Uniqueness of extremals for problems with endpoint and control constraints

JAVIER F ROSENBLUETH
IIMAS, Universidad Nacional Autónma de México
Apartado Postal 20-126, CDMX 01000
MEXICO
jfrl@unam.mx

Abstract: In this paper we study the uniqueness of extremals satisfying first order necessary conditions for optimal control problems involving endpoint and control constraints. In particular we show that, for such problems, a strict Mangasarian-Fromovitz type constraint qualification does imply uniqueness of Lagrange multipliers but, contrary to the corresponding equivalence in mathematical programming, the converse in optimal control may not hold.

Key–Words: Optimal control, uniqueness of Lagrange multipliers, normality, constraint qualifications

1 Introduction

It is well-known that, for nonlinear programming problems involving equality and inequality constraints, a strict version of the Mangasarian-Fromovitz constraint qualification (SMFCQ) is equivalent to the uniqueness of multipliers satisfying the Karush-Kuhn-Tucker conditions (or first order Lagrange multiplier rule). Moreover, that strict constraint qualification implies the satisfaction of second order necessary optimality conditions on a critical cone which takes into account the sign of the multipliers.

For the statement on uniqueness of multipliers we refer the reader to [1, 3, 4, 6, 9], while the result on necessary conditions can be seen with detail in [2, 6, 8, 9].

The equivalence between uniqueness of multipliers and the SMFCQ was first established in [9] and has been widely quoted (see, for example, [1, 3, 4–6, 12–14] and references therein). The proof of that result, found in [9], is strongly based on yet another equivalence between the Mangasarian-Fromovitz constraint qualification (MFCQ) and normality relative to the original set of constraints, a crucial result which has been proved by Hestenes [8] or, as mentioned in [6, 9], by using theorems of alternative (see Motzkin in [6, Theorem 2.4.19]). Here, the notion of *normality* is used in the sense that, if the cost multiplier vanishes in the Fritz John necessary optimality condition, then the only solution of the corresponding first order system is the null solution. Thus, normality of a local minimizer relative to the original set of constraints implies, in the necessary optimality condition, a positive cost multiplier.

Due to this last equivalence, the SMFCQ (which is really the MFCQ applied to a subset of the set of tangential constraints) is equivalent to normality relative to a subset of the original set of constraints which depends on Lagrange multipliers given beforehand, and includes inequalities when the multipliers vanish and equalities otherwise.

In this paper, we shall deal with an optimal control problem involving endpoint and control constraints, and pose the question of uniqueness of multipliers satisfying first order necessary conditions as a consequence of a maximum principle.

As we shall see, the notion of normality follows essentially the same principles as in the finite dimensional case, and there is a natural correspondence with the sets of constraints mentioned above. However, we shall prove that, for this type of optimal control problems, uniqueness of the Lagrange multipliers and normality relative to the corresponding subset of the set of original constraints are no longer equivalent.

2 Nonlinear programming

In this section, we shall give a brief summary of the main results given in [9] relating the SMFCQ with uniqueness of Lagrange multipliers and second order necessary optimality conditions. This will help us to clearly understand some of the main differences between the finite dimensional case and that of optimal control.

Consider the problem, which we shall label (N), of minimizing f on S, where $f, g_i \colon \mathbf{R}^n \to \mathbf{R}$ ($i \in A \cup B$) are given C^1 functions, $A = \{1, \ldots, p\}$, $B =$

$\{p+1,\ldots,m\}$, and

$$S=\{x\in\mathbf{R}^n \mid g_\alpha(x)\le 0\ (\alpha\in A),$$

$$g_\beta(x)=0\ (\beta\in B)\}.$$

Denote by $\Lambda(f,x_0)$ the set of all $\lambda\in\mathbf{R}^m$ (whose components $\lambda_1,\ldots,\lambda_m$ are the *Kuhn-Tucker* or *Lagrange multipliers*) satisfying the Karush-Kuhn-Tucker (KKT) conditions:

i. $\lambda_\alpha\ge 0$ and $\lambda_\alpha g_\alpha(x_0)=0\ (\alpha\in A)$.

ii. If $F(x):=f(x)+\langle\lambda,g(x)\rangle$ then $F'(x_0)=0$.

Here, the function F is the standard *Lagrangian*, g is the function mapping $\mathbf{R}^n$ to $\mathbf{R}^m$ whose components are $g_1,\ldots,g_m$, and $\langle\cdot,\cdot\rangle$ is the standard inner product in $\mathbf{R}^m$ so that $\langle\lambda,g(x)\rangle=\sum_1^m\lambda_i g_i(x)$.

Under a suitable constraint qualification, the KKT conditions hold at a local minimum x_0 of (N), that is, if x_0 affords a local minimum to f on S and a constraint qualification holds, then $\Lambda(f,x_0)\neq\emptyset$. It is important to mention that, as shown in [6–8], if

$$I(x)=\{\alpha\in A \mid g_\alpha(x)=0\}$$

denotes the set of *active indices* at x, and

$$R_S(x_0):=\{h\in\mathbf{R}^n \mid g'_\alpha(x_0;h)\le 0\ (\alpha\in I(x_0)),$$

$$g'_\beta(x_0;h)=0\ (\beta\in B)\}$$

is the set of vectors satisfying the *tangential constraints* at x_0 (or the *linearized tangent cone*), then

$$\Lambda(f,x_0)\neq\emptyset \Leftrightarrow f'(x_0;h)\ge 0 \text{ for all } h\in R_S(x_0).$$

Constraint qualifications can also be seen as conditions which assure the positiveness of the cost multiplier λ_0 in the Fritz John necessary optimality condition which states that, if x_0 solves (N) locally, then there exist $\lambda_0\ge 0$ and $\lambda\in\mathbf{R}^m$, not both zero, such that

i. $\lambda_\alpha\ge 0$ and $\lambda_\alpha g_\alpha(x_0)=0\ (\alpha\in A)$.

ii. If $F_0(x):=\lambda_0 f(x)+\langle\lambda,g(x)\rangle$ then $F_0'(x_0)=0$.

Based on the theory of augmentability, a simple proof of this result is provided in [10], while the proof given in [6] uses Motzkin theorem of the alternative. It yields in a natural way the following constraint qualification.

We shall say that $x\in S$ is *normal relative to* S if $\lambda=0$ is the only solution of

i. $\lambda_\alpha\ge 0$ and $\lambda_\alpha g_\alpha(x)=0\ (\alpha\in A)$.

ii. $\sum_1^m\lambda_i g_i'(x)=0$.

Clearly, the normality condition is a constraint qualification since, in the Fritz John theorem, if x_0 is also a normal point of S, then $\lambda_0>0$ and the multipliers can be chosen so that $\lambda_0=1$, thus implying that $\Lambda(f,x_0)\neq\emptyset$.

As shown in [6, 8], normality of a point x_0 relative to S is equivalent to the *Mangasarian-Fromovitz constraint qualification at* x_0 *with respect to* S, which requires the linear independence of the set

$$\{g'_\beta(x_0) \mid \beta\in B\}$$

and the existence of h such that

$$g'_\alpha(x_0;h)<0\ (\alpha\in I(x_0)),$$

$$g'_\beta(x_0;h)=0\ (\beta\in B).$$

Now, suppose $\lambda\in\Lambda(f,x_0)$. Let

$$S_1(\lambda):=\{x\in S \mid F(x)=f(x)\}.$$

Note that, if $\Gamma=\{\alpha\in A \mid \lambda_\alpha>0\}$, then

$$S_1\ [=S_1(\lambda)\,]\ =$$

$$\{x\in\mathbf{R}^n \mid g_\alpha(x)\le 0\ (\alpha\in A,\ \lambda_\alpha=0),$$

$$g_\beta(x)=0\ (\beta\in\Gamma\cup B)\}=$$

$$\{x\in S \mid g_\alpha(x)=0\ (\alpha\in\Gamma)\}.$$

Therefore

$$R_{S_1}(x_0)=$$

$$\{h\in\mathbf{R}^n \mid g'_\alpha(x_0;h)\le 0\ (\alpha\in I(x_0),\ \lambda_\alpha=0),$$

$$g'_\beta(x_0;h)=0\ (\beta\in\Gamma\cup B)\}=$$

$$\{h\in R_S(x_0) \mid g'_\alpha(x_0;h)=0\ (\alpha\in\Gamma)\}=$$

$$\{h\in R_S(x_0) \mid f'(x_0;h)=0\}.$$

The *strict Mangasarian-Fromovitz constraint qualification at* x_0 corresponds to MFCQ at x_0 with respect to $S_1(\lambda)$. In other words, $x_0\in\mathbf{R}^n$ satisfies the SMFCQ if the set

$$\{g'_\beta(x_0) \mid \beta\in\Gamma\cup B\}$$

is linearly independent, and there exists h such that

$$g'_\alpha(x_0;h)<0\ (\alpha\in I(x_0),\ \lambda_\alpha=0),$$

$$g'_\beta(x_0;h)=0\ (\beta\in\Gamma\cup B).$$

Let us now state and give a (simple) proof of the main result given in [9] relating the SMFCQ with the uniqueness of Lagrange multipliers.

Theorem 1 *Suppose* $\lambda\in\Lambda(f,x_0)$. *Then the following are equivalent:*

a. x_0 *satisfies* SMFCQ.

b. λ *is unique in* $\Lambda(f,x_0)$.

Proof: As mentioned before, (a) is equivalent to normality of x_0 relative to $S_1(\lambda)$, that is, $\mu = 0$ is the only solution of

1. $\mu_\alpha \geq 0$ and $\mu_\alpha g_\alpha(x_0) = 0$ $(\alpha \in A,\ \lambda_\alpha = 0)$.
2. $\sum_1^m \mu_i g_i'(x_0) = 0$.

(a) $\Rightarrow$ (b): Let $\bar\lambda \in \Lambda(f, x_0)$ and set $\mu := \bar\lambda - \lambda$. Then μ satisfies (1) and (2) since

$$\mu_\alpha = \bar\lambda_\alpha \geq 0,$$

$$\mu_\alpha g_\alpha(x_0) = \bar\lambda_\alpha g_\alpha(x_0) = 0 \quad (\alpha \in A,\ \lambda_\alpha = 0)$$

and, if $\bar F(x) := f(x) + \langle \bar\lambda, g(x)\rangle$, then

$$0 = \bar F'(x_0) - F'(x_0) = \sum_1^m \mu_i g_i'(x_0).$$

By (a), $\mu = 0$ and so $\lambda = \bar\lambda$.

(b) $\Rightarrow$ (a): $\sim$(a) $\Rightarrow$ $\sim$(b): Assume x_0 is not a normal point of $S_1(\lambda)$. Then there exists $\mu \in \mathbf{R}^m$, $\mu \neq 0$, satisfying (1) and (2) and such that

$$\max\{|\mu_\alpha| : \alpha \in K\} < \min\{\lambda_\alpha : \alpha \in K\}$$

where

$$K = \{\alpha \in I(x_0) \mid \lambda_\alpha > 0\}.$$

Let $\hat\lambda := \lambda + \mu$. Let us prove that $\hat\lambda \in \Lambda(f, x_0)$ implying $\sim$(b) since $\hat\lambda \neq \lambda$. Indeed, if

$$\hat F(x) := f(x) + \langle \hat\lambda, g(x)\rangle,$$

then we have

$$\hat F'(x_0) = F'(x_0) + \sum_1^m \mu_i g_i'(x_0) = 0$$

and so (ii) in the definition of $\Lambda(f, x_0)$ holds. To prove that also (i) holds, let $\alpha \in A$. If $g_\alpha(x_0) = 0$ then $\hat\lambda_\alpha g_\alpha(x_0) = 0$. If $g_\alpha(x_0) < 0$ then, by (i) with respect to λ, we have $\lambda_\alpha = 0$ and so, by (1),

$$0 = \mu_\alpha g_\alpha(x_0) = \hat\lambda_\alpha g_\alpha(x_0).$$

Finally, if $\lambda_\alpha = 0$, then $\hat\lambda_\alpha = \mu_\alpha \geq 0$. If $\lambda_\alpha > 0$ then

$$\begin{aligned} \hat\lambda_\alpha &= \lambda_\alpha + \mu_\alpha \\ &\geq \min_{i \in K} \lambda_i + \mu_\alpha \\ &> \max_{i \in K} |\mu_i| + \mu_\alpha \geq 0. \ \blacksquare \end{aligned}$$

If we assume that the functions delimiting the problem are C^2, second order necessary conditions can be derived under the assumption of SMFCQ (or normality relative to S_1) at a local minimum of the problem. A proof of this result can be found in [2, 8].

Theorem 2 *Suppose $x_0 \in S$ and $\lambda \in \Lambda(f, x_0)$. If x_0 solves* (N) *locally and is a normal point of $S_1(\lambda)$, then $F''(x_0; h) \geq 0$ for all $h \in R_{S_1}(x_0)$.*

3 Extremals and normality in optimal control

The optimal control problem we shall deal with can be stated as follows. Suppose we are given an interval $T := [t_0, t_1]$ in $\mathbf{R}$, a point $\xi_0 \in \mathbf{R}^n$, and functions

$$(g, h)\colon \mathbf{R}^n \to \mathbf{R} \times \mathbf{R}^k \ (k \leq n),$$

$$(L, f)\colon T \times \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R} \times \mathbf{R}^n,$$

$$\varphi\colon \mathbf{R}^m \to \mathbf{R}^q \ (q \leq m).$$

Denote by X the space of piecewise C^1 functions mapping T to $\mathbf{R}^n$, by $\mathcal{U}_k$ the space of piecewise continuous functions mapping T to $\mathbf{R}^k$ ($k \in \mathbf{N}$), and set $Z := X \times \mathcal{U}_m$,

The set of *endpoint constraints* will be expressed in terms of

$$C := \{x \in \mathbf{R}^n \mid h(x) = 0\},$$

and the set of *control constraints* will be given by

$$U := \{u \in \mathbf{R}^m \mid \varphi_\alpha(u) \leq 0 \ (\alpha \in R),$$
$$\varphi_\beta(u) = 0 \ (\beta \in Q)\}$$

where $R = \{1, \ldots, r\}$ and $Q = \{r+1, \ldots, q\}$.

Define the sets

$$D := \{(x, u) \in Z \mid \dot x(t) = f(t, x(t), u(t)) \ (t \in T),$$
$$x(t_0) = \xi_0,\ x(t_1) \in C\},$$

$$S := \{(x, u) \in D \mid u(t) \in U \ (t \in T)\},$$

and let $I\colon Z \to \mathbf{R}$ be given by

$$I(x, u) := g(x(t_1)) + \int_{t_0}^{t_1} L(t, x(t), u(t))dt.$$

The problem we shall deal with, which we label (P), is that of minimizing I over S.

Elements of Z will be called *processes*, of S *admissible processes*, and a process (x, u) *solves* (P) if (x, u) is admissible and $I(x, u) \leq I(y, v)$ for all admissible processes (y, v).

Given $(x, u) \in Z$ we shall use the notation $(\tilde x(t))$ to represent $(t, x(t), u(t))$, and the symbol '$*$' will denote transpose.

With respect to the functions delimiting the problem, we assume that, if $F := (L, f)$, then $F(t, \cdot, \cdot)$ is C^1 for all $t \in T$ and g, h, φ are C^1; $F(\cdot, x, u)$, $F_x(\cdot, x, u)$ and $F_u(\cdot, x, u)$ are piecewise continuous for all $(x, u) \in \mathbf{R}^n \times \mathbf{R}^m$; and there exists an integrable function $\alpha\colon T \to \mathbf{R}$ such that, at any point $(t, x, u) \in T \times \mathbf{R}^n \times \mathbf{R}^m$,

$$|F(t, x, u)| + |F_x(t, x, u)| + |F_u(t, x, u)| \leq \alpha(t).$$

These assumptions are standard for the derivation of first order necessary conditions (see, for example, [11]). Also, we assume that $h'(x(t_1))$ $(x \in X)$, and the matrix $(\varphi_i'(u(t)))$ $(i \in I(u(t)) \cup Q,\ t \in T)$ are of full rank, where

$$I(u) := \{i \in R \mid \varphi_i(u) = 0\}$$

denotes the set of *active inequality indices*.

For any $x \in \mathbf{R}^n$ let

$$\mathcal{N}(x) := \{p \in \mathbf{R}^n \mid p = h'(x)^*\gamma \text{ for some } \gamma \in \mathbf{R}^k\}.$$

Note that, in view of the full rank assumption on h, the set $\mathcal{N}(x)$ corresponds to the normal cone associated to the (adjacent) tangent cone to C at x.

Denote by $\mathcal{E}$, whose elements will be called *extremals*, the set of all $(x, u, p, \mu) \in Z \times X \times \mathcal{U}_q$ satisfying

i. $\mu_\alpha(t) \geq 0$ with $\mu_\alpha(t)\varphi_\alpha(u(t)) = 0$ $(\alpha \in R,\ t \in T)$;
ii. $\dot{p}(t) = -f_x^*(\tilde{x}(t))p(t) + L_x^*(\tilde{x}(t))$ $(t \in T)$;
iii. $p^*(t)f_u(\tilde{x}(t))) = L_u(\tilde{x}(t))) + \mu^*(t)\varphi'(u(t))$ $(t \in T)$;
iv. $-[p(t_1) + g'(x(t_1))^*] \in \mathcal{N}(x(t_1))$.

As in the finite dimensional case, we shall impose a normality condition in order to have extremality as a necessary condition for optimality.

An admissible process (x, u) will be said to be *normal relative to* S if, given $(p, \mu) \in X \times \mathcal{U}_q$ satisfying

i. $\mu_\alpha(t) \geq 0$ and $\mu_\alpha(t)\varphi_\alpha(u(t)) = 0$ $(\alpha \in R,\ t \in T)$;
ii. $\dot{p}(t) = -f_x^*(\tilde{x}(t))p(t)$ $(t \in T)$;
iii. $f_u^*(\tilde{x}(t))p(t) = \varphi'^*(u(t))\mu(t)$ $(t \in T)$;
iv. $-p(t_1) \in \mathcal{N}(x(t_1))$

then $p \equiv 0$. Note that, in this event, also $\mu \equiv 0$.

Under normality assumptions, we have the following well-known result on first order necessary conditions for problem (P) (see, for example, [7]).

Theorem 3 *Suppose* (x_0, u_0) *solves* (P) *and is a normal process of* S*. Then there exists* $(p, \mu) \in X \times \mathcal{U}_q$ *such that* $(x_0, u_0, p, \mu) \in \mathcal{E}$*.*

Let us now introduce the corresponding set $S_1(\mu)$. Given $\mu \in \mathcal{U}_q$ with $\mu_\alpha(t) \geq 0$ $(\alpha \in R,\ t \in T)$, let

$$S_1 := S_1(\mu) = \{(x, u) \in D \mid$$
$$\varphi_\alpha(u(t)) \leq 0$$
$$(\alpha \in R,\ \mu_\alpha(t) = 0,\ t \in T),$$
$$\varphi_\beta(u(t)) = 0$$
$$(\beta \in R \text{ with } \mu_\beta(t) > 0, \text{ or } \beta \in Q,\ t \in T)\}.$$

Clearly, we have that

$$S_1 = \{(x, u) \in S \mid$$
$$\varphi_\alpha(u(t)) = 0\ (\alpha \in R,\ \mu_\alpha(t) > 0,\ t \in T)\}.$$

Applying the definition of normality to this set of constraints, we obtain the following.

Remark 4 *A process* $(x, u) \in S$ *is normal relative to* $S_1(\mu)$ *if, given* $(q, \nu) \in X \times \mathcal{U}_q$ *satisfying*

i. $\nu_\alpha(t) \geq 0$ *and* $\nu_\alpha(t)\varphi_\alpha(u(t)) = 0$ $(\alpha \in R,\ \mu_\alpha(t) = 0,\ t \in T)$*;*
ii. $\dot{q}(t) = -f_x^*(\tilde{x}(t))q(t)$ $(t \in T)$*;*
iii. $f_u^*(\tilde{x}(t))q(t) = \varphi'^*(u(t))\nu(t)$ $(t \in T)$*;*
iv. $-q(t_1) \in \mathcal{N}(x(t_1))$

then $q \equiv 0$*. In this event, we also have* $\nu \equiv 0$*.*

Let us now prove that, given an extremal, normality relative to S_1 (depending on the extremal) implies uniqueness of the Lagrange multipliers.

Theorem 5 *Let* $(x_0, u_0) \in S$ *and suppose there exists* $(p, \mu) \in X \times \mathcal{U}_q$ *such that* $(x_0, u_0, p, \mu) \in \mathcal{E}$*. If* (x_0, u_0) *is normal relative to* $S_1(\mu)$*, then* (p, μ) *is the unique pair in* $X \times \mathcal{U}_q$ *such that* $(x_0, u_0, p, \mu) \in \mathcal{E}$*.*

Proof: Suppose $(\bar{p}, \bar{\mu}) \in X \times \mathcal{U}_q$ is such that $(x_0, u_0, \bar{p}, \bar{\mu}) \in \mathcal{E}$ and set

$$(q, \nu) := (\bar{p} - p, \bar{\mu} - \mu).$$

Let us show that (q, ν) satisfies the four conditions of Remark 4. If $\alpha \in R$ with $\mu_\alpha(t) = 0$, then

$$\nu_\alpha(t) = \bar{\mu}_\alpha(t) \geq 0,$$
$$\nu_\alpha(t)\varphi_\alpha(u_0(t)) = \bar{\mu}_\alpha(t)\varphi_\alpha(u_0(t)) = 0$$

and so 4(i) holds. Conditions 4(ii) and 4(iii) hold since

$$\dot{q}(t) = \dot{\bar{p}}(t) - \dot{p}(t) = -f_x^*(\tilde{x}_0(t))q(t) \quad (t \in T),$$

$$\begin{aligned} f_u^*(\tilde{x}_0(t))q(t) &= f_u^*(\tilde{x}_0(t))(\bar{p}(t) - p(t)) \\ &= \varphi'^*(u_0(t))\nu(t) \quad (t \in T). \end{aligned}$$

Finally, for some $\bar{\gamma}, \gamma \in \mathbf{R}^k$,

$$-q(t_1) = -\bar{p}(t_1) + p(t_1) = h'(x(t_1))^*(\bar{\gamma} - \gamma)$$

and so $-q(t_1) \in \mathcal{N}(x(t_1))$. Since (x_0, u_0) is normal relative to $S_1(\mu)$, $(q, \nu) \equiv (0, 0)$. ∎

Let us now provide an example showing that, contrary to the nonlinear mathematical problem posed in Section 2, the converse of this result may not hold.

Example 6 Consider the problem of minimizing $I(x,u) = \int_{-1}^{1} tu(t)dt$ subject to

$$\dot{x}(t) = u^3(t) \ \ (t \in [-1,1]), \quad x(-1) = x(1) = 0,$$

$$u^2(t) \leq 1 \ \ (t \in [-1,1]).$$

In this case, we have $R = \{1\}$, $Q = \emptyset$,

$$h(x) = x, \quad f(t,x,u) = u^3,$$

$$L(t,x,u) = tu, \quad \varphi(u) = u^2 - 1.$$

Consider the admissible process $(x_0, u_0) \in Z$ given by

$$x_0(t) := \begin{cases} t+1 & \text{if } t \in [-1,0] \\ 1-t & \text{if } t \in (0,1]. \end{cases}$$

$$u_0(t) := \begin{cases} 1 & \text{if } t \in [-1,0] \\ -1 & \text{if } t \in (0,1]. \end{cases}$$

Suppose $(x_0, u_0, p, \mu) \in \mathcal{E}$. By definition of extremals, we have

$$\mu(t) \geq 0, \quad \dot{p}(t) = 0,$$

$$3p(t) = t + 2u_0(t)\mu(t) \quad (t \in [-1,1]).$$

Thus p is a constant satisfying

$$3p = \begin{cases} t + 2\mu(t) & \text{if } t \in [-1,0] \\ t - 2\mu(t) & \text{if } t \in (0,1]. \end{cases}$$

Since $\mu(t) \geq 0$ for all $t \in [-1,1]$, from the first relation we have $3p \geq 0$ and, from the second, $3p \leq t$ for all $t \in (0,1]$ and so $p \leq 0$. Thus $p \equiv 0$ and therefore

$$\mu(t) = \begin{cases} -t/2 & \text{if } t \in [-1,0] \\ t/2 & \text{if } t \in [0,1]. \end{cases}$$

This implies that (p,μ) is the only pair such that $(x_0, u_0, p, \mu) \in \mathcal{E}$.

Now, consider the pair (q,ν) with $q \equiv 2/3$ and $\nu(t) := u_0(t)$ $(t \in [-1,1])$. It satisfies 4(i) since

$$\nu(t) \geq 0 \text{ and } \nu(t)\varphi(u_0(t)) = 0$$

for all $t \in [-1,1]$ such that $\mu(t) = 0$, that is, for $t = 0$. Moreover, $\dot{q}(t) = 0$ and

$$3q(t) = 2u_0(t)\nu(t) = 2$$

for all $t \in [-1,1]$ and so also 4(ii) and 4(iii) hold. Finally, 4(iv) holds since

$$-q(1) = -2/3 = h'(x_0(1))\gamma = \gamma$$

for some $\gamma \in \mathbf{R}$. Since $(q,\nu) \not\equiv (0,0)$, (x_0,u_0) is not normal relative to $S_1(\mu)$. ▮

References:

[1] J. A. Becerril, K. L. Cortez and J. F. Rosenblueth, Uniqueness of multipliers in optimal control: the missing piece, *IMA J. Math. Control Inform.* 2018, doi.org/10.1093/imamci/dny033

[2] A. Ben-Tal, Second-order and related extremality conditions in nonlinear programming, *J. Optim. Theory Appl.* 31, 1980, pp. 143–165.

[3] G. Bigi and M. Castellani, Uniqueness of KKT multipliers in multiobjective optimization, *Appl. Math. Lett.* 17, 2004, pp. 1285–1290.

[4] K. L. Cortez and J. F. Rosenblueth, Normality and uniqueness of Lagrange multipliers, *Discrete Contin. Dyn. Syst.* 38, 2018, pp. 3169–3188.

[5] K. L. Cortez and J. F. Rosenblueth, The broken link between normality and regularity in the calculus of variations, *Syst. Cont. Lett.* 124, 2019, pp. 27–32.

[6] G. Giorgi, A. Guerraggio and J. Thierfelder, *Mathematics of Optimization: Smooth and Nonsmooth Case*, Elsevier, Amsterdam 2004

[7] M. R. Hestenes, *Calculus of Variations and Optimal Control Theory*, John Wiley, New York 1966

[8] M. R. Hestenes, *Optimization Theory, The Finite Dimensional Case*, John Wiley, New York 1975

[9] J. Kyparisis, On uniqueness of Kuhn-Tucker multipliers in nonlinear programming, *Math. Program.* 32, 1985, pp. 242–246.

[10] E. J. McShane, The Lagrange multiplier rule, *Amer. Math. Monthly.* 80, 1973, pp. 922–925.

[11] J. F. Rosenblueth, Convex cones and conjugacy for inequality control constraints, *J. Convex Anal.* 14, 2007, pp. 361–393.

[12] J. F. Rosenblueth, Normality and quasinormality in nonlinear programming and optimal control, *WSEAS Trans. Syst. Control.* 13, 2018, pp. 510–513.

[13] A. Shapiro, On uniqueness of Lagrange multipliers in optimization problems subject to cone constraints, *SIAM J. Optim.* 7, 1997, pp. 508–518.

[14] G. Wachsmuth, On LICQ and the uniqueness of Lagrange multipliers, *Oper. Res. Lett.* 41, 2013, pp. 78–80.