either the form

$$\text{diag}\{I_1, I_2, I_2, I_2, I_2, I_2\}, \tag{2}$$

or the form $\text{diag}\{I_1, I_1, I_3, I_3, I_3, I_3\}$, or the form $\text{diag}\{I_1, I_1, I_1, I_3, I_3, I_3\}$. In the cases (1) and (2), in the hyperplanes $Dx_2x_3x_4x_5$ and $Dx_2x_3x_4x_5x_6$, respectively, the body is dynamically symmetric.

1.2 Dynamics on so(n) and $\mathbf{R}^n$

The configuratio space of a free n-dimensional rigid body is the direct product of the space $\mathbf{R}^n$ (which describes the coordinates of the center of mass of the body) and the rotation group SO(n) (which describes the rotation of the body about its center of mass):

$$\mathbf{R}^n \times \text{SO}(n) \tag{3}$$

and has dimension $n + n(n-1)/2 = n(n+1)/2$. Therefore, the dimension of the phase space is equal to $n(n+1)$.

In particular, if Ω is the tensor of angular velocity of a f ve-dimensional (respectively, six-dimensional) rigid body (it is a second-rank tensor; see [2, 3, 4]), $\Omega \in \text{so}(5)$ (respectively, $\Omega \in \text{so}(6)$), then the part of dynamical equations of motion corresponding to the Lie algebra so(5) (respectively, so(6)) has the following form:

$$\dot\Omega\Lambda + \Lambda\dot\Omega + [\Omega,\ \Omega\Lambda + \Lambda\Omega] = M, \tag{4}$$

where $\Lambda = \text{diag}\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ (respectively, $\Lambda = \text{diag}\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\}$), $\lambda_1 = (-I_1 + I_2 + I_3 + I_4 + I_5)/2$, $\lambda_2 = (I_1 - I_2 + I_3 + I_4 + I_5)/2$, $\lambda_3 = (I_1 + I_2 - I_3 + I_4 + I_5)/2$, $\lambda_4 = (I_1 + I_2 + I_3 - I_4 + I_5)/2$, $\lambda_5 = (I_1 + I_2 + I_3 + I_4 - I_5)/2$, or, respectively, $\lambda_1 = (-I_1 + I_2 + I_3 + I_4 + I_5 + I_6)/2$, $\lambda_2 = (I_1 - I_2 + I_3 + I_4 + I_5 + I_6)/2$, $\lambda_3 = (I_1 + I_2 - I_3 + I_4 + I_5 + I_6)/2$, $\lambda_4 = (I_1 + I_2 + I_3 - I_4 + I_5 + I_6)/2$, $\lambda_5 = (I_1 + I_2 + I_3 + I_4 - I_5 + I_6)/2$, $\lambda_6 = (I_1 + I_2 + I_3 + I_4 + I_5 - I_6)/2$, where $M = M_F$ is the projection of the moment of exterior forces $\mathbf{F}$ that act on the body in $\mathbf{R}^5$ (respectively, in $\mathbf{R}^6$) to the natural coordinates in the Lie algebra so(5) (respectively, in so(6)), and [ ] is the commutator in so(5) (respectively, in so(6)). The skew-symmetric matrix corresponding to the second-rank tensor $\Omega \in \text{so}(5)$ (respectively, $\Omega \in \text{so}(6)$) can be represented in the form

$$\begin{pmatrix} 0 & -\omega_{10} & \omega_9 & -\omega_7 & \omega_4 \\ \omega_{10} & 0 & -\omega_8 & \omega_6 & -\omega_3 \\ -\omega_9 & \omega_8 & 0 & -\omega_5 & \omega_2 \\ \omega_7 & -\omega_6 & \omega_5 & 0 & -\omega_1 \\ -\omega_4 & \omega_3 & -\omega_2 & \omega_1 & 0 \end{pmatrix}, \tag{5}$$

where $\omega_1,\ \omega_2,\ \ldots,\ \omega_{10}$ are the components of the tensor of angular velocity with respect to the coordinates of the Lie algebra so(5), or, respectively, in the form

$$\begin{pmatrix} 0 & -\omega_{15} & \omega_{14} & -\omega_{12} & \omega_9 & -\omega_5 \\ \omega_{15} & 0 & -\omega_{13} & \omega_{11} & -\omega_8 & \omega_4 \\ -\omega_{14} & \omega_{13} & 0 & -\omega_{10} & \omega_7 & -\omega_3 \\ \omega_{12} & -\omega_{11} & \omega_{10} & 0 & -\omega_6 & \omega_2 \\ -\omega_9 & \omega_8 & -\omega_7 & \omega_6 & 0 & -\omega_1 \\ \omega_5 & -\omega_4 & \omega_3 & -\omega_2 & \omega_1 & 0 \end{pmatrix}, \tag{6}$$

where $\omega_1,\ \omega_2,\ \ldots,\ \omega_{15}$ are the components of the tensor of angular velocity with respect to the coordinates of the Lie algebra so(6).

Obviously, the following equalities hold for all $i, j = 1, \ldots, 5$ (respectively, $i, j = 1, \ldots, 6$)

$$\lambda_i - \lambda_j = I_j - I_i. \tag{7}$$

For the calculation of the moment of the exterior force acting on the body, we must construct the mapping

$$\mathbf{R}^n \times \mathbf{R}^n \longrightarrow \text{so}(n), \tag{8}$$

that to each pair of vectors

$$(\mathbf{DN}, \mathbf{F}) \in \mathbf{R}^n \times \mathbf{R}^n \tag{9}$$

from $\mathbf{R}^n \times \mathbf{R}^n$

$$\mathbf{DN} = \{0, x_{2N}, \ldots, x_{nN}\},\ \mathbf{F} = \{F_1, \ldots, F_n\}, \tag{10}$$

where $\mathbf{F}$ is the exterior force acting on the body, puts in correspondence an element of the Lie algebra so(n), $n = 5, 6$, determined by the auxiliary matrix

$$\begin{pmatrix} 0 & x_{2N} & \ldots & x_{nN} \\ F_1 & F_2 & \ldots & F_n \end{pmatrix}. \tag{11}$$

Then the right-hand side of the system (4) takes the form

$$M = \{x_{4N}F_5 - x_{5N}F_4, x_{5N}F_3 - x_{3N}F_5,$$
$$x_{2N}F_5 - x_{5N}F_2, x_{5N}F_1,$$
$$x_{3N}F_4 - x_{4N}F_3, x_{4N}F_2 - x_{2N}F_4, -x_{4N}F_1,$$
$$x_{2N}F_3 - x_{3N}F_2, x_{3N}F_1, -x_{2N}F_1\}, \tag{12}$$

for $n = 5$ and the form

$$M = \{x_{5N}F_6 - x_{6N}F_5, x_{6N}F_4 - x_{4N}F_6,$$
$$x_{3N}F_6 - x_{6N}F_3, x_{6N}F_2 - x_{2N}F_6,$$
$$-x_{6N}F_1, x_{4N}F_5 - x_{5N}F_4, x_{5N}F_3 - x_{3N}F_5,$$
$$x_{2N}F_5 - x_{5N}F_2, x_{5N}F_1, x_{3N}F_4 - x_{4N}F_3,$$

$$x_{4N}F_2 - x_{2N}F_4, -x_{4N}F_1,$$
$$x_{2N}F_3 - x_{3N}F_2, x_{3N}F_1, -x_{2N}F_1\}, \qquad (13)$$

for $n = 6$.

Generally speaking, the dynamical systems considered below are nonconservative and belongs to the class of systems with variable dissipation with zero mean (see [7, 8, 9]). We must to examine a part of the fundamental equation of dynamics, namely, the Newton equation. In the case considered, this equation describes the motion of the center of mass, i.e., corresponds to the space $\mathbf{R}^n,\ n = 5, 6$:

$$m\mathbf{w}_C = \mathbf{F}, \qquad (14)$$

where $\mathbf{w}_C$ is the acceleration of the center of mass C of the body and m is its mass. Using the multidimensional Rivals formula (note that it can be obtained by using the operator method) we arrive at the following equalities:

$$\mathbf{w}_C = \mathbf{w}_D + \Omega^2\mathbf{DC} + E\mathbf{DC},\ \mathbf{w}_D = \dot{\mathbf{v}}_D + \Omega\mathbf{v}_D,\ E = \dot{\Omega}, \qquad (15)$$

where $\mathbf{w}_D$ is the acceleration of the point D, $\mathbf{F}$ is the exterior force acting on the body (in our case $\mathbf{F} = \mathbf{S}$), and E is the tensor of angular acceleration (it is a second-rank tensor).

Thus, the system of equations (4), (14) (its order is 15 for $n = 5$ and 21 for $n = 6$) on the manifold $\mathbf{R}^n \times \mathrm{so}(n)$ determines a closed system of dynamical equations of motion of a free f ve-dimensional (respectively, six-dimensional) rigid body under the action of an exterior force F. This system can be segregated from the kinematic part of the equations of motion on the manifold (3) and can be examined separately.

2 General Problem on the Motion with a Tracking Force

Consider the motion of a homogeneous, dynamically symmetric (cases (1) and (2)) rigid body with four-dimensional (respectively, f ve-dimensional) plane front end (disk) interacting with a medium that fill up the f ve-dimensional (respectively, six-dimensional) space in the fiel of a resistance force **S** under the quasi-stationary conditions (see [9, 10, 11]).

Let $(v, \alpha, \beta_1, \beta_2, \beta_3)$ (respectively, $(v, \alpha, \beta_1, \beta_2, \beta_3, \beta_4)$) be the (generalized) spherical coordinates of the velocity vector of a certain characteristic point D of the rigid body (let D be the center of the disk lying on the symmetry axis of the body), Ω be the tensor of angular velocity of the body, and $Dx_1x_2x_3x_4x_5$ (respectively, $Dx_1x_2x_3x_4x_5x_6$) be the coordinate system attached to the body such that the symmetry axis CD coincides with the axis Dx_1 (here C is the center of mass) and the axes Dx_2, Dx_3, Dx_4, Dx_5 (and $Dx_2, Dx_3, Dx_4, Dx_5, Dx_6$ in the six-dimensional case) lie in the hyperplane of the disk, and $I_1,\ I_2,\ I_3 = I_2,\ I_4 = I_2,\ I_5 = I_2$, and m (and $I_1,\ I_2,\ I_3 = I_2,\ I_4 = I_2,\ I_5 = I_2,\ I_6 = I_2$, and m in the six-dimensional case) are the principal moments of inertia and the mass of the body.

We introduce the following notation for the components with respect to coordinate system $Dx_1x_2x_3x_4x_5$ $\mathbf{E}^5$: $\mathbf{DC} = \{-\sigma, 0, 0, 0, 0\}$,

$$\mathbf{v}_D = \{v\cos\alpha, v\sin\alpha\cos\beta_1, v\sin\alpha\sin\beta_1\cos\beta_2,$$
$$v\sin\alpha\sin\beta_1\sin\beta_2\cos\beta_3, v\sin\alpha\sin\beta_1\sin\beta_2\sin\beta_3\} \qquad (16)$$

(similar relations can be written for $\mathbf{E}^6$).

In the case (1) (and (2)) $\mathbf{S} = \{-S, 0, 0, 0, 0\}$, i.e., in the case considered we have $\mathbf{F} = \mathbf{S}$.

Then the part of the dynamical equations of motion of the body corresponding to the motion of the center of mass (in the space $\mathbf{R}^5$) under the assumption that tangent forces vanish can be written in the form

$$\dot{v}\cos\alpha - \dot{\alpha}v\sin\alpha - \omega_{10}v\sin\alpha\cos\beta_1 +$$
$$+\omega_9 v\sin\alpha\sin\beta_1\cos\beta_2 -$$
$$-\omega_7 v\sin\alpha\sin\beta_1\sin\beta_2\cos\beta_3 +$$
$$+\omega_4 v\sin\alpha\sin\beta_1\sin\beta_2\sin\beta_3 +$$
$$+\sigma(\omega_{10}^2 + \omega_9^2 + \omega_7^2 + \omega_4^2) = -S/m, \qquad (17)$$
$$\dot{v}\sin\alpha\cos\beta_1 + \dot{\alpha}v\cos\alpha\cos\beta_1 - \dot{\beta}_1 v\sin\alpha\sin\beta_1 +$$
$$+\omega_{10}v\cos\alpha - \omega_8 v\sin\alpha\sin\beta_1\cos\beta_2 +$$
$$+\omega_6 v\sin\alpha\sin\beta_1\sin\beta_2\cos\beta_3 -$$
$$-\omega_3 v\sin\alpha\sin\beta_1\sin\beta_2\sin\beta_3 -$$
$$-\sigma(\omega_9\omega_8 + \omega_6\omega_7 + \omega_3\omega_4) - \sigma\dot{\omega_{10}} = 0, \qquad (18)$$
$$\dot{v}\sin\alpha\sin\beta_1\cos\beta_2 + \dot{\alpha}v\cos\alpha\sin\beta_1\cos\beta_2 +$$
$$+\dot{\beta}_1 v\sin\alpha\cos\beta_1\cos\beta_2 -$$
$$-\dot{\beta}_2 v\sin\alpha\sin\beta_1\sin\beta_2 - \omega_9 v\cos\alpha + \omega_8 v\sin\alpha\cos\beta_1 -$$
$$-\omega_5 v\sin\alpha\sin\beta_1\sin\beta_2\cos\beta_3 +$$
$$+\omega_2 v\sin\alpha\sin\beta_1\sin\beta_2\sin\beta_3 -$$
$$-\sigma(\omega_8\omega_{10} - \omega_5\omega_7 - \omega_2\omega_4) + \sigma\dot{\omega}_9 = 0, \qquad (19)$$
$$\dot{v}\sin\alpha\sin\beta_1\sin\beta_2\cos\beta_3 +$$
$$+\dot{\alpha}v\cos\alpha\sin\beta_1\sin\beta_2\cos\beta_3 +$$
$$+\dot{\beta}_1 v\sin\alpha\cos\beta_1\sin\beta_2\cos\beta_3 +$$

$$+\dot{\beta}_2 v \sin\alpha \sin\beta_1 \cos\beta_2 \cos\beta_3-$$
$$-\dot{\beta}_3 v \sin\alpha \sin\beta_1 \sin\beta_2 \sin\beta_3 + \omega_7 v \cos\alpha-$$
$$-\omega_6 v \sin\alpha \cos\beta_1+$$
$$+\omega_5 v \sin\alpha \sin\beta_1 \cos\beta_2-$$
$$-\omega_1 v \sin\alpha \sin\beta_1 \sin\beta_2 \sin\beta_3+$$
$$+\sigma(\omega_6\omega_{10} + \omega_5\omega_9 - \omega_1\omega_4) - \sigma\dot{\omega}_7 = 0, \qquad (20)$$

$$\dot{v} \sin\alpha \sin\beta_1 \sin\beta_2 \sin\beta_3+$$
$$+\dot{\alpha} v \cos\alpha \sin\beta_1 \sin\beta_2 \sin\beta_3+$$
$$+\dot{\beta}_1 v \sin\alpha \cos\beta_1 \sin\beta_2 \sin\beta_3+$$
$$+\dot{\beta}_2 v \sin\alpha \sin\beta_1 \cos\beta_2 \sin\beta_3+$$
$$+\dot{\beta}_3 v \sin\alpha \sin\beta_1 \sin\beta_2 \cos\beta_3-$$
$$-\omega_4 v \cos\alpha + \omega_3 v \sin\alpha \cos\beta_1-$$
$$-\omega_2 v \sin\alpha \sin\beta_1 \cos\beta_2+$$
$$+\omega_1 v \sin\alpha \sin\beta_1 \sin\beta_2 \cos\beta_3-$$
$$-\sigma(\omega_3\omega_{10} + \omega_2\omega_9 + \omega_1\omega_7) + \sigma\dot{\omega}_4 = 0, \qquad (21)$$

where $S = s(\alpha)v^2,\ \sigma = CD,\ v > 0$.

Similar equations can be also obtained for the six-dimensional case.

Further, the auxiliary matrix (11) for the calculation of the moment of the resistance force has the form

$$\begin{pmatrix} 0 & x_{2N} & x_{3N} & x_{4N} & x_{5N} \\ -S & 0 & 0 & 0 & 0 \end{pmatrix}. \qquad (22)$$

Then the part of dynamical equations of motion corresponding to the rotation of the body about its center of mass (in the Lie algebra so(5)) can be written in the form

$$(\lambda_4+\lambda_5)\dot{\omega}_1 + (\lambda_4-\lambda_5)(\omega_4\omega_7+\omega_3\omega_6+\omega_2\omega_5) = 0, \qquad (23)$$
$$(\lambda_3+\lambda_5)\dot{\omega}_2 + (\lambda_5-\lambda_3)(\omega_1\omega_5-\omega_3\omega_8-\omega_4\omega_9) = 0, \qquad (24)$$
$$(\lambda_2+\lambda_5)\dot{\omega}_3 + (\lambda_2-\lambda_5)(\omega_4\omega_{10}-\omega_2\omega_8-\omega_1\omega_6) = 0, \qquad (25)$$
$$(\lambda_1+\lambda_5)\dot{\omega}_4 + (\lambda_5-\lambda_1)(\omega_3\omega_{10}+\omega_2\omega_9+\omega_1\omega_7) =$$
$$= -x_{5N}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right) s(\alpha)v^2, \qquad (26)$$
$$(\lambda_3+\lambda_4)\dot{\omega}_5 + (\lambda_3-\lambda_4)(\omega_7\omega_9+\omega_6\omega_8+\omega_1\omega_2) = 0, \qquad (27)$$
$$(\lambda_2+\lambda_4)\dot{\omega}_6 + (\lambda_4-\lambda_2)(\omega_5\omega_8-\omega_7\omega_{10}-\omega_1\omega_3) = 0, \qquad (28)$$
$$(\lambda_1+\lambda_4)\dot{\omega}_7 + (\lambda_1-\lambda_4)(\omega_1\omega_4-\omega_6\omega_{10}-\omega_5\omega_9) =$$
$$= x_{4N}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right) s(\alpha)v^2, \qquad (29)$$
$$(\lambda_2+\lambda_3)\dot{\omega}_8 + (\lambda_2-\lambda_3)(\omega_9\omega_{10}+\omega_5\omega_6+\omega_2\omega_3) = 0, \qquad (30)$$
$$(\lambda_1+\lambda_3)\dot{\omega}_9 + (\lambda_3-\lambda_1)(\omega_8\omega_{10}-\omega_5\omega_7-\omega_2\omega_4) =$$
$$= -x_{3N}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right) s(\alpha)v^2, \qquad (31)$$
$$(\lambda_1+\lambda_2)\dot{\omega}_{10} + (\lambda_1-\lambda_2)(\omega_8\omega_9+\omega_6\omega_7+\omega_3\omega_4) =$$
$$= x_{2N}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right) s(\alpha)v^2. \qquad (32)$$

Similar equation can be written for $\mathbf{E}^6$.

Thus, the phase space of the system (17)–(21), (23)–(32) of order 15 is the direct product of the f ve-dimensional manifold and the Lie algebra so(5):

$$\mathbf{R}^1 \times \mathbf{S}^4 \times \mathrm{so}(5). \qquad (33)$$

Note that the system (17)–(21), (23)–(32), due to the dynamical symmetry

$$I_2 = I_3 = I_4 = I_5, \qquad (34)$$

possesses the following cyclic firs integrals:

$$\omega_1 \equiv \omega_1^0,\ \omega_2 \equiv \omega_2^0,\ \omega_3 \equiv \omega_3^0,$$
$$\omega_5 \equiv \omega_5^0,\ \omega_6 \equiv \omega_6^0,\ \omega_8 \equiv \omega_8^0. \qquad (35)$$

In the sequel, we consider the dynamics of the system on zero levels:

$$\omega_1^0 = \omega_2^0 = \omega_3^0 = \omega_5^0 = \omega_6^0 = \omega_8^0 = 0. \qquad (36)$$

In the case of a six-dimensional body, we note that, due to the dynamical symmetry

$$I_2 = I_3 = I_4 = I_5 = I_6, \qquad (37)$$

the system possesses the following cyclic firs integrals:

$$\omega_1 \equiv \omega_1^0,\ \omega_2 \equiv \omega_2^0,\ \omega_3 \equiv \omega_3^0,\ \omega_4 \equiv \omega_4^0,\ \omega_6 \equiv \omega_6^0,$$
$$\omega_7 \equiv \omega_7^0,\ \omega_8 \equiv \omega_8^0,\ \omega_{10} \equiv \omega_{10}^0, \qquad (38)$$
$$\omega_{11} \equiv \omega_{11}^0,\ \omega_{13} \equiv \omega_{13}^0.$$

In this case, we also consider the dynamics of the system on zero levels:

$$\omega_1^0 = \omega_2^0 = \omega_3^0 = \omega_4^0 = \omega_6^0 =$$
$$= \omega_7^0 = \omega_8^0 = \omega_{10}^0 = \omega_{11}^0 = \omega_{13}^0. \qquad (39)$$

If we consider a more general problem on the motion of a body under a tracking force $\mathbf{T}$ acting along the straight line $CD = Dx_1$ and providing the equality

$$v \equiv \mathrm{const}, \qquad (40)$$

then the system (17)–(21), (23)–(32) for a f ve-dimensional body (and the corresponding system for a six-dimensional body) contains the value $T - s(\alpha)v^2$, $\sigma = DC$, instead of F_1.

Choosing the value T of the tracking force appropriately, we can achieve (40). Indeed, formally expressing T from the system (17)–(21), (23)–(32) in the case where $\cos\alpha \neq 0$ we obtain

$$T = T_v(\alpha, \beta_1, \beta_2, \beta_3, \Omega) = m\sigma(\omega_4^2+\omega_7^2+\omega_9^2+\omega_{10}^2)+$$
$$+s(\alpha)v^2\left[1 - \frac{m\sigma}{3I_2}\frac{\sin\alpha}{\cos\alpha}\Gamma_v\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right)\right], \quad (41)$$

$$\Gamma_v\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right) =$$
$$= x_{5N}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right)\sin\beta_1\sin\beta_2\sin\beta_3+$$
$$+x_{4N}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right)\sin\beta_1\sin\beta_2\cos\beta_3+$$
$$+x_{3N}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right)\sin\beta_1\cos\beta_2+$$
$$+x_{2N}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right)\cos\beta_1. \quad (42)$$

To deduce Eq. (41), we have used the conditions (35)–(40).

For a six-dimensional body, Eq. (41) has the form

$$T = T_v(\alpha, \beta_1, \beta_2, \beta_3, \beta_4, \Omega) =$$
$$= m\sigma(\omega_5^2 + \omega_9^2 + \omega_{12}^2 + \omega_{14}^2 + \omega_{15}^2)+$$
$$+s(\alpha)v^2\left[1 - \frac{m\sigma}{4I_2}\frac{\sin\alpha}{\cos\alpha}\Gamma_v\left(\alpha, \beta_1, \beta_2, \beta_3, \beta_4, \frac{\Omega}{v}\right)\right]. \quad (43)$$

This procedure can be interpreted as follows. First, we have transformed the system by using the tracking force (control) that guarantees that the motion belongs to the class (40). Second, this procedure allows one to reduce the order of the system. Indeed, the system (17)–(21), (23)–(32) generates the following independent system of eighth order:

$$\dot\alpha v\cos\alpha\cos\beta_1 - \dot\beta_1 v\sin\alpha\sin\beta_1+$$
$$+\omega_{10}v\cos\alpha - \sigma\dot{\omega_{10}} = 0, \quad (44)$$

$$\dot\alpha v\cos\alpha\sin\beta_1\cos\beta_2 + \dot\beta_1 v\sin\alpha\cos\beta_1\cos\beta_2-$$
$$-\dot\beta_2 v\sin\alpha\sin\beta_1\sin\beta_2-$$
$$-\omega_9 v\cos\alpha + \sigma\dot\omega_9 = 0, \quad (45)$$

$$\dot\alpha v\cos\alpha\sin\beta_1\sin\beta_2\cos\beta_3+$$
$$+\dot\beta_1 v\sin\alpha\cos\beta_1\sin\beta_2\cos\beta_3+$$
$$+\dot\beta_2 v\sin\alpha\sin\beta_1\cos\beta_2\cos\beta_3-$$
$$-\dot\beta_3 v\sin\alpha\sin\beta_1\sin\beta_2\sin\beta_3+$$
$$+\omega_7 v\cos\alpha - \sigma\dot\omega_7 = 0, \quad (46)$$

$$\dot\alpha v\cos\alpha\sin\beta_1\sin\beta_2\sin\beta_3+$$
$$+\dot\beta_1 v\sin\alpha\cos\beta_1\sin\beta_2\sin\beta_3+$$
$$+\dot\beta_2 v\sin\alpha\sin\beta_1\cos\beta_2\sin\beta_3+$$
$$+\dot\beta_3 v\sin\alpha\sin\beta_1\sin\beta_2\cos\beta_3-$$
$$-\omega_4 v\cos\alpha + \sigma\dot\omega_4 = 0, \quad (47)$$

$$3I_2\dot\omega_4 = -x_{5N}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right)s(\alpha)v^2, \quad (48)$$

$$3I_2\dot\omega_7 = x_{4N}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right)s(\alpha)v^2, \quad (49)$$

$$3I_2\dot\omega_9 = -x_{3N}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right)s(\alpha)v^2, \quad (50)$$

$$3I_2\dot\omega_{10} = x_{2N}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right)s(\alpha)v^2, \quad (51)$$

which, in addition to the constant parameters listed above, also contains the parameter v.

The system (44)–(51) is equivalent to the following system:

$$\dot\alpha v\cos\alpha+$$
$$+v\cos\alpha\{\omega_{10}\cos\beta_1 + [(\omega_7\cos\beta_3-$$
$$-\omega_4\sin\beta_3)\sin\beta_2 - \omega_9\cos\beta_2]\sin\beta_1\}+$$
$$+\sigma\{-\dot{\omega_{10}}\cos\beta_1 + [\dot\omega_9\cos\beta_2-$$
$$-(\dot\omega_7\cos\beta_3 - \dot\omega_4\sin\beta_3)\sin\beta_2\sin\beta_1\} = 0, \quad (52)$$

$$\dot\beta_1 v\sin\alpha+$$
$$+v\cos\alpha\{[(\omega_7\cos\beta_3 - \omega_4\sin\beta_3)\sin\beta_2-$$
$$-\omega_9\cos\beta_2]\cos\beta_1 - \omega_{10}\sin\beta_1\}+$$
$$+\sigma\{[\dot\omega_9\cos\beta_2 - (\dot\omega_7\cos\beta_3-$$
$$-\dot\omega_4\sin\beta_3)\sin\beta_2]\cos\beta_1 + \dot{\omega_{10}}\sin\beta_1\} = 0, \quad (53)$$

$$\dot\beta_2 v\sin\alpha\sin\beta_1+$$
$$+v\cos\alpha\{[\omega_7\cos\beta_3-$$
$$-\omega_4\sin\beta_3]\cos\beta_2 + \omega_9\sin\beta_2\}+$$
$$+\sigma\{-[\dot\omega_7\cos\beta_3 - \dot\omega_4\sin\beta_3]\cos\beta_2-$$
$$-\dot\omega_9\sin\beta_2\} = 0, \quad (54)$$

$$\dot\beta_3 v\sin\alpha\sin\beta_1\sin\beta_2+$$
$$+v\cos\alpha\{-\omega_4\cos\beta_3 - \omega_7\sin\beta_3\}+$$
$$+\sigma\left\{\dot\omega_4\cos\beta_3 + \dot\omega_7\sin\beta_3\right\} = 0, \quad (55)$$

$$\dot\omega_4 = -\frac{v^2}{3I_2}x_{5N}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right)s(\alpha), \quad (56)$$

$$\dot{\omega}_7 = \frac{v^2}{3I_2} x_{4N}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right) s(\alpha), \quad (57)$$

$$\dot{\omega}_9 = -\frac{v^2}{3I_2} x_{3N}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right) s(\alpha), \quad (58)$$

$$\dot{\omega}_{10} = \frac{v^2}{3I_2} x_{2N}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right) s(\alpha). \quad (59)$$

Similar systems can be also obtained for a six-dimensional body.

We introduce the new quasi-velocities in the system. For this, we transform the values $\omega_4, \omega_7, \omega_9$, and ω_{10} by the compositions of three rotations as follows:

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = T_{3,4}(-\beta_1) \circ T_{2,3}(-\beta_2) \circ T_{1,2}(-\beta_3) \begin{pmatrix} \omega_4 \\ \omega_7 \\ \omega_9 \\ \omega_{10} \end{pmatrix}, \quad (60)$$

$$T_{3,4}(\beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\beta & -\sin\beta \\ 0 & 0 & \sin\beta & \cos\beta \end{pmatrix},$$

$$T_{2,3}(\beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta & 0 \\ 0 & \sin\beta & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$T_{1,2}(\beta) = \begin{pmatrix} \cos\beta & -\sin\beta & 0 & 0 \\ \sin\beta & \cos\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus, the following relations are valid:

$$\begin{array}{c} z_1 = \omega_4 \cos\beta_3 + \omega_7 \sin\beta_3, \\ z_2 = (\omega_7 \cos\beta_3 - \omega_4 \sin\beta_3)\cos\beta_2 + \\ +\omega_9 \sin\beta_2, \\ z_3 = [(-\omega_7 \cos\beta_3 + \omega_4 \sin\beta_3)\sin\beta_2 + \\ +\omega_9 \cos\beta_2]\cos\beta_1 + \omega_{10}\sin\beta_1, \\ z_4 = [(\omega_7 \cos\beta_3 - \omega_4 \sin\beta_3)\sin\beta_2 - \\ -\omega_9 \cos\beta_2]\sin\beta_1 + \omega_{10}\cos\beta_1. \end{array} \quad (61)$$

For the case of a six-dimensional body, the new quasi-velocities in the system are introduced as follows. We transform the values $\omega_5, \omega_9, \omega_{12}, \omega_{14}$, and ω_{15} by the compositions of four rotations as follows:

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{pmatrix} = T_{4,5}(-\beta_1) \circ T_{3,4}(-\beta_2) \circ \quad (62)$$

$$\circ T_{2,3}(-\beta_3) \circ T_{1,2}(-\beta_4) \begin{pmatrix} \omega_4 \\ \omega_9 \\ \omega_{12} \\ \omega_{14} \\ \omega_{15} \end{pmatrix}.$$

We see from (52)–(59) that on the manifold

$$O_1 = \Big\{(\alpha, \beta_1, \beta_2, \beta_3, \omega_4, \omega_7, \omega_9, \omega_{10}) \in \mathbf{R}^8 : \alpha = \frac{\pi}{2}k,\ \beta_1 = \pi l_1,\ \beta_2 = \pi l_2,\ k, l_1, l_2 \in \mathbf{Z}\Big\} \quad (63)$$

the system cannot be uniquely solved with respect $\dot{\alpha},\ \dot{\beta}_1,\ \dot{\beta}_2$, and $\dot{\beta}_3$. Therefore, on the manifold (63) the uniqueness theorem is formally violated. Moreover, for even k and any l_1, and l_2, the unambiguity appears due to the degeneration of the spherical coordinates $(v, \alpha, \beta_1, \beta_2, \beta_3)$, whereas for odd k the uniqueness theorem is explicitly violated due to the degeneration of the firs equation in (52).

This implies that the system (52)–(59) outside the manifold (63) is equivalent to the following system:

$$\dot{\alpha} = -z_4 + \frac{\sigma v}{3I_2}\frac{s(\alpha)}{\cos\alpha}\Gamma_v\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right), \quad (64)$$

$$\begin{array}{c} \dot{z}_4 = \dfrac{v^2}{3I_2} s(\alpha)\Gamma_v\left(\alpha, \beta_1, \beta_2, \beta_3, \dfrac{\Omega}{v}\right) - \\ -(z_1^2 + z_2^2 + z_3^2)\dfrac{\cos\alpha}{\sin\alpha} + \\ +\dfrac{\sigma v}{3I_2}\dfrac{s(\alpha)}{\sin\alpha}\{-z_3\Delta_{v,1}\left(\alpha, \beta_1, \beta_2, \beta_3, \dfrac{\Omega}{v}\right) + \\ +z_2\Delta_{v,2}\left(\alpha, \beta_1, \beta_2, \beta_3, \dfrac{\Omega}{v}\right) - \\ -z_1\Delta_{v,3}\left(\alpha, \beta_1, \beta_2, \beta_3, \dfrac{\Omega}{v}\right)\}, \end{array} \quad (65)$$

$$\begin{array}{c} \dot{z}_3 = z_3 z_4 \dfrac{\cos\alpha}{\sin\alpha} + (z_1^2 + z_2^2)\dfrac{\cos\alpha}{\sin\alpha}\dfrac{\cos\beta_1}{\sin\beta_1} + \\ +\dfrac{\sigma v}{3I_2}\dfrac{s(\alpha)}{\sin\alpha}\{z_4\Delta_{v,1}\left(\alpha, \beta_1, \beta_2, \beta_3, \dfrac{\Omega}{v}\right) - \\ -z_2\Delta_{v,2}\left(\alpha, \beta_1, \beta_2, \beta_3, \dfrac{\Omega}{v}\right)\dfrac{\cos\beta_1}{\sin\beta_1} + \end{array}$$

$$+z_1\Delta_{v,3}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right)\frac{\cos\beta_1}{\sin\beta_1}\}-$$

$$-\frac{v^2}{3I_2}s(\alpha)\Delta_{v,1}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right), \quad (66)$$

$$\dot{z_2}=z_2z_4\frac{\cos\alpha}{\sin\alpha}-z_2z_3\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1}-$$

$$-z_1^2\frac{\cos\alpha}{\sin\alpha}\frac{1}{\sin\beta_1}\frac{\cos\beta_2}{\sin\beta_2}+$$

$$+\frac{\sigma v}{3I_2}\frac{s(\alpha)}{\sin\alpha}\Delta_{v,2}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right)\times$$

$$\times\left\{-z_4+z_3\frac{\cos\beta_1}{\sin\beta_1}\right\}+$$

$$+\frac{\sigma v}{3I_2}\frac{s(\alpha)}{\sin\alpha}\Delta_{v,3}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right)\times$$

$$\times\left\{-z_1\frac{1}{\sin\beta_1}\frac{\cos\beta_2}{\sin\beta_2}\right\}+$$

$$+\frac{v^2}{3I_2}s(\alpha)\Delta_{v,2}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right), \quad (67)$$

$$\dot{z_1}=z_1z_4\frac{\cos\alpha}{\sin\alpha}-z_1z_3\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1}+$$

$$+z_1z_2\frac{\cos\alpha}{\sin\alpha}\frac{1}{\sin\beta_1}\frac{\cos\beta_2}{\sin\beta_2}+$$

$$+\frac{\sigma v}{3I_2}\frac{s(\alpha)}{\sin\alpha}\Delta_{v,3}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right)\times$$

$$\times\left\{z_4-z_3\frac{\cos\beta_1}{\sin\beta_1}+z_2\frac{1}{\sin\beta_1}\frac{\cos\beta_2}{\sin\beta_2}\right\}-$$

$$-\frac{v^2}{3I_2}s(\alpha)\Delta_{v,3}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right), \quad (68)$$

$$\dot{\beta_1}=z_3\frac{\cos\alpha}{\sin\alpha}+$$

$$+\frac{\sigma v}{3I_2}\frac{s(\alpha)}{\sin\alpha}\Delta_{v,1}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right), \quad (69)$$

$$\dot{\beta_2}=-z_2\frac{\cos\alpha}{\sin\alpha\sin\beta_1}+$$

$$+\frac{\sigma v}{3I_2}\frac{s(\alpha)}{\sin\alpha\sin\beta_1}\Delta_{v,2}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right), \quad (70)$$

$$\dot{\beta_3}=z_1\frac{\cos\alpha}{\sin\alpha\sin\beta_1\sin\beta_2}+$$

$$+\frac{\sigma v}{3I_2}\frac{s(\alpha)}{\sin\alpha\sin\beta_1\sin\beta_2}\Delta_{v,3}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right), \quad (71)$$

$$\Delta_{v,1}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right)=$$

$$=-x_{2N}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right)\sin\beta_1+$$

$$+x_{3N}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right)\cos\beta_1\cos\beta_2+$$

$$+x_{4N}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right)\cos\beta_1\sin\beta_2\cos\beta_3+$$

$$+x_{5N}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right)\cos\beta_1\sin\beta_2\sin\beta_3,$$

$$\Delta_{v,2}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right)=$$

$$=-x_{3N}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right)\sin\beta_2+ \quad (72)$$

$$+x_{4N}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right)\cos\beta_2\cos\beta_3+$$

$$+x_{5N}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right)\cos\beta_2\sin\beta_3,$$

$$\Delta_{v,3}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right)=$$

$$=-x_{4N}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right)\sin\beta_3+$$

$$+x_{5N}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right)\cos\beta_3,$$

and the function $\Gamma_v(\alpha,\beta_1,\beta_2,\beta_3,\Omega/v)$ can be represented in the form (42).

Here and in the sequel, the dependence on the groups of variables $(\alpha,\beta_1,\beta_2,\beta_3,\Omega/v)$ is considered as the composite dependence on $(\alpha,\beta_1,\beta_2,\beta_3,z_1/v,z_2/v,z_3/v,z_4/v)$ due to (61).

A similar system can be obtained for a six-dimensional rigid body.

The violation of the uniqueness theorem for the system (52)–(59) on the manifold (63) for odd k can be interpreted as follows: for odd k and for almost all points of the manifold (63), there exists a nonsingular phase trajectory of the system (52)–(59) that intersects the manifold (63) orthogonally and also there exists a phase trajectory that completely coincides with this points at all moments of time. However, these trajectories are distinct since they correspond to different values of the tracking force.

3 Case where the Moment of Nonconservative Forces Is Independent of the Angular Velocity Tensor

3.1 Reduced system

Similarly to the choice of Chaplygin analytic functions (see [12, 13]), we choose the dynamical functions s, x_{2N}, x_{3N}, x_{4N}, and x_{5N} in the following form:

$$s(\alpha) = B\cos\alpha,$$

$$x_{2N}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right) =$$

$$= x_{2N0}(\alpha, \beta_1, \beta_2, \beta_3) = A\sin\alpha\cos\beta_1,$$

$$x_{3N}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right) =$$

$$= x_{3N0}(\alpha, \beta_1, \beta_2, \beta_3) = A\sin\alpha\sin\beta_1\cos\beta_2, \quad (73)$$

$$x_{4N}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right) =$$

$$= x_{4N0}(\alpha, \beta_1, \beta_2, \beta_3) = A\sin\alpha\sin\beta_1\sin\beta_2\cos\beta_3,$$

$$x_{5N}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right) =$$

$$= x_{5N0}(\alpha, \beta_1, \beta_2, \beta_3) = A\sin\alpha\sin\beta_1\sin\beta_2\sin\beta_3,$$

where $A, B > 0$ and $v \neq 0$. This representation shows that in the system considered, the moment of nonconservative forces is independent of the angular velocity (it depends only on the angles α, β_1, β_2, and β_3).

In this case, the functions $\Gamma_v(\alpha, \beta_1, \beta_2, \beta_3, \Omega/v)$, and $\Delta_{v,s}(\alpha, \beta_1, \beta_2, \beta_3, \Omega/v)$, $s = 1, 2, 3$, in the system (64)–(71), have the following form:

$$\Gamma_v\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right) = A\sin\alpha,$$

$$\Delta_{v,s}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right) \equiv 0, \; s = 1, 2, 3. \quad (74)$$

In the six-dimensional case, the dynamical functions x_{2N}, x_{3N}, x_{4N}, x_{5N}, and x_{6N} have the form

$$x_{2N}\left(\alpha, \beta_1, \beta_2, \beta_3, \beta_4, \frac{\Omega}{v}\right) =$$

$$= x_{2N0}(\alpha, \beta_1, \beta_2, \beta_3, \beta_4) = A\sin\alpha\cos\beta_1,$$

$$x_{3N}\left(\alpha, \beta_1, \beta_2, \beta_3, \beta_4, \frac{\Omega}{v}\right) =$$

$$= x_{3N0}(\alpha, \beta_1, \beta_2, \beta_3, \beta_4) = A\sin\alpha\sin\beta_1\cos\beta_2,$$

$$x_{4N}\left(\alpha, \beta_1, \beta_2, \beta_3, \beta_4, \frac{\Omega}{v}\right) =$$

$$= x_{4N0}(\alpha, \beta_1, \beta_2, \beta_3, \beta_4) = \quad (75)$$

$$= A\sin\alpha\sin\beta_1\sin\beta_2\cos\beta_3,$$

$$x_{5N}\left(\alpha, \beta_1, \beta_2, \beta_3, \beta_4, \frac{\Omega}{v}\right) =$$

$$= x_{5N0}(\alpha, \beta_1, \beta_2, \beta_3, \beta_4) =$$

$$= A\sin\alpha\sin\beta_1\sin\beta_2\sin\beta_3, \cos\beta_4,$$

$$x_{6N}\left(\alpha, \beta_1, \beta_2, \beta_3, \beta_4, \frac{\Omega}{v}\right) =$$

$$= x_{6N0}(\alpha, \beta_1, \beta_2, \beta_3, \beta_4) =$$

$$= A\sin\alpha\sin\beta_1\sin\beta_2\sin\beta_3\sin\beta_4.$$

Then, due to the nonintegrable constraint (40), outside the manifold (63) the dynamical part of the equations of motion (the system (64)–(71)) takes the form of the analytic system

$$\alpha' = -z_4 + \frac{\sigma ABv}{3I_2}\sin\alpha, \quad (76)$$

$$z_4' = \frac{ABv^2}{3I_2}\sin\alpha\cos\alpha - (z_1^2 + z_2^2 + z_3^2)\frac{\cos\alpha}{\sin\alpha}, \quad (77)$$

$$z_3' = z_3 z_4\frac{\cos\alpha}{\sin\alpha} + (z_1^2 + z_2^2)\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1}, \quad (78)$$

$$z_2' = z_2 z_4\frac{\cos\alpha}{\sin\alpha} - z_2 z_3\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1} - z_1^2\frac{\cos\alpha}{\sin\alpha}\frac{1}{\sin\beta_1}\frac{\cos\beta_2}{\sin\beta_2}, \quad (79)$$

$$z_1' = z_1 z_4\frac{\cos\alpha}{\sin\alpha} - z_1 z_3\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1} = + z_1 z_2\frac{\cos\alpha}{\sin\alpha}\frac{1}{\sin\beta_1}\frac{\cos\beta_2}{\sin\beta_2}, \quad (80)$$

$$\beta_1' = z_3\frac{\cos\alpha}{\sin\alpha}, \quad (81)$$

$$\beta_2' = -z_2\frac{\cos\alpha}{\sin\alpha\sin\beta_1}, \quad (82)$$

$$\beta_3' = z_1\frac{\cos\alpha}{\sin\alpha\sin\beta_1\sin\beta_2}. \quad (83)$$

We introduce the dimensionless variables, parameters, and differentiation as follows:

$$z_k \mapsto n_0 v z_k, \; k = 1, 2, 3, 4, \; n_0^2 = \frac{AB}{3I_2}, \quad (84)$$

$$b = \sigma n_0, \; < \cdot >= n_0 v <'> .$$

We reduce the system (76)–(83) to the form

$$\alpha' = -z_4 + b\sin\alpha, \quad (85)$$

$$z_4' = \sin\alpha\cos\alpha - (z_1^2 + z_2^2 + z_3^2)\frac{\cos\alpha}{\sin\alpha}, \quad (86)$$

$$z_3' = z_3 z_4\frac{\cos\alpha}{\sin\alpha} + (z_1^2 + z_2^2)\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1}, \quad (87)$$

$$z_2' = z_2 z_4\frac{\cos\alpha}{\sin\alpha} - z_2 z_3\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1} - z_1^2\frac{\cos\alpha}{\sin\alpha}\frac{1}{\sin\beta_1}\frac{\cos\beta_2}{\sin\beta_2}, \quad (88)$$

$$z_1' = z_1 z_4 \frac{\cos\alpha}{\sin\alpha} - z_1 z_3 \frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1} + $$

$$+ z_1 z_2 \frac{\cos\alpha}{\sin\alpha}\frac{1}{\sin\beta_1}\frac{\cos\beta_2}{\sin\beta_2}, \quad (89)$$

$$\beta_1' = z_3 \frac{\cos\alpha}{\sin\alpha}, \quad (90)$$

$$\beta_2' = -z_2 \frac{\cos\alpha}{\sin\alpha\sin\beta_1}, \quad (91)$$

$$\beta_3' = z_1 \frac{\cos\alpha}{\sin\alpha\sin\beta_1\sin\beta_2}. \quad (92)$$

We see that the eighth-order system (85)–(92), which can be considered on the tangent bundle $T\mathbf{S}^4$ which can be considered on the tangent bundle $\mathbf{S}^4$, contains an independent seventh-order system (85)–(91) on its own seven-dimensional manifold.

In the case of a six-dimensional body, the corresponding system of dynamical equations takes the following form:

$$\alpha' = -z_5 + b\sin\alpha, \quad (93)$$

$$z_5' = \sin\alpha\cos\alpha - (z_1^2 + z_2^2 + z_3^2 + z_4^2)\frac{\cos\alpha}{\sin\alpha}, \quad (94)$$

$$z_4' = z_4 z_5 \frac{\cos\alpha}{\sin\alpha} + (z_1^2 + z_2^2 + z_3^2)\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1}, \quad (95)$$

$$z_3' = z_3 z_5 \frac{\cos\alpha}{\sin\alpha} - z_3 z_4 \frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1} - $$

$$-(z_1^2 + z_2^2)\frac{\cos\alpha}{\sin\alpha}\frac{1}{\sin\beta_1}\frac{\cos\beta_2}{\sin\beta_2}, \quad (96)$$

$$z_2' = z_2 z_5 \frac{\cos\alpha}{\sin\alpha} - z_2 z_4 \frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1} + $$

$$+ z_2 z_3 \frac{\cos\alpha}{\sin\alpha}\frac{1}{\sin\beta_1}\frac{\cos\beta_2}{\sin\beta_2} + $$

$$+ z_1^2 \frac{\cos\alpha}{\sin\alpha}\frac{1}{\sin\beta_1}\frac{1}{\sin\beta_2}\frac{\cos\beta_3}{\sin\beta_3}, \quad (97)$$

$$z_1' = z_1 z_5 \frac{\cos\alpha}{\sin\alpha} - z_1 z_4 \frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1} + $$

$$+ z_1 z_3 \frac{\cos\alpha}{\sin\alpha}\frac{1}{\sin\beta_1}\frac{\cos\beta_2}{\sin\beta_2} - $$

$$- z_1 z_2 \frac{\cos\alpha}{\sin\alpha}\frac{1}{\sin\beta_1}\frac{1}{\sin\beta_2}\frac{\cos\beta_3}{\sin\beta_3}, \quad (98)$$

$$\beta_1' = z_4 \frac{\cos\alpha}{\sin\alpha}, \quad (99)$$

$$\beta_2' = -z_3 \frac{\cos\alpha}{\sin\alpha\sin\beta_1}, \quad (100)$$

$$\beta_3' = z_2 \frac{\cos\alpha}{\sin\alpha\sin\beta_1\sin\beta_2}, \quad (101)$$

$$\beta_4' = -z_1 \frac{\cos\alpha}{\sin\alpha\sin\beta_1\sin\beta_2\sin\beta_3}. \quad (102)$$

For the complete integration of the system (85)–(92), we need, in the general case, seven independent firs integrals. However, after the substitution

$$w_4 = z_4, \ w_3 = \sqrt{z_1^2 + z_2^2 + z_3^2},$$

$$w_2 = \frac{z_2}{z_1}, \ w_1 = \frac{z_3}{\sqrt{z_1^2 + z_2^2}}, \quad (103)$$

the system (85)–(92) splits as follows:

$$\alpha' = -w_4 + b\sin\alpha, \quad (104)$$

$$w_4' = \sin\alpha\cos\alpha - w_3^2\frac{\cos\alpha}{\sin\alpha}, \quad (105)$$

$$w_3' = w_3 w_4 \frac{\cos\alpha}{\sin\alpha}, \quad (106)$$

$$\begin{array}{c} w_2' = d_2(w_4, w_3, w_2, w_1; \alpha, \beta_1, \beta_2, \beta_3)\times \\ \times \frac{1+w_2^2}{w_2}\frac{\cos\beta_2}{\sin\beta_2}, \\ \beta_2' = d_2(w_4, w_3, w_2, w_1; \alpha, \beta_1, \beta_2, \beta_3), \end{array} \quad (107)$$

$$\begin{array}{c} w_1' = d_1(w_4, w_3, w_2, w_1; \alpha, \beta_1, \beta_2, \beta_3)\times \\ \times \frac{1+w_1^2}{w_1}\frac{\cos\beta_1}{\sin\beta_1}, \\ \beta_1' = d_1(w_4, w_3, w_2, w_1; \alpha, \beta_1, \beta_2, \beta_3), \end{array} \quad (108)$$

$$\beta_3' = d_3(w_4, w_3, w_2, w_1; \alpha, \beta_1, \beta_2, \beta_3), \quad (109)$$

$$\begin{array}{c} d_1(w_4, w_3, w_2, w_1; \alpha, \beta_1, \beta_2, \beta_3) = \\ = Z_3(w_4, w_3, w_2, w_1)\frac{\cos\alpha}{\sin\alpha}, \\ d_2(w_4, w_3, w_2, w_1; \alpha, \beta_1, \beta_2, \beta_3) = \\ = -Z_2(w_4, w_3, w_2, w_1)\frac{\cos\alpha}{\sin\alpha\sin\beta_1}, \\ d_3(w_4, w_3, w_2, w_1; \alpha, \beta_1, \beta_2, \beta_3) = \\ = Z_1(w_4, w_3, w_2, w_1)\frac{\cos\alpha}{\sin\alpha\sin\beta_1\sin\beta_2}. \end{array} \quad (110)$$

Moreover,

$$z_k = Z_k(w_4, w_3, w_2, w_1), \ k = 1, 2, 3, \quad (111)$$

due to the substitution (103).

We see that the eighth-order system splits into independent subsystems of lower orders: the system (104)–(106) of third order and the systems (107) and (108) of second order (of course, after a change of the independent variable). Thus, for the complete integrability of the system (104)–(109) we need two independent firs integrals of the system (104)–(106), one firs integral of each of the systems (107) and (108), and an additional firs integral that "attaches" Eq. (109).

Note that the system (104)–(106) can be considered on the tangent bundle $T\mathbf{S}^2$ of the twodimensional sphere $\mathbf{S}^2$.

In the case of a six-dimensional rigid body, the corresponding change of variables has the form

$$\begin{gathered} w_5 = z_5, \\ w_4 = \sqrt{z_1^2 + z_2^2 + z_3^2 + z_4^2},\ w_3 = \frac{z_2}{z_1}, \\ w_2 = \frac{z_3}{\sqrt{z_1^2+z_2^2}},\ w_1 = \frac{z_4}{\sqrt{z_1^2+z_2^2+z_3^2}}. \end{gathered} \quad (112)$$

The system (93)–(102) splits as follows:

$$\alpha' = -w_5 + b\sin\alpha, \quad (113)$$

$$w_5' = \sin\alpha\cos\alpha - w_4^2\frac{\cos\alpha}{\sin\alpha}, \quad (114)$$

$$w_4' = w_4 w_5\frac{\cos\alpha}{\sin\alpha}, \quad (115)$$

$$\begin{gathered} w_3' = \\ = d_3(w_5, w_4, w_3, w_2, w_1; \alpha, \beta_1, \beta_2, \beta_3, \beta_4)\times \\ \times\frac{1+w_3^2}{w_3}\frac{\cos\beta_3}{\sin\beta_3}, \\ \beta_3' = d_3(w_5, w_4, w_3, w_2, w_1; \alpha, \beta_1, \beta_2, \beta_3, \beta_4), \end{gathered} \quad (116)$$

$$\begin{gathered} w_2' = \\ = d_2(w_5, w_4, w_3, w_2, w_1; \alpha, \beta_1, \beta_2, \beta_3, \beta_4)\times \\ \times\frac{1+w_2^2}{w_2}\frac{\cos\beta_2}{\sin\beta_2}, \\ \beta_2' = d_2(w_5, w_4, w_3, w_2, w_1; \alpha, \beta_1, \beta_2, \beta_3, \beta_4), \end{gathered} \quad (117)$$

$$\begin{gathered} w_1' = \\ = d_1(w_5, w_4, w_3, w_2, w_1; \alpha, \beta_1, \beta_2, \beta_3, \beta_4)\times \\ \times\frac{1+w_1^2}{w_1}\frac{\cos\beta_1}{\sin\beta_1}, \\ \beta_1' = d_1(w_5, w_4, w_3, w_2, w_1; \alpha, \beta_1, \beta_2, \beta_3, \beta_4), \end{gathered} \quad (118)$$

$$\beta_4' = d_4(w_5, w_4, w_3, w_2, w_1; \alpha, \beta_1, \beta_2, \beta_3, \beta_4), \quad (119)$$

$$\begin{gathered} d_1(w_5, w_4, w_3, w_2, w_1; \alpha, \beta_1, \beta_2, \beta_3, \beta_4) = \\ = Z_4(w_5, w_4, w_3, w_2, w_1)\frac{\cos\alpha}{\sin\alpha}, \\ d_2(w_5, w_4, w_3, w_2, w_1; \alpha, \beta_1, \beta_2, \beta_3, \beta_4) = \\ = -Z_3(w_5, w_4, w_3, w_2, w_1)\frac{\cos\alpha}{\sin\alpha\sin\beta_1}, \\ d_3(w_5, w_4, w_3, w_2, w_1; \alpha, \beta_1, \beta_2, \beta_3, \beta_4) = \\ = Z_2(w_5, w_4, w_3, w_2, w_1)\frac{\cos\alpha}{\sin\alpha\sin\beta_1\sin\beta_2}, \\ d_4(w_5, w_4, w_3, w_2, w_1; \alpha, \beta_1, \beta_2, \beta_3, \beta_4) = \\ = -Z_1(w_5, w_4, w_3, w_2, w_1)\frac{\cos\alpha}{\sin\alpha\sin\beta_1\sin\beta_2\sin\beta_3}, \end{gathered} \quad (120)$$

and

$$z_k = Z_k(w_5, w_4, w_3, w_2, w_1),\ k = 1, 2, 3, 4, \quad (121)$$

due to the substitution (112).

3.2 Complete list of invariant relations

In this section, we present results for a f ve-dimensional rigid body; for six-dimensional bodies, results are similar.

The system (104)–(106) is similar to the system of equations of dynamics of a three-dimensional rigid body in a nonconservative force field

First, to the third-order system (104)–(106), we put in correspondence the following nonautonomous second-order system:

$$\begin{gathered} \frac{dw_4}{d\alpha} = \frac{\sin\alpha\cos\alpha - w_3^2\cos\alpha/\sin\alpha}{-w_4+b\sin\alpha}, \\ \frac{dw_3}{d\alpha} = \frac{w_3 w_4\cos\alpha/\sin\alpha}{-w_4+b\sin\alpha}. \end{gathered} \quad (122)$$

Using the substitution $\tau = \sin\alpha$, we rewrite the system (122) in the algebraic form

$$\begin{gathered} \frac{dw_4}{d\tau} = \frac{\tau - w_3^2/\tau}{-w_4+b\tau}, \\ \frac{dw_3}{d\tau} = \frac{w_3 w_4/\tau}{-w_4+b\tau}. \end{gathered} \quad (123)$$

Further, introducing the homogeneous variables by the fornulas

$$w_3 = u_1\tau,\ w_4 = u_2\tau, \quad (124)$$

we reduce the system (123) to the following form:

$$\tau\frac{du_2}{d\tau} + u_2 = \frac{1-u_1^2}{-u_2+b},\ \tau\frac{du_1}{d\tau} + u_1 = \frac{u_1 u_2}{-u_2+b}, \quad (125)$$

which is equivalent to the following:

$$\tau\frac{du_2}{d\tau} = \frac{1-u_1^2+u_2^2-bu_2}{-u_2+b},\ \tau\frac{du_1}{d\tau} = \frac{2u_1u_2-bu_1}{-u_2+b}. \quad (126)$$

To the second-order system (126), we put in correspondence the nonautonomous first-orde equation

$$\frac{du_2}{du_1} = \frac{1 - u_1^2 + u_2^2 - bu_2}{2u_1u_2 - bu_1}, \quad (127)$$

which can be easily reduced to the complete differential form:

$$d\left(\frac{u_2^2 + u_1^2 - bu_2 + 1}{u_1}\right) = 0. \quad (128)$$

Thus, Eq. (127) has the following firs integral:

$$\frac{u_2^2 + u_1^2 - bu_2 + 1}{u_1} = C_1 = \text{const}, \quad (129)$$

which in the old variables has the form

$$\frac{w_4^2 + w_3^2 - bw_4\sin\alpha + \sin^2\alpha}{w_3\sin\alpha} = C_1 = \text{const}. \quad (130)$$

Remark 1 *Consider the system (104)–(106) with variable dissipation with zero mean (see [14, 15, 16]), which becomes conservative for* $b = 0$*:*

$$\begin{gathered} \alpha' = -w_4,\ w_4' = \sin\alpha\cos\alpha - w_3^2\frac{\cos\alpha}{\sin\alpha}, \\ w_3' = w_3 w_4\frac{\cos\alpha}{\sin\alpha}. \end{gathered} \quad (131)$$

It possesses the following two analytic first integrals:

$$w_4^2 + w_3^2 + \sin^2\alpha = C_1^* = const, \qquad (132)$$

$$w_3 \sin\alpha = C_2^* = const. \qquad (133)$$

Obviously, the ratio of two first integrals (132) and (133) is also a first integral of the system (131). However, for $b \neq 0$, the functions

$$w_4^2 + w_3^2 - bw_4 \sin\alpha + \sin^2\alpha \qquad (134)$$

and (133) are not first integrals of the system (104)–(106), but their ratio is a first integral of the system (104)–(106) for any b.

Further, we fin an explicit form of the additional firs integral of the third-order system (104)–(106). We transform the invariant relation (129) for $u_1 \neq 0$ as follows:

$$\left(u_2 - \frac{b}{2}\right)^2 + \left(u_1 - \frac{C_1}{2}\right)^2 = \frac{b^2 + C_1^2}{4} - 1. \qquad (135)$$

We see that the parameters of this invariant relations must satisfy the condition

$$b^2 + C_1^2 - 4 \geq 0, \qquad (136)$$

and the phase space of the system (104)–(106) splits into the family of surfaces determined by Eq. (135).

Thus, due to the relation (129), the firs equation of the system (126) has the form

$$\tau \frac{du_2}{d\tau} = \frac{2(1 - bu_2 + u_2^2) - C_1 U_1(C_1, u_2)}{-u_2 + b}, \qquad (137)$$

$$U_1(C_1, u_2) = \frac{1}{2}\{C_1 \pm \sqrt{C_1^2 - 4(u_2^2 - bu_2 + 1)}\}; \qquad (138)$$

here the integration constant C_1 is define by the condition (136).

Therefore, the quadrature that determines the additional firs integral of the system (104)–(106) has the form

$$\int \frac{d\tau}{\tau} = \int \frac{(b - u_2)du_2}{2(1 - bu_2 + u_2^2) - C_1\{W\}/2}, \qquad (139)$$

$$W = C_1 \pm \sqrt{C_1^2 - 4(u_2^2 - bu_2 + 1)}.$$

Obviously, the left-hand side (up to an additive constant) is equal to $\ln|\sin\alpha|$. If

$$u_2 - \frac{b}{2} = r_1, \; b_1^2 = b^2 + C_1^2 - 4, \qquad (140)$$

then the right-hand side of Eq. (139) becomes

$$-\frac{1}{4}\int \frac{d(b_1^2 - 4r_1^2)}{(b_1^2 - 4r_1^2) \pm C_1\sqrt{b_1^2 - 4r_1^2}} -$$

$$-b\int \frac{dr_1}{(b_1^2 - 4r_1^2) \pm C_1\sqrt{b_1^2 - 4r_1^2}} =$$

$$= -\frac{1}{2}\ln\left|\frac{\sqrt{b_1^2 - 4r_1^2}}{C_1} \pm 1\right| \pm \frac{b}{2} I_1, \qquad (141)$$

$$I_1 = \int \frac{dr_3}{\sqrt{b_1^2 - r_3^2}(r_3 \pm C_1)}, \; r_3 = \sqrt{b_1^2 - 4r_1^2}. \qquad (142)$$

In the calculation of the integral (142), the following three cases are possible.

I. $b > 2$.

$$I_1 = -\frac{1}{2\sqrt{b^2 - 4}} \times$$

$$\times \ln\left|\frac{\sqrt{b^2 - 4} + \sqrt{b_1^2 - r_3^2}}{r_3 \pm C_1} \pm \frac{C_1}{\sqrt{b^2 - 4}}\right| +$$

$$+ \frac{1}{2\sqrt{b^2 - 4}} \times$$

$$\times \ln\left|\frac{\sqrt{b^2 - 4} - \sqrt{b_1^2 - r_3^2}}{r_3 \pm C_1} \mp \frac{C_1}{\sqrt{b^2 - 4}}\right| + \qquad (143)$$

$$+ \text{const}.$$

II. $b < 2$.

$$I_1 = \frac{1}{\sqrt{4 - b^2}} \arcsin \frac{\pm C_1 r_3 + b_1^2}{b_1(r_3 \pm C_1)} + \text{const}. \qquad (144)$$

III. $b = 2$.

$$I_1 = \mp \frac{\sqrt{b_1^2 - r_3^2}}{C_1(r_3 \pm C_1)} + \text{const}. \qquad (145)$$

Returning to the variable

$$r_1 = \frac{w_4}{\sin\alpha} - \frac{b}{2}, \qquad (146)$$

we obtain the fina formulas for I_1:

I. $b > 2$.

$$I_1 = -\frac{1}{2\sqrt{b^2 - 4}} \ln\left|\frac{\sqrt{b^2 - 4} \pm 2r_1}{\sqrt{b_1^2 - 4r_1^2} \pm C_1} \pm \frac{C_1}{\sqrt{b^2 - 4}}\right| +$$

$$+\frac{1}{2\sqrt{b^2-4}}\ln\left|\frac{\sqrt{b^2-4}\mp 2r_1}{\sqrt{b_1^2-4r_1^2}\pm C_1}\mp\frac{C_1}{\sqrt{b^2-4}}\right|+ \tag{147}$$

$$+\text{const.}$$

II. $b<2$.

$$I_1=\frac{1}{\sqrt{4-b^2}}\arcsin\frac{\pm C_1\sqrt{b_1^2-4r_1^2}+b_1^2}{b_1(\sqrt{b_1^2-4r_1^2}\pm C_1)}+\text{const.} \tag{148}$$

III. $b=2$.

$$I_1=\mp\frac{2r_1}{C_1(\sqrt{b_1^2-4r_1^2}\pm C_1)}+\text{const.} \tag{149}$$

Thus, we have found the additional firs integral for the third-order system (104)–(106) and hence we have a complete list of firs integrals, which are transcendental functions of their phase variables.

Remark 2 *In the expression of the first integral we must formally substitute the left-hand side of the first integral (129) instead of* C_1.

Then the additional firs integral takes the following structure (which is similar to the transcendental firs integral in the fla dynamics):

$$\ln|\sin\alpha|+G_2\left(\sin\alpha,\frac{w_4}{\sin\alpha},\frac{w_3}{\sin\alpha}\right)=C_2=\text{const.} \tag{150}$$

Thus, for the eighth-order system (104)–(109), we have found two independent firs integrals. As was said above, for its complete integrability, we need one firs integral for the separated systems (107) and (108) and an additional firs integral that "attaches" Eq. (109).

To fin a firs integral of separated systems (107) and (108), we consider the following first-orde nonautonomous equations:

$$\frac{dw_s}{d\beta_s}=\frac{1+w_s^2}{w_s}\frac{\cos\beta_s}{\sin\beta_s},\ s=1,2. \tag{151}$$

The last equalities lead to the required invariant relations

$$\frac{\sqrt{1+w_s^2}}{\sin\beta_s}=C_{s+2}=\text{const},\ s=1,2. \tag{152}$$

Further, to fin the additional firs integral that "attaches" Eq. (109), we put in correspondence to Eqs. (109) and (107) the nonautonomous equation

$$\frac{dw_2}{d\beta_3}=-(1+w_2^2)\cos\beta_2. \tag{153}$$

Since, due to (152),

$$C_4\cos\beta_2=\pm\sqrt{C_4^2-1-w_2^2}, \tag{154}$$

we have

$$\frac{dw_2}{d\beta_3}=\mp\frac{1}{C_4}(1+w_2^2)\sqrt{C_4^2-1-w_2^2}. \tag{155}$$

Then, integrating the last equality, we arrive at the following quadrature:

$$\mp(\beta_3+C_5)=\int\frac{C_4dw_2}{(1+w_2^2)\sqrt{C_4^2-1-w_2^2}}, \tag{156}$$

$$C_5=\text{const.}$$

Integrating this, we obtain the equality

$$\mp\text{tg}(\beta_3+C_5)=\frac{C_4w_2}{\sqrt{C_4^2-1-w_2^2}}, \tag{157}$$

$$C_5=\text{const.}$$

Finally, we obtain the additional firs integrals that "attaches" Eq. (109):

$$\text{arctg}\frac{C_4w_2}{\sqrt{C_4^2-1-w_2^2}}\pm\beta_3=C_5,\ C_5=\text{const.} \tag{158}$$

Thus, in the case considered, the system of dynamical equations (17)–(21), (23)–(32) under the condition (73) has 12 invariant relation: the analytic nonintegrable constraint (40), the cyclic firs integrals (35) and (36), the firs integral (130), the firs integral expressed by the relations (143)–(150), which is a transcendental function of the phase variables (it is expressed as a finit combination of elementary functions), and the transcendental firs integrals (152) and (158).

Theorem 3 *The system (17)–(21), (23)–(32) under the conditions (40), (73), (36) possesses 12 invariant relations (a complete set). Five of these relations are transcendental functions (from the point of view of complex analysis). All these relations are expressed as finite combinations of elementary functions.*

A similar theorem is also valid for the six-dimensional case.

4 Conclusion

Activity contains a review of results on the integrability of equations of motions in the dynamics of f ve- and six-dimensional rigid bodies in nonconservative force fields Such problems are governed by dynamical systems with variable dissipation with zero mean. Moreover, such systems often possess a complete list of firs integrals expressed through elementary functions.

We also presented a method of reduction of systems with right-hand sides containing polynomial of trigonometric functions to systems with polynomial right-hand sides, which allows one to fin firs integrals (or prove their absence) for systems of a more general form, not only those having specifie symmetries (see also [9, 10, 12]).

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