Pole assignment by constant output feedback for linear time invariant systems

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Abstract: - In this paper the pole assignment problem by constant output feedback for a class of linear time invariant systems is studied. In particular, explicit necessary conditions are given for the existence of a constant output feedback which places the poles of closed-loop system at desired locations.

Key-Words: - arbitrary pole assignment, constant output feedback, linear systems.

1 Introduction

Can the poles of closed-loop system obtained from a given linear time invariant system by constant output feedback, be placed at desired positions? This simple question is known in linear systems theory as pole assignment by constant output feedback and has been intensively studied in the last forty nine years.

In [1] the pole assignment problem by constant output feedback for linear time invariant systems with m inputs, p outputs, and McMillan degree n is studied for the first time. In particular in [1] has been proved that p poles of closed-loop system are assignable almost arbitrarily by constant output feedback. In [2] and [3] it was proved that, if the McMillan degree n does not exceed ($m+ p-1$), then an almost arbitrary set of distinct closed-loop poles is assignable by real constant output feedback. Alternative proofs of the above condition also are given in [4-6]. A method for assigning $min(n, p + m - 1)$ (or more) closed-loop poles by linear output feedback is presented in [7]. In [8], by using the dominant morphism theorem has been proved that the condition $mp > n$ is a sufficient condition for the pole assignment map to be almost surjective. [9] proved that arbitrary pole assignment is not possible in general for real linear time invariant systems with $m = p = 2$ and $n = 4$. Furthermore, [9] established the necessity of condition $mp > n$ for the solution of arbitrary pole placement by constant output feedback over the field of real numbers. In [10] it was proved that, if $mp = n$ the complex pole assignment map is surjective for generic systems. For some special cases, constructive procedures for pole assignment by constant output feedback were developed by in [11]. In [12] has been proved that, if the system is generic and $mp > n$, the pole assignment map is surjective. An alternative proof of the above result is given in [13]. Later, it was proved in [14] and [15, 16] that the geometric techniques used in [12] actually are based on a linearization of the pole assignment map around the dependent compensator. In [17] has been proved that, if $mp = n$ and both m and p are even, there exists an open subset of such systems where the real pole placement map is not surjective. [18] proved that, if $mp = n$, $min(m, p) = 2$ and $max(m, p)$ is even, there is a non-empty subset of such systems where arbitrary pole placement by real constant output feedback is impossible. In [19] it was proved, that arbitrary pole placement by constant output feedback of unitary rank is generically not possible even if $(m + p) > n$ holds true. In [20] have been derived new expressions for the characteristic polynomial of a linear system subject to constant output feedback. [21] established a necessary condition for pole assignment by constant output feedback; this condition is simply checked by computing the Smith-McMillan form of the open-loop system. Furthermore, the necessary condition in [21] shows that the solution of the pole assignment problem by constant output feedback depends on the finite zero structure of the open-loop system. Necessary and sufficient conditions for pole assignment by complex output feedback for symmetric and Hamiltonian systems have been derived in [22]. A new analytical solution to the problem of pole assignment via constant output feedback under the condition $(m + p) > n$, is presented in [23]. A necessary and sufficient condition and a new algorithm for the solution of pole assignment by constant output feedback were derived in [24]. New sufficient conditions as well as a linear method for the solution of the output feedback pole assignment problem were presented in [25]. [26] proved that the problem of pole placement via constant output feedback is NP-hard. [27] proved that for the class of linear time invariant systems with m inputs, p outputs, and McMillan degree n, the condition $(m + p) > n$ is necessary for the solution of the arbitrary pole assignment problem by constant output feedback. Numerical methods for the solution of pole placement by constant output feedback have been derived in [14], [16], [28], [29], [30], and [31], and the papers cited therein. Many other papers have been published in the past in this area; for more complete references, we refer the reader to the survey articles [32], [33], and [34]. The problem of pole placement by constant output feedback is difficult [35] and challenging; it remains an open problem in linear systems theory.

In this paper, are established explicit necessary for the solution of arbitrary pole assignment problem by constant output feedback for the class of linear time invariant systems.

2 Problem Formulation

Consider a linear controllable and observable system described by the following state-space equations

$$\dot{\mathbf{x}}(t)=\mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \qquad (1)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \qquad (2)$$

and the control low

$$\mathbf{u}(t) = -\mathbf{F}\mathbf{y}(t) + \mathbf{v}(t) \qquad (3)$$

where $\mathbf{F}$ is an $m \times p$ real matrix, $\mathbf{v}(t)$ is the reference input vector of dimensions $mx1$, $\mathbf{x}(t)$ is the state vector of dimensions $n \times 1$, $\mathbf{u}(t)$ is the vector of inputs of dimensions $m \times 1$ and $\mathbf{y}(t)$ is the vector of outputs of dimensions $p \times 1$ and $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are real matrices of dimensions $n \times n$, $n \times m$ and $n \times p$, respectively.The transfer function matrix of system (1) and (2) is given by

$$\mathbf{T}(s) = \mathbf{C}(\mathbf{I}s - \mathbf{A})^{-1}\mathbf{B} \qquad (4)$$

By applying the constant output feedback (3) to the system (1) and (2) the state-space equations of closed–loop system are

$$\dot{\mathbf{x}}(t)=(\mathbf{A}-\mathbf{BFC})\,\mathbf{x}(t) + \mathbf{B}\mathbf{v}(t) \qquad (5)$$

$$\mathbf{y}(t)=\mathbf{C}\mathbf{x}(t) \qquad (6)$$

Let $\mathcal{R}$ be the field of real numbers. Also let $\mathcal{R}[s]$ be the ring of polynomials with coefficients in $\mathcal{R}$. Let c(s) be a given arbitrary monic polynomial over $\mathcal{R}[s]$ of degree n. The pole assignment problem considered in this paper can be stated as follows: Does there exist a constant output feedback (3) such that

$$\det[\mathbf{I}s - \mathbf{A} + \mathbf{BFC}] = c(s) \qquad (7)$$

if so, give conditions for existence.

3 Basic concepts and preliminary results

Let us first introduce some notations that are used throughout the paper. Let $\mathbf{D}(s)$ be a nonsingular matrix over $\mathcal{R}[s]$ of dimensions $m \times m$, write $\deg_{ci}$ for the degree of column i of $\mathbf{D}(s)$. if

$$\deg_{ci}\mathbf{D}(s) \geq \deg_{cj}\mathbf{D}(s),\ i< j \qquad (8)$$

the matrix $\mathbf{D}(s)$ is said to be column degree ordered.

Denote $\mathbf{D}_n$ the highest column degree coefficient matrix of $\mathbf{D}(s)$.The matrix $\mathbf{D}(s)$ is said to be column reduced if the real matrix $\mathbf{D}_n$ is nonsingular. The matrix $\mathbf{D}(s)$ is said to be column monic if its highest column degree coefficient matrix is the identity matrix. A polynomial matrix $\mathbf{U}(s)$ of dimensions $k x k$ is said to be unimodular if and only if has polynomial inverse. Two polynomial matrices $\mathbf{A}(s)$ and $\mathbf{B}(s)$ having the same numbers of columns are said to be relatively right prime if and only if there are matrices $\mathbf{X}(s)$ and $\mathbf{Y}(s)$ over $\mathcal{R}[s]$ such that

$$\mathbf{X}(s)\mathbf{A}(s) + \mathbf{Y}(s)\mathbf{B}(s)=\mathbf{I} \qquad (9)$$

where $\mathbf{I}$ is the identity matrix of dimensions $l \times l$, l is the number of columns of the polynomial matrices $\mathbf{A}(s)$ and $\mathbf{B}(s)$. Let $\mathbf{D}(s)$ be a nonsingular matrix over $\mathcal{R}[s]$ of dimensions $m \times m$, then there exist unimodular matrices $\mathbf{U}(s)$ and $\mathbf{V}(s)$ over $\mathcal{R}[s]$ such that

$$\mathbf{D}(s)=\mathbf{U}(s)\ \mathrm{diag}\,[a_1(s), a_2(s), \ldots a_m(s)]\mathbf{V}(s) \qquad (10)$$

where the polynomials $a_i(s)$ for $i=1,2, .., m$ are termed invariant polynomials of $\mathbf{D}(s)$ and have the following property

$$a_i(s) \text{ divides } a_{i+1}(s), \text{ for } i=1,2,\ldots m-1 \qquad (11)$$

Furthermore we have that

$$a_i(s) = \frac{d_i(s)}{d_{i-1}(s)},\ \text{for } i= 1,2,\ldots m \qquad (12)$$

where $d_o(s)=1$ by definition and $d_i(s)$ is the monic greatest common divisor of all minors of order i in $\mathbf{D}(s)$, for $i=1,2,\ldots, m$. Two polynomial matrices $\mathbf{A}(s)$ and $\mathbf{B}(s)$ of appropriate dimensions are equivalent over $\mathcal{R}[s]$ if and only if there exist unimodular matrices $\mathbf{P}_1(s)$ and $\mathbf{P}_2(s)$ over $\mathcal{R}[s]$, such that $\mathbf{A}(s) = \mathbf{P}_1(s)\ \mathbf{B}(s)\ \mathbf{P}_2(s)$. If two polynomial matrices are equivalent over $\mathcal{R}[s]$, then they have the same invariant polynomials. The relationship (10) is known as the Smith – McMillan form of $\mathbf{D}$(s) over $\mathcal{R}$[s]. The system (1) and (2) is controllable if and only if

$$\text{rank}\,[\mathbf{I}s - \mathbf{A}\,,\, \mathbf{B}] = n \qquad (13)$$

for all complex numbers s.

The system (1) and (2) is observable if and only if

$$\text{rank}\begin{bmatrix} \mathbf{C} \\ \mathbf{I}s - \mathbf{A} \end{bmatrix} = n \qquad (14)$$

for all complex numbers s.

If the system (1) and (2) is controllable, then the minimal column indices of singular pencil (13) are the controllability indices of system (1) and (2). If the system (1) and (2) is observable, then the minimal row indices of singular pencil (14) are the observability indices of system (1) and (2).

Let $v_1 \geq v_2 \geq \ldots\ldots \geq v_m$ be the ordered list of reachability indices of system (1). Then $v = max\, v_i$ for $i = 1, 2, \ldots, m$ is called reachability index of system (1), [36]. Let $\mu_1 \geq \mu_2 \geq \ldots\ldots \geq \mu_p$ be the ordered list of observability indices of system (1). Then $\mu = max\, \mu_i$ for $i = 1, 2, \ldots, p$ is called observability index of system (1), [36]. Let $\text{rank}[\mathbf{B}] = m$ and $\text{rank}[\mathbf{C}] = p$. Then the reachability index v and the observability index μ of system (1) and (2) satisfy the following relationships [37]

$$n/m \leq v \leq (n - m + 1) \qquad (15)$$

$$n/p \leq \mu \leq (n - p + 1) \qquad (16)$$

Definiton 1. Relatively right prime polynomials matrices $\mathbf{D}$(s) and $\mathbf{N}$(s) of dimensions $m \times m$ and $p \times m$ respectively with $\mathbf{D}$(s) to be column reduced and column degree ordered such that

$$\mathbf{C}(\mathbf{I}s - \mathbf{A})^{-1}\mathbf{B} = \mathbf{N}(s)\mathbf{D}^{-1}(s) \qquad (17)$$

are said to form a standard right matrix fraction description of system (1) and (2).

The column degrees of the matrix $\mathbf{D}$(s) are the controllability indices of system (1) and (2).

Let $\mathbf{A}$ be a matrix over $\mathcal{R}$ of dimensions n x n. The quantity

$$\text{tr}[\mathbf{A}] \;= \Sigma_{i=1}^{n} a_{ii} \qquad (18)$$

is called trace of the matrix $\mathbf{A}$. Let $\mathbf{A}$ be a matrix over $\mathcal{R}$ of size n x n and let a(s) be the characteristic polynomial of the matrix $\mathbf{A}$ given by

$$a(s) = s^n + a_{n-1}s^{n-1} + \ldots. + a_1 s + a_0 \qquad (19)$$

Then the coefficient a_{n-1} of the characteristic polynomial $a(s)$ is given by [38]

$$a_{n-1} = -\text{tr}[\mathbf{A}] = -\Sigma_{i=1}^{n} a_{ii} \qquad (20)$$

Every column monic polynomial matrix $\mathbf{M}$(s) of dimensions $m \times m$ with column degrees $v_1, v_2, \ldots, v_m$ such that

$$v_1 = v_2 =, \ldots, = v_m = v \qquad (21)$$

can be written as follows [39]

$$\mathbf{M}(s) = \mathbf{I}\, s^v + \Sigma_{i=0}^{v-1} \mathbf{M}_i\, s^i \qquad (22)$$

where $\mathbf{M}_i$ for $i=1,2, .., m$ are real matrices of dimensions $m \times m$.

For every column monic polynomial matrix is defined the following matrix over $\mathcal{R}$ [39]

$$\mathbf{N} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{K} & \mathbf{L} \end{bmatrix} \qquad (23)$$

where $\mathbf{I}$ is the identity matrix of size $m(v\text{-}1) \times m(v\text{-}1)$. The real matrices $\mathbf{K}$ and $\mathbf{L}$ of appropriate dimensions are given by

$$\mathbf{K} = -\mathbf{M}_0 \;\text{ and }\; \mathbf{L} = [-\mathbf{M}_1, \ldots., -\mathbf{M}_{v-1}] \qquad (24)$$

The matrix (23) is termed first companion form of polynomial matrix $\mathbf{M}$(s). In [39] is proven that the polynomial matrices [$\mathbf{M}$(s), $\mathbf{I}$] and [$\mathbf{I}$s – $\mathbf{N}$] are equivalent over $\mathcal{R}$[s] and therefore

$$\det[\mathbf{I}s - \mathbf{N}] = \det[\mathbf{M}(s)] \qquad (25)$$

From (25) it follows that the zeros of the column monic polynomial matrix $\mathbf{M}$(s) and the eigenvalues of the matrix $\mathbf{N}$ are the same. From the structure of matrix $\mathbf{N}$ and (18) it follows that

$$\text{tr}[\mathbf{N}] = \text{tr}\,[-\mathbf{M}_{v-1}] \qquad (26)$$

The following Lemmas are needed to prove the main theorem of this paper.

Lemma 1, [40]. Let $\mathbf{D}$(s), $\mathbf{N}$(s)be a standard right matrix fraction description of system (1). Also let v_i for $i=1,2,\ldots.m$ be the controllability indices of (1) and (2). Then for every $m \times p$ real matrix $\mathbf{F}$ we have:

(a)The polynomial matrices $\mathbf{N}$(s) and $[\mathbf{D}(s) + +\mathbf{FN}(s)]$ are relatively right prime over $\mathcal{R}$[s].

(b)The matrix $[\mathbf{D}(s) + \mathbf{FN}(s)]$ is column reduced and column degree ordered and its column degrees are the numbers v_i for $i=1,2,\ldots m$.

(c)The open-loop system (1) and (2) and the closed – loop system (5) and (6) have the same controllability indices.

Proof: Let $\mathbf{D}$(s) and $\mathbf{N}$(s) be a standard matrix fraction description of (1) and (2). Then for the transfer function of closed – loop system (5), (6) we have that

$$\mathbf{C}[\mathbf{I}s-\mathbf{A}+\mathbf{BFC}]^{-1}\mathbf{B}=\mathbf{N}(s)[\mathbf{D}(s)+ +\mathbf{FN}(s)]^{-1} \qquad (27)$$

We can write

$$\begin{bmatrix}\mathbf{N}(s)\\ \mathbf{D}(s)+\mathbf{FN}(s)\end{bmatrix}=\begin{bmatrix}\mathbf{I}_p & 0\\ \mathbf{F} & \mathbf{I}_m\end{bmatrix}\begin{bmatrix}\mathbf{N}(s)\\ \mathbf{D}(s)\end{bmatrix} \qquad (28)$$

Since the matrix

$$\begin{bmatrix}\mathbf{I}_p & 0\\ \mathbf{F} & \mathrm{I}_m\end{bmatrix} \qquad (29)$$

is unimodular and the matrices **N**(s) and **D**(s) are relatively right prime over $\mathcal{R}$[s], we have from (28) that the matrices $[\mathbf{D}(s)+\mathbf{FN}(s)]$ and **N**(s) are relatively right prime over $\mathcal{R}$[s]. This is condition (a) of the Lemma.
Since the open – loop system (1) and (2) is strictly proper with controllability indices v_i for $i = 1, 2 \ldots, m$ we have that

$$\deg_{ci}\mathbf{N}(s) < \deg_{ci}\mathbf{D}(s) = v_i, \text{ for } i=1,2,\ldots, m \qquad (30)$$

Since **F** is real matrix it follows from (30) that

$$\deg_{ci}\mathbf{FN}(s) < \deg_{ci}\mathbf{D}(s) = v_i,\ i= 1,2,\ldots, m \quad (31)$$

Since by definition the matrix **D**(s) is column reduced and column and degree ordered, it follows from (28) that the matrix $[\mathbf{D}(s)+\mathbf{FN}(s)]$ is column reduced and column degree ordered and its column degrees are the numbers v_i for $i=1,2,\ldots m$. This is condition (b) of the Lemma.
Since the matrices $[\mathbf{D}(s)+\mathbf{FN}(s)]$ and **N**(s) are relatively right prime over $\mathcal{R}$[s] and the matrices **D**(s) and $[\mathbf{D}(s)+\mathbf{FN}(s)]$ are column reduced with the same column degrees, we conclude that the open – loop system (1) and (2) the closed – loop system (5) and (6) have the same controllability indices. This is condition (c) of the Lemma and the proof is complete.

Lemma 2, [41]. Let **D**(s), **N**(s) be a standard right matrix fraction description of system (1) and (2). Then for every $m \times p$ real matrix **F** the polynomial matrices [**I**s – **A**+**BFC**] and [**D**(s)+**FN**(s)] have the same nonunit invariant polynomials.

Proof. Let **D**(s) and $\mathbf{N}_1$(s) are relatively right prime polynomial matrices over $\mathcal{R}$[s] of respective dimensions $m \times m$ and $n \times m$ such that

$$(\mathbf{I}s-\mathbf{A})^{-1}\mathbf{B}=\mathbf{N}_1(s)\mathbf{D}^{-1}(s) \qquad (32)$$

We have that

$$[\mathbf{I}s-\mathbf{A}]\mathbf{N}_1(s)=\mathbf{BD}(s) \qquad (33)$$

We add $\mathbf{BFCN}_1(s)$ to both sides of the above identity and rearrange to get

$$[\mathbf{I}s-\mathbf{A}+\mathbf{BFC}]^{-1}\mathbf{B}=\mathbf{N}_1(s)[\mathbf{D}(s)+\mathbf{FCN}_1(s)]^{-1} \quad 34)$$

Since $[\mathbf{I}s-\mathbf{A}]$and **B** are relatively left prime over $\mathcal{R}$[s] by controllability of (1) and (2) and since

$$[\mathbf{I}s-\mathbf{A}+\mathbf{BFC},\mathbf{B}]=[\mathbf{I}s-\mathbf{A},\mathbf{B}]\begin{bmatrix}\mathbf{I}_n & \mathbf{0}\\ \mathbf{FC} & \mathbf{I}_m\end{bmatrix} \qquad (35)$$

it follows that $[\mathbf{I}s-\mathbf{A}+\mathbf{BFC}]$ and **B** are relatively left prime over $\mathcal{R}$[s]. On the other hand **D**(s) and $\mathbf{N}_1$(s) are relatively right prime over $\mathcal{R}$[s] and

$$\begin{bmatrix}\mathbf{N}_1(s)\\ \mathbf{D}(s)+\mathbf{FCN}_1(s)\end{bmatrix}= =\begin{bmatrix}\mathbf{I}_n & \mathbf{0}\\ \mathbf{FC} & \mathbf{I}_m\end{bmatrix}\begin{bmatrix}\mathbf{N}_1(s)\\ \mathbf{D}(s)\end{bmatrix} \qquad (36)$$

Hence $[\mathbf{D}(s)+\mathbf{FCN_1}(s)]$and $\mathbf{N}_1(s)$ are relatively right prime over $\mathcal{R}$[s]. It follows that the matrices $[\mathbf{I}s-\mathbf{A}+\mathbf{BFC}]$ and $[\mathbf{D}(s)+\mathbf{FCN}_1(s)]$ or equivalently the matrices $[\mathbf{I}s-\mathbf{A}+\mathbf{BFC}]$ and $[\mathbf{D}(s)+\mathbf{FN}(s)]$ must share the same nonunit invariant polynomials. This completes the proof of the Lemma.

Lemma 3. Let **D**(s), **N**(s) be a standard right matrix fraction description of system (1) and (2) with **D**(s) to be column monic. Then the characteristic polynomial c(s) of closed-loop system (5) and (6) is given by

$$\det[\mathbf{D}(s)+\mathbf{FN}(s)] = c(s) \qquad (37)$$

Proof. The characteristic polynomial c(s) of closed-loop system (5) and (6) is given by

$$\det[\mathbf{I}s-\mathbf{A}+\mathbf{BFC}] = \Pi_{i=1}^{k}a_i(s)] = c(s) \qquad (38)$$

where $a_i(s)$ for $i=1,2,\ldots k$ are the nonunit invariant polynomials of the matrix **[I**s – **A**+**BFC]** [38]. By Lemma 2, the polynomial matrices [**I**s – **A**+**BFC**] and [**D**(s)+**FN**(s)] have the same nonunit invariant polynomials and therefore from (10) we have

$$\det[\mathbf{D}(s)+\mathbf{FN}(s)] = \Pi_{i=1}^{k}a_i(s)] \qquad (39)$$

Relationship (37) follows from (38) and (39) and the proof is complete.

4 Problem Solution

Let **D**(s), **N**(s) be a standard right matrix fraction description of completely controllable and observable system (1) and (2) with **D**(s) to be column monic. Also let $v_1 \geq v_2 \geq \ldots\ldots \geq v_m$ be the ordered list of controllability indices of (1) and (2). By Lemma 1 the column degrees of polynomial matrix $\mathbf{D}$(s) are $\nu_1, \nu_2, \ldots, \nu_m$. In what follows, we consider the special class of systems for which

$$\nu_1 = \nu_2 =, \ldots, = \nu_m = \nu \qquad (40)$$

Using (22) the column monic polynomial matrix $\mathbf{D}$(s) and the polynomial matrix $\mathbf{N}$(s) can be written as follows

$$\mathbf{D}(s) = \mathbf{I}\, s^{\nu} + \Sigma_{i=0}^{\nu-1} \mathbf{D}_i\, s^i \qquad (41)$$

$$\mathbf{N}(s) = \Sigma_{i=0}^{\nu-1} \mathbf{N}_i\, s^i \qquad (42)$$

where $\mathbf{D}_i$ and $\mathbf{N}_i$ for $i=0,1,\ldots,\nu-1$ are real matrices of dimensions $m \times m$ and $p \times m$ respectively. Let

$$\mathbf{Y} = [\mathbf{D}_0^T \quad \mathbf{N}_0^T]^T \text{ and } \mathbf{Z} = [\mathbf{I} \quad \mathbf{N}_0^T\,]^T \qquad (43)$$

The following theorem is the main result of this paper and gives explicit necessary conditions for the solution of the pole assignment problem by constant output feedback for a class of completely controllable and observable linear time invariant systems.

Theorem 1. Let $\mathbf{D}$(s), $\mathbf{N}$(s) be a standard right matrix fraction description of controllable and observable system (1) and (2) with $\mathbf{D}$(s) to be column monic. Also let $\nu_1 \geq \nu_2 \geq \ldots\ldots \geq \nu_m$ be the ordered list of controllability indices of system (1) and (2). Suppose that relationship (40) is satisfied. Then the pole assignment problem by constant output feedback has a solution only if

(a) $\mathbf{N}_{\nu-1} \neq \mathbf{0}$

(b) The rows of the matrices $\mathbf{Y}$ and $\mathbf{Z}$ span the same linear space over $\mathcal{R}$.

Proof. Let $\mathbf{D}$(s), and $\mathbf{N}$(s) be a standard right matrix fraction description of controllable and observable system (1) and (2) with $\mathbf{D}$(s) to be column monic. Also let c(s) be arbitrary monic polynomial over $\mathcal{R}$[s] of degree n. Suppose that the pole assignment problem by constant output feedback has a solution. From Lemma 3 it follows that

$$\det[\mathbf{D}(s)+\mathbf{F}\mathbf{N}(s)] = c(s) \qquad (44)$$

Since by assumption the polynomial matrix $\mathbf{D}$(s) is column monic and column degree ordered from Lemma 1 it follows that the polynomial matrix $\mathbf{D}$(s)+$\mathbf{F}\mathbf{N}$(s) is also column monic and column degree ordered and its column degrees are the numbers ν_i for $i=1,2,\ldots m$. Since by assumption relationship (40) is satisfied, the first companion form of the polynomial matrix $\mathbf{D}$(s)+$\mathbf{F}\mathbf{N}$(s) is given by

$$\mathbf{N} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{K} & \mathbf{L} \end{bmatrix} \qquad (45)$$

where $\mathbf{I}$ is the identity matrix of size $m(\nu-1) \times m(\nu-1)$. The real matrices $\mathbf{K}$ and $\mathbf{L}$ of appropriate dimensions are given by

$$\mathbf{K} = -(\mathbf{D}_0+\mathbf{F}\mathbf{N}_0) \qquad (46)$$

$$\mathbf{L} = [-(\mathbf{D}_1+\mathbf{F}\mathbf{N}_1), \ldots, -(\mathbf{D}_{\nu-1}+\mathbf{F}\mathbf{N}_{\nu-1})] \qquad (47)$$

Since the polynomial matrices [$\mathbf{D}$(s)+$\mathbf{F}\mathbf{N}$(s)], $\mathbf{I}$] and [$\mathbf{I}$s – $\mathbf{N}$] are equivalent over $\mathcal{R}$[s] [39] we have that

$$\det[\mathbf{D}(s)+\mathbf{F}\mathbf{N}(s)]] = \det[\mathbf{I}s-\mathbf{N}] = c(s) \qquad (48)$$

From (45), (46), (47), (48), (20) and (26) it follows that the coefficient c_{n-1} of c(s) is given by

$$c_{n-1} = -\operatorname{tr}[-(\mathbf{D}_{\nu-1}+\mathbf{F}\mathbf{N}_{\nu-1})] \qquad (49)$$

Since by assumption c(s) is arbitrary monic polynomial of degree n over $\mathcal{R}$[s], c_{n-1} is arbitrary real number. This implies that

$$\mathbf{N}_{\nu-1} \neq \mathbf{0} \qquad (50)$$

This is condition (a) of Theorem 1. Since c(s) is by assumption arbitrary monic polynomial of degree n over $\mathcal{R}$[s], we assume without any loss of generality that all roots of c(s) are nonzero. This implies that the columns of the matrix $\mathbf{N}$ are linearly independent over $\mathcal{R}$. From the above it follows that the matrix ($\mathbf{D}_0$+$\mathbf{F}\mathbf{N}_0$) is nonsingular. Let

$$\mathbf{D}_0+\mathbf{F}\mathbf{N}_0 = \mathbf{T} \qquad (51)$$

where $\mathbf{T}$ is nonsingular matrix of appropriate dimensions. (51) is rewritten as follows

$$\mathbf{T}^{-1}\mathbf{D}_0 + \mathbf{T}^{-1}\mathbf{F}\mathbf{N}_0 = \mathbf{I} \qquad (52)$$

Let $\mathbf{E} = \mathbf{T}^{-1}\mathbf{F}$. Using (43) relationship (52) can be rewritten as follows

$$\begin{bmatrix} \mathbf{T}^{-1} & \mathbf{E} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \mathbf{Y} = \mathbf{Z} \qquad (53)$$

Since by assumption, matrix $\mathbf{T}^{-1}$ is nonsingular the transformation matrix in (53) is also nonsingular and therefore the rows of the matrix $\mathbf{Y}$ span the same linear space over $\mathcal{R}$ as those of $\mathbf{Z}$. This is condition (b) of Theorem 1 and the proof is complete.

Corollary 1. Let $\mathbf{D}$(s), $\mathbf{N}$(s) be a standard right matrix fraction description of controllable and observable system (1) and (2) with $\mathbf{D}$(s) to be column monic. Also let $\nu_1 \geq \nu_2 \geq \ldots\ldots \geq \nu_m$ be the ordered list of controllability indices of system (1) and (2). Further, let $\mathbf{D}_0 = \mathbf{0}$. Suppose that relationship (40) is satisfied. Then the pole assignment problem by constant output feedback has a solution only if $p \geq m$.

Proof. Suppose that the pole assignment problem by constant output feedback has a solution. Since by assumption $\mathbf{D}_0 = \mathbf{0}$, it follows from (51) and (52) that

$$\mathbf{T}^{-1}\,\mathbf{F}\mathbf{N_0} = \mathbf{I} \qquad (54)$$

From (54) it follows that $(\mathbf{T}^{-1}\,\mathbf{F})$ is the left inverse of $p \times m$ real matrix $\mathbf{N_0}$. This implies that

$$\text{rank}\,[\mathbf{N_0}] = m \qquad (55)$$

From (55) it follows that $p \geq m$ and the proof is complete.

Remark. The arbitrary pole assignment problem by constant output feedback for linear time invariant systems is probably one of the most prominent open questions in linear systems theory [32], [33] and [44]. Besides the trivial conditions of controllability and observability there are only a few explicit necessary conditions in the literature for the solvability of arbitrary pole assignment problem by constant output feedback for linear time invariant systems. The main theorem of this paper adds yet two explicit necessary conditions to existing ones. This clearly demonstrates the originality of the contribution of main theorem of this paper with respect to existing results.
It is well known that the solution of the arbitrary pole assignment problem by constant output feedback depends on the relationship between the number of inputs and outputs and the McMillan degree of the open–loop system [9] and [27]. Corollary 1 of this paper shows that the solution of the arbitrary pole assignment problem by constant output feedback depends on the relationship between the number of inputs and the number of outputs. This demonstrates the originality of the contribution of Corollary 1 of Theorem 1, with respect to existing results.

Remark 2. The problem of arbitrary pole placement by constant output feedback in its full generality is NP-hard [26]; therefore it is not polynomial-time solvable [42, 43]. This means that it is extremely difficult [42] to obtain an efficient algorithm to correctly solve all instances of the problem [28]. This justifies investigation of the solvability of the arbitrary pole placement problem by constant output feedback for special classes of linear time invariant systems (1) and (2).

Example: Consider a completely controllable and observable system (1) specified by [2]

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$m=2, p=2$ and $n=4$

The task is to check if the pole assignment problem by constant output feedback has a solution. The transfer function of the given system is given by

$$\mathbf{T}(s) = \begin{bmatrix} 1/s^2 & 0 \\ 0 & 1/s^2 \end{bmatrix}$$

We define the matrices

$$\mathbf{D}(s) = \text{diag}[s^2,\ s^2]$$

$$\mathbf{N}(s) = [1, 1]$$

It is obvious that matrices $\mathbf{D}(s)$ and $\mathbf{N}(s)$ are relatively right prime over $\mathcal{R}[s]$ and the matrix $\mathbf{D}(s)$ is column reduced and column degree ordered. Since

$$\mathbf{T}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s)$$

we conclude that the polynomial matrices $\mathbf{D}(s)$, $\mathbf{N}(s)$ form a standard right matrix fraction description of the given system. The column degrees of the polynomial matrix $\mathbf{D}(s)$ are

$$v_1 = v_2 = 2$$

Assumption (40) is satisfied and therefore the polynomial matrices $\mathbf{D}(s)$ and $\mathbf{N}(s)$ can be rewritten as follows

$$\mathbf{D}(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} s^2 + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} s + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{N}(s) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} s + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We have that

$$\mathbf{D_1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \mathbf{D}_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{N_1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \mathbf{N}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since

$$\mathbf{N_1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

from Theorem 1 it follows that arbitrary pole assignment by constant output feedback is impossible.

5 Conclusions

The pole assignment problem by constant output feedback for linear time invariant systems is very broadly studied problem and is probably one of the most prominent open questions in linear systems theory. Considerable progress has been achieved over the years but we are still far from finding exact necessary and sufficient conditions that guarantee the solvability of the problem. In this paper, are established explicit necessary conditions for the solution of arbitrary pole assignment problem by constant output feedback for a class of linear time invariant systems. The main results obtained for linear continuous -time systems hold also for linear discrete-time systems. We believe that our results are useful for further understanding of this important and longstanding open problem.

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