

Absolute and uniform convergence of spectral expansion in the eigenfunctions of a third-order ordinary differential operator

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Abstract: - In this paper studied the convergence of spectral expansions of functions of the class $W_p^1(G)$, $G = (0, 1)$ in eigenfunctions of an ordinary differential operator of third order with integral coefficients. Sufficient conditions for absolute and uniform convergence are obtained and the rate of uniform convergence of these expansions on the interval \overline{G} is found.

Key-Words: - absolute and uniform convergence, eigenfunction, spectral expansion

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1 Introduction

It is well known that any function in the domain of a self-adjoint differential operator can be expanded in a uniformly convergent series in the eigenfunctions of this operator [1, p.90]. For functions that do not belong to the domain of self-adjoint Sturm-Liouville operator, the problems of absolute and uniform convergence have been studied in [2-5]. In [2,3] the Sturm-Liouville operator $Lu = -u'' + q(x)u$, $x \in G = (0, 1)$, with two point self-adjoint boundary conditions (the coefficients in the boundary conditions are real) was considered, and under the condition ρ , the absolute and uniform convergence on the interval \overline{G} of the expansions of functions $f(x) \in W_p^1(G)$, $1 < p \in \mathbb{Z}$, $f(0) = f(1) = 0$, in orthonormal eigenfunctions of this operator was proved. The operator L with a real potential $q(x) \in L_1(G)$ independent of the specific boundary conditions (in particular, self-adjoint boundary conditions with complex coefficients are also allowed) was considered in [4,5]. The results obtained in [2-5] (for the case of the Sturm-Liouville operator) were generalized in [6] (for the one-dimensional Schrödinger operator).

In the present paper, we study the problems of absolute and uniform convergence of expansions of functions of the class $W_p^1(G)$ in the eigenfunctions of a third-order differential operator [7]. Sufficient conditions for the absolute and uniform convergence of these expansions are obtained, and the rate of

uniform convergence is estimated. This study are based on Ilin's spectral method [8].

On the interval $G = (0, 1)$, consider the differential operator $L_u = u^{(3)} + P_2(x)u^{(1)} + P_3(x)u$

with coefficients $P_l(x) \in L_1(G)$, $l = 2, 3$

By $D(G)$ we denote the class of functions absolutely continuous together with their derivatives of order ≤ 2 on the interval $\overline{G} = [0, 1]$.

An eigenfunction of operator L corresponding to an eigenvalue λ is understood as a function $u(x) \in D(G)$ that is not identically zero and satisfies the equation $L_u + \lambda u = 0$ almost everywhere in G [8].

Let $\{u_k(x)\}_{k=1}^{\infty}$ be a complete orthonormal system in $L_2(G)$ consisting of eigenfunctions of the operator L , and let $\{\lambda_k\}_{k=1}^{\infty}$ be the corresponding system of eigenvalues, $\text{Re } \lambda_k = 0$. By $W_p^1(G)$, $p \geq 1$, we denote the class of functions $f(x)$ absolutely continuous on the interval \overline{G} for which $f'(x) \in L_p(G)$. We write $m_k(-i/\lambda_k)^{1/3}$, $\text{Im } \lambda_k \geq 0$; $m_k = (i/\lambda_k)^{1/3}$, if $\text{Im } \lambda_k < 0$, and introduce a partial sum of the spectral expansion of

the function $f(x) \hat{=} W_1^A(G)$ with respect to the system

$$\{u_k(x)\}_{k=1}^\infty : \delta_\nu(x, f) = \sum_{\mu_k \leq \nu} f_k u_k(x), \nu > 0,$$

where $f_k = (f, u_k) = \int_0^1 f(x) \overline{u_k(x)} dx$. Study the behavior of the difference $R_n(x, f) = f(x) - d_n(x, f)$.

In this paper, we prove the following results.

Theorem. Assume that a function $f(x) \in W_1^1(G)$ and a system $\{u_k(x)\}_{k=1}^\infty$ satisfy the conditions

$$\left| f(x) \overline{u_k^{(2)}(x)} \right|_0 \in C(f) m_k^2 \|u_k\|_\infty, 0 \in a < 2, m_k \geq 1 \quad (1)$$

and

$$\sum_{k=2}^\infty k^{-1} w_1(f', k^{-1}) < \infty. \quad (2)$$

Then the spectral expansion of the function $f(x)$ with respect to the system $\{u_k(x)\}_{k=1}^\infty$ converges absolutely and uniformly on the interval $\overline{G} = [0, 1]$, and the estimate

$$\begin{aligned} & \|R_\nu(\cdot, f)\|_{C[0,1]} \leq \\ & \leq \text{const} \left\{ C(f) \nu^{\alpha-2} + \sum_{k=[\nu]}^\infty k^{-1} w_1(f', k^{-1}) + w_1(f', \nu^{-1}) + \right. \\ & \left. + \nu^{-1} \left(\|f'\|_1 + (\|f\|_\infty + \|f'\|_1) \sum_{l=2}^3 \nu^{2-l} \|p_l\|_1 \right) \right\} \end{aligned} \quad (3)$$

holds, where $n \geq n_0 = 8\rho$, $W(x, d)$ is the modulus of continuity on the space $L_1(G)$, $\|\cdot\|_p = \|\cdot\|_{L_p(G)}$, and the constant is independent of the function $f(x)$.

Corollary 1. If the function $f(x) \hat{=} W_1^A(G)$ in theorem satisfies the conditions $f(0) = f(1) = 0$, then condition (1) is necessarily satisfied (with the constant $C(f) = 0$), its spectral expansion with

respect to the system $\{u_k(x)\}_{k=1}^\infty$ convergence absolutely and uniformly on the interval $\overline{G} = [0, 1]$, and the following estimate holds:

$$\begin{aligned} \|R_\nu(\cdot, f)\|_{C[0,1]} & \leq \text{const} \{ t w_1(f, \nu^{-1}) + \sum_{k=[\nu]}^\infty k^{-1} w_1(f', k^{-1}) \\ & + \nu^{-1} \|f'\|_1 \left(1 + \sum_{l=2}^3 \nu^{2-l} \|p_l\|_1 \right) \} \end{aligned}$$

Corollary 2. If the function $f(x) \hat{=} W_1^A(G)$ in Theorem satisfies the relations $f(0) = f(1) = 0$ and $f'(x) \in H_1^\beta(G)$, $0 < \beta \in [1, 2)$ ($H_1^\beta(G)$ is the Nikolski class), then conditions (1) and (2) are necessarily satisfied, its spectral expansion converges absolutely and uniformly on the interval $\overline{G} = [0, 1]$, and the following estimate holds:

$$\|R_\nu(\cdot, f)\|_{C[0,1]} \leq \text{const} \nu^{-\beta} \|f'\|_1^\beta, \nu \geq \nu_0$$

where $\|f\|_1^\beta = \|f\|_1 + \sup_{d>0} d^{-\beta} w_1(f, d)$ and the constant is independent of the function $f(x)$.

Corollary 3. If the function $f(x) \hat{=} W_1^A(G)$ in Theorem satisfies the relations $f(0) = f(1) = 0$ and $w_1(f', \delta) = O(\ln^{-(1+\gamma)} \delta^{-1})$ as $\delta \rightarrow +0$ for $\gamma > 0$ then conditions (1) and (2) are necessarily satisfied, its spectral expansion converges absolutely and uniformly on $\overline{G} = [0, 1]$, and the following estimate holds: $\|R_n(\cdot, f)\|_{C[0,1]} = O(\ln^{-\gamma} n)$ as $n \rightarrow +\infty$.

2 Some auxiliary lemmas

To prove the Theorem, we first prove several assertions. The following lemma was proved in [9, Th.3].

Lemma 1. The following estimates hold for the system $\{u_k(x)\}_{k=1}^\infty$ and the sequence $\{m_k\}_{k=1}^\infty$ for any $t \geq 0$:

$$\sum_{0 \leq \mu_k \leq \tau} \|u_k\|_\infty^2 \leq \text{const} (1 + \tau) \text{ and } \sum_{\tau \leq \mu_k \leq \tau+1} 1 \leq \text{const}. \quad (4)$$

Lemma2. (see [7]) The following estimate hold for the system $\{u_k(x)\}_{k=1}^m$ for any $m \geq 2$:

$$\sum_{\mu_k \geq \mu} \mu_k^{-(1+\theta)} \|u_k\|_{\infty}^2 \leq C_1(\theta), \theta > 0, \quad (5)$$

where $C_1(\theta)$ is a positive constant.

Lemma 3. Assume that a function $f(x) \in W_1^l(G)$ satisfies condition (2). Then the estimate

$$\begin{aligned} & \sum_{\mu_k \geq \rho} \mu_k^{-1} w_1(f', \mu_k^{-1}) \|u_k\|_{\infty}^2 \leq \\ & \leq C_2 \left\{ w_1(f', \rho^{-1}) + \sum_{k=[\rho]}^{\infty} k^{-1} w_1(f', \rho^{-1}) \right\} \end{aligned} \quad (6)$$

holds, where $C_2 \geq 8\rho$ and C_2 is a positive constant independent of ρ and the function $f(x)$.

Proof. Take a positive integer m . By the estimates (4), using the Abel transformation, we obtain the chain of inequalities

$$\begin{aligned} & \sum_{\rho \leq \mu_k \leq [\rho]+m} \mu_k^{-1} w_1(f', \mu_k^{-1}) \|u_k\|_{\infty}^2 \leq \\ & \sum_{n=[\rho]}^{[\rho]+m} n^{-1} w_1(f', n^{-1}) \left(\sum_{n \leq \mu_k \leq n+1} \|u_k\|_{\infty}^2 \right) \leq \\ & \sum_{n=[\rho]}^{[\rho]+m-1} \left(\sum_{1 \leq \mu_k \leq n+1} \|u_k\|_{\infty}^2 \right) \left\{ n^{-1} w_1(f', n^{-1}) - \right. \\ & \quad \left. - (n+1)^{-1} w_1(f', (n+1)^{-1}) \right\} + \\ & \left(\sum_{1 \leq \mu_k < [\rho]+m-1} \|u_k\|_{\infty}^2 \right) ([\rho]+m)^{-1} w_1(f', ([\rho]+m)^{-1}) + \\ & \quad + \left(\sum_{1 \leq \mu_k < [\rho]} \|u_k\|_{\infty}^2 \right) [\rho]^{-1} w_1(f', [\rho]^{-1}) \leq \\ & \leq \text{const} \sum_{n=[\rho]}^{[\rho]+m-1} (n+1) \left\{ n^{-1} w_1(f', n^{-1}) - (n+1)^{-1} w_1(f', (n+1)^{-1}) \right\} + \\ & + \text{const} ([\rho]+m) ([\rho]+m)^{-1} w_1(f', ([\rho]+m)^{-1}) + \\ & \quad + \text{const} [\rho] [\rho]^{-1} w_1(f', [\rho]^{-1}) \leq \\ & \leq \text{const} \left\{ \sum_{n=[\rho]}^{[\rho]+m-1} n^{-1} w_1(f', n^{-1}) + w_1(f', [\rho]^{-1}) - w_1(f', ([\rho]+m)^{-1}) \right\} + \end{aligned}$$

$$\begin{aligned} & + \text{const} w_1(f', ([\rho]+m)^{-1}) + \text{const} w_1(f', [\rho]^{-1}) \leq \\ & \leq \text{const} \left\{ \sum_{n=[\rho]}^{[\rho]+m-1} n^{-1} w_1(f', n^{-1}) + w_1(f', [\rho]^{-1}) + w_1(f', ([\rho]+m)^{-1}) \right\}. \end{aligned}$$

sense the number m is arbitrary, this together with inequality (2), implies the estimate (6).

Lemma 4. The following estimates hold for the Fourier coefficients f_k of a function

$$\begin{aligned} & f(x) \in W_1^l(G) \text{ satisfying condition (1): } (m_k \geq n_0) \\ & |f_k| \leq \text{const} \{ C(f) \mu_k^{\alpha-3} + \mu_k^{-1} w_1(f', \mu_k^{-1}) + \\ & + \mu_k^{-2} \left(\|f'\|_1 + (\|f\|_{\infty} + \|f'\|_1) \sum_{l=2}^3 \mu_k^{2-l} \|P_l\|_1 \right) \} \|u_k\|_{\infty}. \end{aligned} \quad (7)$$

The Lemma 4 was proved in [10, see Lemma 2], in the case $\|u_k\|_{\infty} \in \text{const}$, $k=1, 2, \dots$. In general case this lemma is proved with similarly way.

3 Proof of the results

Proof of theorem. Under the assumptions of theorem, we prove the uniform convergence of the series

$\sum_{k=1}^{\infty} |f_k| |u_k(x)|$ on the interval \bar{G} . To this end, we represent this series as

$$\sum_{k=1}^{\infty} |f_k| |u_k(x)| = \sum_{0 \leq \mu_k < \nu_0} |f_k| |u_k(x)| + \sum_{\mu_k \geq \nu_0} |f_k| |u_k(x)| \quad (8)$$

To estimate the first sum on the right-hand side in (8), we apply the first estimate in (4),

$$\sum_{0 \leq \mu_k < \nu_0} |f_k| |u_k(x)| \leq \text{const} \|f\|_1 \sum_{0 \leq \mu_k \leq \nu_0} \|u_k\|_{\infty}^2 \leq \text{const} \|f\|_1$$

To estimate the second sum in (8), we use Lemma 4 and the estimates (5) and (6),

$$\begin{aligned} & \sum_{\mu_k \geq \nu_0} |f_k| |u_k(x)| \leq \text{const} \left\{ C(f) \sum_{\mu_k \geq \nu_0} \mu_k^{\alpha-3} \|u_k\|_\infty^2 + \right. \\ & + \sum_{\mu_k \geq \nu_0} \mu_k^{-1} w_1(f', \mu_k^{-1}) \|u_k\|_\infty^2 + \|f'\|_1 \sum_{\mu_k \geq \nu_0} \mu_k^{-2} \|u_k\|_\infty^2 + \\ & \left. + (\|f\|_\infty + \|f'\|_1) \sum_{l=2}^3 \left(\sum_{\mu_k \geq \nu_0} \mu_k^{-2} \|u_k\|_\infty^2 \right) \right\} \leq \\ & \leq \text{const} \left\{ C(f) \nu_0^{\alpha-2} + \sum_{n=[\nu_0]} n^{-1} w_1(f', n^{-1}) + w_1(f', [\nu_0]^{-1}) \right. \\ & \left. + \|f'\|_1 [\nu_0]^{-1} + (\|f\|_\infty + \|f'\|_1) \sum_{l=2}^3 \|P_l\|_1 [\nu_0]^{1-l} \right\} < \infty \end{aligned}$$

Thus, the series (8) converges uniformly on the interval \bar{G} . Therefore, the expansion $\sum_{k=1}^{\infty} f_k u_k(x)$ converges absolutely and uniformly on this interval. By the completeness of the system $\{u_k(x)\}_{k=1}^{\infty}$ in $L_2(G)$ and the absolute continuity of the function $f(x)$, we have the identity $f(x) = \sum_{k=1}^{\infty} f_k u_k(x)$, $x \in \bar{G}$.

To prove the estimate (3), we use Lemmas 2, 3 and 4.

$$\begin{aligned} & \|R_\nu(\cdot, f)\|_{C[0,1]} = \|f - \delta_\nu(\cdot, f)\|_{C[0,1]} = \\ & \left\| \sum_{k=1}^{\infty} f_k u_k(\cdot) - \sum_{\mu_k \leq \nu} f_k u_k(\cdot) \right\|_{C[0,1]} = \\ & \left\| \sum_{\mu_k > \nu} f_k u_k(\cdot) \right\|_{C[0,1]} \leq \sum_{\mu_k \geq \nu} |f_k| \|u_k\| \leq \\ & \leq \text{const} \sum_{\mu_k \geq \nu} \{ C(f) \mu_k^{\alpha-3} + \mu_k^{-1} w_1(f', \mu_k^{-1}) + \mu_k^{-2} \|f'\|_1 + \\ & + \mu_k^{-2} (\|f\|_\infty + \|f'\|_1) \sum_{l=2}^3 \|P_l\|_1 \mu_k^{2-l} \} \|u_k\|_\infty^2 \leq \\ & \leq \text{const} \left\{ C(f) \nu^{\alpha-2} + \sum_{k=[\nu]} k^{-1} w_1(f', k^{-1}) + w_1(f', \nu^{-1}) + \right. \\ & \left. + \nu^{-1} \left(\|f'\|_1 + (\|f\|_\infty + \|f'\|_1) \sum_{l=2}^3 \nu^{2-l} \|P_l\|_1 \right) \right\}. \end{aligned}$$

The proof of theorem is complete. Corollary 2 follows from the definition of norm on the space $H_1^b(G)$ and Theorem with regard to the

inequality $\|f\|_{\mathcal{W}} \leq \|f\|_1$, which holds for any function $f(x) \in \mathcal{W}_1^a(G)$ satisfying the relations $f(0) = f(1) = 0$ indeed, if $f(0) = f(1) = 0$ and $f(x) \in H_1^b(G)$, then we have $C(f) = 0$, and the following chain of inequalities is satisfied ($n \geq n_0$):

$$\begin{aligned} & C(f) \nu^{\alpha-2} + \sum_{k=[\nu]}^{\infty} k^{-1} w_1(f', k^{-1}) + w_1(f', \nu^{-1}) + \nu^{-1} \times \\ & \times \left(\|f'\|_1 + (\|f\|_\infty + \|f'\|_1) \sum_{l=2}^3 \nu^{2-l} \|P_l\|_1 \right) \leq \sup_{\delta > 0} (\delta^{-\beta} w_1(f', \delta)) \\ & \left(\sum_{k=[\nu]}^{\infty} k^{-(1+\beta)} + \nu^{-\beta} \right) + \nu^{-1} \|f'\|_1 + 2\nu^{-1} \|f'\|_1 \sum_{l=2}^3 \|P_l\|_1 \leq \\ & \leq \text{const} \left\{ \|f'\|_1 + \sup_{\delta > 0} (\delta^{-\beta} w_1(f', \delta)) \right\} [\nu]^{-\beta} \leq \text{const} \nu^{-\beta} \|f'\|_1^\beta. \end{aligned}$$

To prove corollary 3, it suffices to note that in this case

$$\begin{aligned} & \sum_{k=[\nu]}^{\infty} k^{-1} w_1(f', k^{-1}) \leq \\ & \leq C_1(f') \sum_{k=[\nu]}^{\infty} k^{-1} \ln^{-(1+\beta)} k = O(\ln^{-\beta} \nu), \beta > 0 \end{aligned}$$

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