Considerations on State Space Analysis

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Abstract: - This paper presents some considerations on the mathematical description as well as the assay and synthesis systems in state space. The mathematical model of the systems consists of a set of equations describing the system trajectory and how the system behaves from some points of view. After obtaining the mathematical model and transfer matrix, we analysed the system both in terms of controllability and observability, as well as its response in time and frequency. This analysis was facilitated by the use of the Matlab programming environment.

Key-Words: - state space, mathematical model, controllability, observability, time response, frequency response

1 Introduction

The output of a system is affected by the inputs before the time t_0. Taking into account the inputs from the time point t=-∞ is difficult to achieve, so a new concept, namely the state variable, will be introduced. Status variables represent a group of sizes that completely define the status of the system at a time. These variables also fulfil the role of initial conditions for the evolution of the previous system. By definition, the state $x(t_0)$ of the system at the time t_0 is the information available that together with the input u(t) for $t \geq t_0$, uniquely determines the output y(t) of the system for $t \geq t_0$. Thus, if the state of the system is known at the time t_0 of its determination, y(t) to $t \geq t_0$, it is no longer necessary to know the inputs applied before the time t_0. State variables are selected as output signal, along with its derivatives in relation to time.

We will write the general form of the system input-state-output equations with matrices A, B, C and D independent of time with the meaning of matrices.

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \tag{1}$$

where:
A - the system matrix (nxn)
B - the input matrix (nxm)
C - the output matrix (pxn)
C - the direct connection matrix (pxn)

2 Mathematical model for the circuit RLC

Using the transformed Laplace for the system (1) we obtain:

$$\begin{aligned}
&\begin{cases} sX(s) - x(0) = AX(s) + BU(s) \\ Y(s) = CX(s) + DU(s) \end{cases} \Rightarrow \\
&\begin{cases} sX(s) - AX(s) = BU(s) + x(0) \\ Y(s) = CX(s) + DU(s) \end{cases} \Rightarrow \\
&\Rightarrow \begin{cases} X(s) = \dfrac{BU(s) + x(0)}{sI - A} \\ Y(s) = CX(s) + DU(s) \end{cases} \Rightarrow \\
&\begin{cases} X(s) = (sI - A)^{-1} BU(s) + (sI - A)^{-1} x(0) \\ Y(s) = CX(s) + DU(s) \end{cases} \Rightarrow
\end{aligned} \tag{2}$$

Replace equations X(s) in equation Y(s):

$$\begin{aligned}
&\Rightarrow Y(s) = C\left[(sI - A)^{-1} BU(s) + (sI - A)^{-1} x(0)\right] + DU(s) \Rightarrow \\
&\Rightarrow Y(s) = C(sI - A)^{-1}\left[BU(s) + x(0)\right] + DU(s) \Rightarrow \\
&\Rightarrow Y(s) = C(sI - A)^{-1} x(0) + \left[C(sI - A)^{-1} B + D\right]U(s)
\end{aligned} \tag{3}$$

If we consider x(0)=0 and know $G(s) = \dfrac{Y(s)}{U(s)}$ that we will determine the transfer function or the transfer matrix (it is also called because it is calculated on the basis of matrix operations).

$$G(s) = C(sI - A)^{-1} B + D \tag{4}$$

where: I - is the unit matrix

In RLC circuits, capacitors can store energy and can therefore be associated with state variables. Exit state input equations and equivalent transfer function are determined. The voltage across the capacitor can be expressed based on the physical law of its operation: $u_C = \frac{1}{C}\int i(t)dt$ deriving this relationship we will get: $\dot{u}_C = \frac{1}{C} i(t) \Rightarrow i(t) = C\dot{u}_C$. If the capacitor voltage can be considered as a state variable x, then the current passing through the capacitor will be $C\dot{x}$, where C is capacitor capacitance.

For the coil we will have: $u_L(t) = L\frac{di(t)}{dt}$. If the current through the coil is associated with a state x variable, then the voltage on the coil is $L\dot{x}$ where L is the inductance of the coil. It is worth mentioning that the resistors have elements without memory, the voltage on them and the current flowing through them cannot be considered state variables. Having the assigned status variables, Kirchhoff's laws are used on RLC circuits to obtain status equations.

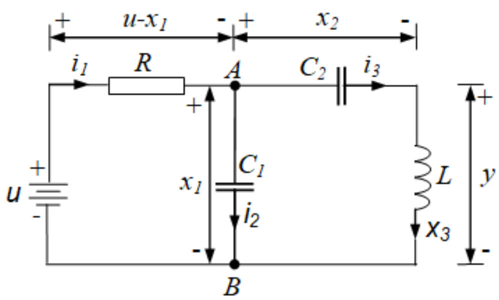

Fig.1 RLC circuit

So for the circuit in figure 1 we will have state variables voltage on the capacitors C1, C2, and the voltage at the terminals of the resistor R, and the output size will be the current passing through the coil L.

$$i_1 = \frac{u - x_1}{R}, \quad i_2 = C_1\dot{x}_1 = x_2, \quad i_3 = C_2\dot{x}_2 = x_3, \qquad (5)$$
$$y = L\dot{x}_3$$

Applying Kirchhoff's law for points A and B we will have:

$$i_3 = C_2\dot{x}_2 = x_3$$
$$i_1 = i_2 + i_3 \Rightarrow \frac{u - x_1}{R} = C_1\dot{x}_1 + C_2\dot{x}_2 = C_1\dot{x}_1 + x_3 \qquad (6)$$

From equations (6) we obtain:

$$u - x_1 = R(C_1\dot{x}_1 + x_3) = RC_1\dot{x}_1 + Rx_3 \Rightarrow -RC_1\dot{x}_1 = Rx_3 + x_1 - u$$
$$\dot{x}_1 = -\frac{Rx_3}{RC_1} - \frac{x_1}{RC_1} + \frac{u}{RC_1} = -\frac{x_3}{C_1} - \frac{x_1}{RC_1} + \frac{u}{RC_1}$$
$$x_3 = C_2\dot{x}_2 \qquad (7)$$
$$\dot{x}_2 = \frac{x_3}{C_2}$$

If we apply Kirchoff's law for the circuit loop containing the coil, we get:

$$L\dot{x}_3 = x_1 - x_2$$
$$\dot{x}_3 = \frac{x_1 - x_2}{L} = \frac{x_1}{L} - \frac{x_2}{L} \qquad (8)$$

The output will be:

$$y = L\dot{x}_3 = x_1 - x_2 \qquad (9)$$

Writing in matrix form we have the following system of state equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{1}{RC_1} & 0 & -\frac{1}{C_1} \\ 0 & 0 & \frac{1}{C_2} \\ \frac{1}{L} & -\frac{1}{L} & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{RC_1} \\ 0 \\ 0 \end{bmatrix}}_{B} u \qquad (10)$$
$$y = \underbrace{\begin{bmatrix} 1 & -1 & 0 \end{bmatrix}}_{C}$$

3 Determining the transfer matrix and testing system controllability and observability

To determine the equivalent transfer function we will use the following Matlab code sequence:

```
syms s R C1 C2 L
A=[-1/(R*C1) 0 -1/C1; 0 0 1/C2; 1/L -1/L 0]
B=[1/(R*C1); 0; 0]
C=[1 -1 0]
D=[0]
Phi=inv(s*eye(3)-A)
G=C*Phi*B+D
pretty(simple(G))
```

We will get matrices for the input-state-output system and transfer function:

$$A = \begin{bmatrix} -0.0333 & 0 & -3.3333 \\ 0 & 0 & 10 \\ 5 & -5 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.0333 \\ 0 \\ 0 \end{bmatrix}, \qquad (11)$$
$$C = [1 \quad -1 \quad 0], \quad D = [0]$$

$$G(s) = \frac{LC_2 s^2}{RLC_1C_2s^3 + LC_2s^2 + (RC_1 + RC_2)s + 1} \qquad (12)$$

or

syms s
R=100
C1=0.3
C2=0.1
L=0.2
*num=[L*C2 0 0]*
*den=[R*L*C1*C2 L*C2 R*(C1+C2) 1]*
G=tf(num, den)

we obtain:

$$G(s) = \frac{0.02s^2}{0.6s^3 + 0.02s^2 + 40s + 1} \qquad (13)$$

3.1 System controllability

A system is controllable if, for any initial state, an input vector can be found to determine the evolution of the state to any desired final value.

It is considered a system described by the equations (1). Because the output of the system has nothing to do with the controllability property, only the first equation will be referred. A system is fully controllable if there is an input vector u (t) that transfers the system from the initial state x (t0) to the final state, whatever the initial state x (t0) = x0 and the final state x (t1) x (t0) in a finite time. A system is a partially controllable or uncontrollable state if the x1 component of the state is controllable, and the x2 component of the state is uncontrollable.

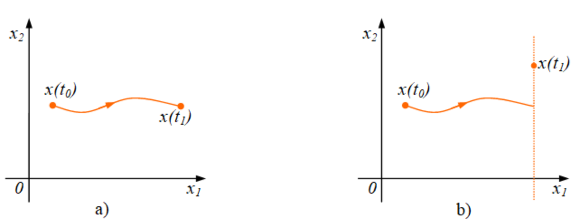

Fig. 2 System controllability.
a) fully controllable state system, b) partially controllable state system

Testing the controllability with formula (14).

$$C = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \qquad (14)$$

where C is the controllability matrix formed by the sub-matrices A^kB, k = 0,1, ..., n-1

The system is fully controllable if and only if rank C=n or det C≠0.

3.1 System observability

A system is observable if the evolution of the inputs and outputs is known over a time interval and the state function can be deduced over the time interval considered. In other words, observability is the property of dynamic systems that highlight the possibility of estimating the state of the system by knowing its output.

It is considered a system described by the equations (1). A system is fully observable if, for any t0, the state vector x(t0) can be fully determined based on the knowledge of the input vector u (t) and the output vector y(t) on the interval [t0, t1] with t1> t0≥0.

Testing for a system observability is performed with formula (15).

$$O = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \qquad (15)$$

where O is the observability matrix formed by subclasses CA^k, k = 0,1, ..., n-1.

A system is fully observable and only if the rank O=n or det O ≠ 0.

We tested the system controllability and observability with the following Matlab program:

syms s t
A = [-0.0333 0 -3.3333; 0 0 10; 5 -5 0];
B=[0.0333; 0;0]
C=[1 -1 0];
CO=ctrb(A,B)
OB=obsv(A,C)
d1=det(CO)
d2=det(OB)
if ((d1==0)&(d2==0))
disp ('The system not is controllable and nor is observable')
elseif ((d1==0)&(d2~=0))

disp('The system is observable, but not is controllable')
elseif ((d1~=0)&(d2==0))
disp('The system is controllable, but not is observable')
elseif ((d1~=0)&(d2~=0))
disp('The system is controllable and observable')
end

we will obtain d_1=-0.0092 and d_2=0.0111 and matrices that controllability and observability (16):

$$CO = \begin{bmatrix} 0.0333 & 0 & -0.555 \\ 0 & 0 & 1.665 \\ 0 & 0.1665 & -0.0055 \end{bmatrix} \quad and$$

$$OB = \begin{bmatrix} 1 & -1 & 0 \\ -0.0333 & 0 & -13.3333 \\ -66.6654 & 66.6665 & 0.111 \end{bmatrix} \qquad (16)$$

Resulting: *The system is controllable and observable*

4 System analysis in time domain and frequency domain

4.1 System response in time domain

Considering the above system, to see the system time response we use the following Matlab code sequence:

```
syms s
R=100
C1=0.3
C2=0.1
L=0.2
num=[L*C2 0 0]
den=[R*L*C1*C2 L*C2 R*(C1+C2) 1]
G=tf(num, den)
figure(1)
step(G)
grid
figure(2)
impulse(G)
grid
```

we obtain:

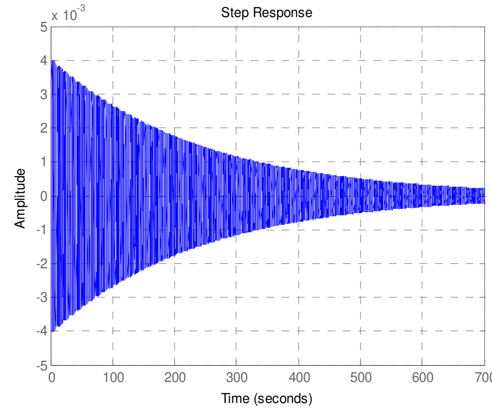

Fig. 3 Step response

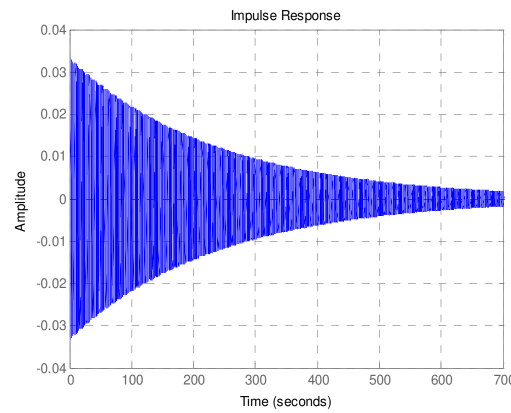

Fig. 4 Impulse response

4.1 System response in frequency domain

For the above system, to see the frequency response of the system, we use the following Matlab code sequence:

```
syms s
R=100
C1=0.3
C2=0.1
L=0.2
num=[L*C2 0 0]
den=[R*L*C1*C2 L*C2 R*(C1+C2) 1]
G=tf(num, den)
figure(1)
nyquist(G)
grid
figure(2)
bode(G)
grid
```

we obtain:

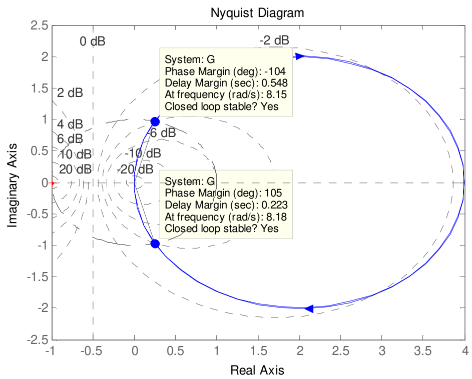

Fig. 5 Nyquist diagram

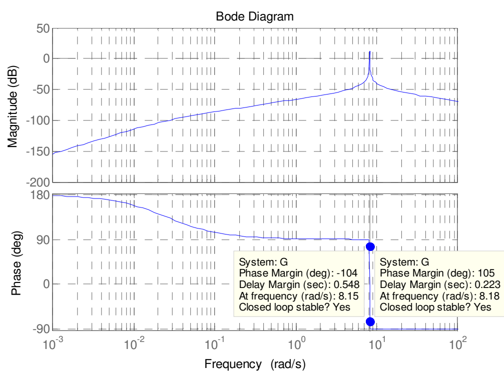

Fig. 6 Bode diagram

5 Conclusion

The analysis of the trajectory of the electrical systems with the Matlab matrix calculation program has functions dedicated to the system analysis using the controllability matrix - ctrb and observability matrix - obsv. The ranges of these matrices will give us information about the performance of the system, and the correlation with the time and frequency domain analysis will complement the overall image of the system under consideration as well as the boundaries of its simulation domain. Thus, prototyping a system will have to go through the steps of analyzing the space of the states listed above in order to shorten the start-up time of the analyzed system.

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