Optimal Control Based on Fuzzy Estimation of Takagi-Sugeno Model for the Furuta Pendulum: Experimental Results

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Abstract: The aim of this paper is an experimental study of a discrete-time fuzzy optimal controller based on blending of Takagi-Sugeno (T-S) fuzzy modeling and the Linear Quadratic Regulator (LQR) for an Underactuated Mechanical System. The proposed scheme combines the optimality of the LQR in terms of a desired behavior dynamic of the system with admissible control actions and, the approximation capability of nonlinear functions of T-S fuzzy model. From an input-output data a T-S fuzzy model based on an improved approach of fuzzy identification is estimated and, an extended optimal state feedback control is used in order to control stabilization and to guarantee reference tracking of Furuta's Pendulum. For the purpose of validating the controller scheme proposed experimental tests on QNET Rotary Inverted Pendulum Trainer for NI ELVIS are carried out. The performance of designed controller is compared against the classical LQR using an Integral Square Error (ISE) index.

Key–Words: Takagi-Sugeno Fuzzy Model, Nonlinear Control, Pendulum of Furuta, Underactuated Mechanical System, LQR.

1 Introduction

An underactuated mechanical system (UMS) has more degrees of freedom to be controlled than independent control actuators [1]. The effects of unmodeled dynamics, external forces and, non-holonomic behavior of non-directly actuated variables represent a challenge when experimental implementation of a synthesized control scheme is required [2]. UMS are being used in a lot applications, such as: mobile robot systems, aerial vehicle systems, underwater vehicle, [1, 3–5].

Rotational inverted pendulum also called Furuta's Pendulum (FP) [6] is the most popular UMS, which has been widely studied by control researchers. The FP consists of an actuated arm which rotates in a horizontal plane, and a non-actuated rotating pendulum in the vertical plane and it only has an input actuator that provides torque at the arm. The pendulum of Furuta has been used for testing linear and non-linear control techniques and has become a benchmark challenge problem for designing of control systems. Due to it is a multivariable system and, it has a highly nonlinear and unstable dynamic behavior [7, 8].

The classical control objectives studied in the literature for FP are mainly: (1) balancing up the inverted pendulum until it reaches the unstable position (swing-up), (2) maintaining the stabilization at this position and trajectory tracking of the inverted pendulum arm. Thus, several linear and non-linear control techniques have been tested. For instance, in [9], a nonlinear optimal (H-∞) control to FP using an approximate linearization around temporary equilibrium is developed. Using a Lyapunov stability theorem for stabilization control of rotary inverting pendulum is designed in [10]. The swing-up problem has been addressed using different approaches based on energy control [11, 12]. In order to solve the tracking problem of FP several new interesting approach have been presented, for example, a composite scheme based on input-output linearization and energy function of the system is proposed in [13], feedback linearization controller techniques have been introduced in [14, 15], an adaptive network-based control scheme is proposed in [16].

On the other hand, the optimal control has been studied and was satisfactory applied on UMS [17, 18]. Linear Quadratic Regulator is an optimal control and it has been successfully used for the stabilization of a rotating inverted pendulum [19–21], where LQR provides a compromise between the performance of the variables to be controlled and the control action effort. However, designing a LQR requires a linear model of the system, which limits the performance of this controller against highly non-linear systems.

An interesting method of fuzzy systems identification using input/output data is presented in [22]. Takagi-Sugeno (T-S) fuzzy modeling has been wide used on non-linear control systems [23–25], due to its ability to approximate any non-linear function by suitable linear subsystems. It has presented versatility when it has been blended with different kinds of controllers, such as: Sliding mode controllers [26], optimal [27, 28],and predictive model controllers [29].

Some approaches to T-S fuzzy modeling have been implemented in FP, in [30] the inverted rotary pendulum is modeled using robust fuzzy Takagi-Sugeno descriptors, from this model a stabilization control based on a LQR is designed. A T-S fuzzy model is constructed from a simplified system based on nonlinear sector is proposed in [31], then a Parallel Distributed Compensation (PDC) controller is designed with control input constraint using linear matrix inequality (LMI) toolbox of MatLab. In these two works the T-S fuzzy model is derived from simplified nonlinear model , however in practical applications is difficult to obtain an analytic model because only input/output data of the process is accessible.

This article contains two main parts, firstly an experimental study is carried out based on several tests on the FP, then from input/output data a Takagi-Sugeno model is estimated by fuzzy identification system [35, 36]. Secondly, the T-S fuzzy model is blending with an optimal controller based on an extended LQR controller (FCL-LQR) in order to guarantee a stabilization and reference tracking control of the system.

The article is organized as follows, in section 2 a brief description and nonlinear model of FP is presented. An identification T-S fuzzy modeling method for multivariable system is presented in section 3. In section 4 an optimal controller based on an extended LQR controller is designed based on T-S fuzzy model. In section 5 a T-S fuzzy model and LQR extended control laws are obtained using MatLab, and experimental results based on step change references, parametric variation and external disturbance tests are carried out. In section 6 conclusions are presented.

2 Furuta Pendulum Nonlinear Model

In this work, QNET Rotary Inverted Pendulum Trainer for NI ELVIS [32] is used. In Fig.1, the scheme of FP is illustrated; this is composed of a pillar with moment of inertia J, connected to a horizontal arm of length l_a of mass m_a and a pendulum of length l_p of mass m_p. By considering that the weights are evenly distributed.

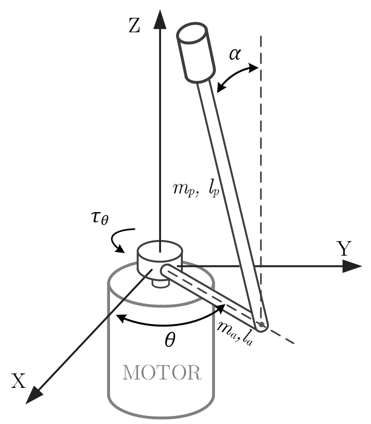

Figure 1: Furuta's pendulum scheme

where, α is pendulum angular position and θ is arm angular position. Taking the states vector as follows:

$$[x_1 \quad x_2 \quad x_3 \quad x_4]^T = \begin{bmatrix}\theta & \dot{\theta} & \alpha & \dot{\alpha}\end{bmatrix}^T \tag{1}$$

the space-state representation of nonlinear FP model [20] is given by:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{\beta\eta x_2{}^2\left(\sin^2 x_3 - 1\right)\sin x_3}{\varphi\beta - \eta^2 + (\beta^2+\eta^2)\sin^2\alpha} \\
&\quad - \frac{2\beta^2 x_2 x_4 \cos x_3 \sin x_3}{\varphi\beta - \eta^2 + (\beta^2+\eta^2)\sin^2\alpha} \\
&\quad + \frac{\beta\eta\dot{x}_3{}^2 \sin x_3 - \eta\delta\cos x_3 \sin x_3 + \beta\tau_\theta}{\varphi\beta - \eta^2 + (\beta^2+\eta^2)\sin^2\alpha} \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{\beta x_1{}^2\left(\varphi + \beta\sin^2 x_3\right)\cos x_3 \sin x_3}{\varphi\beta - \eta^2 + (\beta^2+\eta^2)\sin^2\alpha} \\
&\quad + \frac{2\beta\eta\dot{x}_1\dot{x}_3\left(1 - \sin^2 x_3\right)\sin x_3}{\varphi\beta - \eta^2 + (\beta^2+\eta^2)\sin^2\alpha} \\
&\quad - \frac{\eta^2\dot{x}_3{}^2\cos x_3 \sin x_3}{\varphi\beta - \eta^2 + (\beta^2+\eta^2)\sin^2\alpha} \\
&\quad + \frac{\delta\left(\varphi + \beta\sin^2 x_3\right)\sin x_3 - \eta\tau_\theta\cos x_3}{\varphi\beta - \eta^2 + (\beta^2+\eta^2)\sin^2\alpha}
\end{aligned} \tag{2}$$

with:

$$\beta = \frac{1}{3}m_p l_p^2, \quad \eta = \frac{1}{2}m_p l_a l_p, \quad \delta = \frac{1}{2}m_p g l_p$$

$$\tau_\theta = \frac{K_e}{R}V - \frac{K_e K_t}{R}\dot{\theta},\ \varphi = J + \left(\frac{m_a}{3} + m_p\right)l_a^2$$

Table 1 Furuta′s Pendulum Parameters [20]

Parameter	Unit	Description
$m_p = 0.027$	$[Kg]$	Mass of pendulum and weight combined
$m_a = 0.008$	$[Kg]$	Mass of the horizontal arm
$l_p = 0.191$	$[m]$	Total pendulum length
$l_a = 0.0826$	$[m]$	Length of horizontal arm
$J = 1.8e^{-6}$	$[kg \cdot m^2]$	Motor-rotor moment of inertia
$g = 9.8$	$[m/s^2]$	Gravity
$K_t = 0.03334$	$[N \cdot m]$	Motor current-torque constant
$K_e = 0.03334$	$[N \cdot m]$	Motor back-emf constant
$R_m = 8.6$	$[\Omega]$	Motor armature resistance

The nonlinear FP model (2) considers the input of system is the DC motor voltage (V). The physical parameters and its values of Quanser QNET Rotary Inverted Pendulum are listed in the Table 1, [32].

3 Identification of Takagi-Sugeno (T-S) Fuzzy Model

In [22] a fuzzy identification method of systems from data is presented. The idea of fuzzy system identification is to estimate of a multivariable nonlinear system parameters by minimizing a quadratic performance index. The identification of vectorial function is equivalent to m scalar functions:

$$y_j = f_j\left(x_1, x_2, ..., x_n\right) \quad y \,\epsilon\, \Re, \quad j = 1, .., m \quad (3)$$

where y_j is each one system output. By choosing $[r_1, r_2, ..., r_n]$ numbers of fuzzy sets of $[x_1, x_2, ..., x_n]$ measurable variables of multivariable nonlinear system. A multivarible fuzzy system $Sys_j^{(i_1 ... i_n)}$ can be defined for nth order system as follows:

$Sys_j^{(i_1 ... i_n)}$:

$IF\ x_1(k)\ is\ M_{j1}^{i_1}\ and\ x_2(k)\ is\ M_{j2}^{i_2}\ and \cdots x_n(k)\ is\ M_{jn}^{i_n}\quad THEN:$

$$\hat{y}_j(k) = p_{j0}^{(i_1..i_n)} + p_{j1}^{(i_1..i_n)} x_1 + p_{j2}^{(i_1..i_n)} x_2 + \cdots + p_{jn}^{(i_1..i_n)} x_n \quad (4)$$

where $\hat{y}_j$ is each system output estimated. For each IF-THEN rule, a discrete-time state-space linear model with affine term is represented by the application of equation (4) as:

$Sys_j^{(i_1 ... i_n)}$:

$IF\ x_1(k)\ is\ M_{j1}^{i_1}\ x_2(k)\ is\ M_{j2}^{i_2}\ and \cdots x_n(k)\ is\ M_{jn}^{i_n}\quad THEN:$

$$x(k+1) = a_0{}^{(i_1..i_n)} + A^{(i_1..i_n)} x(k) + B^{(i_1..i_n)} u(k) \quad (5)$$

where: $a_0{}^{(i_1..i_n)}\ \epsilon\ \Re^n$, $A^{(i_1..i_n)}\ \epsilon\ \Re^{nxn}$, $B^{(i_1..i_n)}\ \epsilon\ \Re^{nxm}$ and $M_1^{i_1}\ (i_1 = 1, 2, \cdots, r_1)$ are the fuzzy sets of $x_1(k)$, $M_n^{i_1}\ (i_1 = 1, 2, \cdots, r_1)$ are the fuzzy sets of $x_n(k)$ and, $u(k)$ is the input system.

The fuzzy system is described in (6):

$$x(k+1) = \sum_{i_1}^{r_1} \cdots \sum_{i_n=1}^{r_n} \beta^{(i_1..i_n)}(x)\left[a_0{}^{(i_1..i_n)} + A^{(i_1..i_n)} x(k) + B^{(i_1..i_n)} u(k)\right] \quad (6)$$

where:

$$\beta^{(i_1..i_n)}\left(x_{(i_1..i_n)}(k)\right) = \frac{u_{1i_1}(x_1) \cdots u_{ni_n}(x_n)}{\sum_{i_1}^{r_1} \cdots \sum_{i_n=1}^{r_n} \left(u_{1i_1}(x_1) \cdots u_{ni_n}(x_n)\right)} \quad (7)$$

where $u_{ji_j}(x_j)$ is the membership function that corresponds to the fuzzy set of $M_j^{i_j}$.

3.1 Estimation of T-S fuzzy model parameters

The traditional method based on the estimation of T-S fuzzy model parameters [22] presents problems and it can not be applied to the most common case where triangular membership functions of a fuzzy system are overlapping by pairs. For this case, it is easy to demonstrate that the T-S matrix is not of full rank, therefore it is not invertible [35]. Thus, in [35], [36] an efficient method to improve the identification and estimation of parameters T-S fuzzy model was developed, this approach is simple, of low computational cost and it is based on the tuning and weighting of parameters.

In order to tune T-S model parameters; an input/output data set $\{x_{1k}, \cdots, x_{nk}, y_{1k}, , \cdots, y_{1m}\}$ from k samples are used to obtain affine linear model parameters (8). These are estimated around the equilibrium point using for instance the least squares method:

$$p_0 = \begin{bmatrix} p_0^0 & p_1^0 & p_2^0 & \cdots & p_n^0 \end{bmatrix}^t \quad (8)$$

This first approximation could be used as reference for all subsystems. Then, the monovariable T-S fuzzy model can be obtained minimizing [28]:

$$J = \sum_{k=1}^{m} (y_k - \hat{y}_k)^2 + \left(p_j^0 - p_j^{(i_1..i_n)}\right)^2 \sum_{i_1=1}^{r_1} \cdot\cdot \sum_{i_n=1}^{r_n} \sum_{j=0}^{n} \gamma_j^{(i_1..i_n)2} = \|Y - XP\|^2 + \|\Gamma(p_0 - p)\|^2 \tag{9}$$

where γ is the factor that represents the degree of confidence of the linear estimated parameters. In this case weights factor $\gamma_j^{(i_1..i_n)}$ are the same for initial parameters p_j^0 in each rule IF-ELSE. The Γ is a diagonal matrix with weights factor $\gamma_j^{(i_1..i_n)}$. For a multivariable system the parameters tuning is given by the index:

$$J_j = \left\| \begin{bmatrix} Y_j \\ \Gamma_j p_{j0} \end{bmatrix} - \begin{bmatrix} X_j \\ \Gamma_j \end{bmatrix} P_j \right\|^2 \tag{10}$$

where: Y_j is the output, X_j is the input/output fuzzy data, P_j is the fuzzy T-S model parameters. p_0 is the linear estimated parameters, which are repeated as many times as fuzzy rules. It is usual in T-S fuzzy model that the premise of fuzzy sets of all subsystems are the same, in this case:

$$X_1 = X_2 = \cdots X_m = X \tag{11}$$

Thus, all subsystems can be grouped as:

$$J = \left\| \begin{bmatrix} Y_1 & Y_2 & Y_m \\ \Gamma p_{10} & \Gamma p_{20} & \Gamma p_{m0} \end{bmatrix} - \begin{bmatrix} X \\ \Gamma \end{bmatrix} \begin{bmatrix} P_1 & P_2 & P_m \end{bmatrix} \right\|^2 = \|Y_a - X_a P\|^2 \tag{12}$$

Finally, it should be noted that the X_a matrix is of full rank, therefore the problem of traditional T-S model is solved. The parameters of this model can be calculated by [35]:

$$P = \left(X_a^t X_a\right)^t X_a^t Y_a \tag{13}$$

where,the matrix P contains the matrices of the T-S fuzzy model for each IF-THEN rule.

4 Design of a Fuzzy Optimal Controller

In order to get a control law $u(k)$ that allows the system going from an initial state $x(k_0)$ to a final state $(x(k_1) = 0)$ with a dynamic behavior, a discrete optimal state feedback controller based on Linear Quadratic Regulator (LQR) is chosen [20]. The LQR method is optimal for linear systems, however in the case of nonlinear systems it is complex to propose a minimization of an objective function for the global system, for which reason the solution will be suboptimal. In this methodology, global stability is not guaranteed and which needs be analyzed a posteriori [27].

By considering any subsystem represented by equation (5) the affine term $a_0{}^{(i_1..i_n)}$ can be considered as a perturbation and in order to guarantee a zero steady state error, an integral action control based on error signal is added:

$$w(k+1) = w(k) + x_r - x(k+1) \tag{14}$$

which allows to rewrite an extended system with N new states as:

$$\begin{bmatrix} w(k+1) \\ x(k+1) \end{bmatrix} = \begin{bmatrix} I & -A \\ 0 & A \end{bmatrix} \begin{bmatrix} w(k) \\ x(k) \end{bmatrix} + \begin{bmatrix} -B \\ B \end{bmatrix} u(k) + \begin{bmatrix} I \\ 0 \end{bmatrix} x_r \tag{15}$$

where I is identity matrix and 0 is null matrix. Extended system is considered according to the following nomenclature:

$$x_a = \begin{bmatrix} w(k+1) \\ x(k+1) \end{bmatrix}; \quad A_a = \begin{bmatrix} I & -A \\ 0 & A \end{bmatrix}; \quad B_a = \begin{bmatrix} -B \\ B \end{bmatrix}$$

In order to determine the feedback optimal gains, where the goal is to minimize the cost index J_e with extended states reference $x_{ra} = \begin{bmatrix} 0 & x_r \end{bmatrix}$:

$$J_e = \sum_{k=0}^{\infty} \left[(x_{ra} - x_a(k))^t \begin{bmatrix} Q_w & 0 \\ 0 & Q_x \end{bmatrix} (x_{ra} - x_a(k)) \right] + \sum_{k=0}^{\infty} \left[u(k)^t R u(k) \right] \tag{16}$$

where Q_w and Q_x are non-negative define matrices that penalize the state and state of the error respectively, R is a positive define matrix that penalize the input.

It is well known that state feedback control law is given by:

$$u(k) = K_a (x_{ra} - x_a) \tag{17}$$

where the K_a can be obtained by solving the discrete-time algebraic Riccati equation.

In order to obtain a control law for each IF-ELSE rule $C^{(i_1..i_n)}$ T-S fuzzy model, the feedback optimal gain matrices $K_a{}^{(i_1..i_n)} = \begin{bmatrix} K_w{}^{(i_1..i_n)} & K_x{}^{(i_1..i_n)} \end{bmatrix}$ are obtained for each system $Sys^{(i_1..i_n)}$. The control law of Fuzzy Logic Controller based on T-S fuzzy model and extended LQR is given by:

$C^{(i_1..i_n)}$:

$IF\ \ x_1(k)\ \ is\ \ M_{j1}^{i_1}\ \ and\ \ x_2(k)\ \ is\ \ M_{j2}^{i_2}\ \ and\cdots$
$x_n(k)\ \ is\ M_{jn}^{i_n}\ \ \ THEN:$

$$u(k) = K_w w(k) - K_x (x_r - x(k)) \tag{18}$$

where:

$$\begin{aligned} K_w &= \textstyle\sum_{i_1}^{r_1}\cdots\sum_{i_n=1}^{r_n}\beta^{(i_1..i_n)}K_w{}^{(i_1..i_n)} \\ K_x &= \textstyle\sum_{i_1}^{r_1}\cdots\sum_{i_n=1}^{r_n}\beta^{(i_1..i_n)}K_x{}^{(i_1..i_n)} \end{aligned} \tag{19}$$

and $\beta^{(i_1..i_n)}$ is calculated from equation (7).

In Fig. 2 a general scheme of optimal fuzzy logic controller based on T-S model is shown.

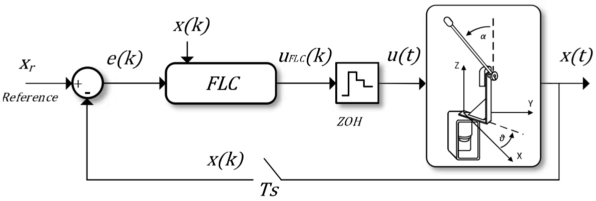

Figure 2: General scheme of fuzzy logic controller based on T-S model

In Fig. 3 an internal scheme of optimal fuzzy logic controller based on T-S model is shown.

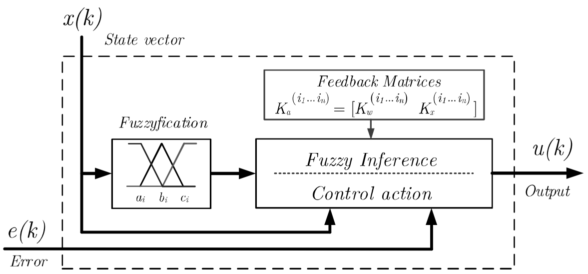

Figure 3: Internal scheme of optimal fuzzy logic controller

5 Results

This section is divided in two parts; in the first one, it is concerned the T-S fuzzy identification and optimal control laws obtained from the methodology described in section (3) and (4). Second part a brief description of the real system is presented and the experimental results are obtained by the proposed controller implemented on Quanser QNET Rotary Inverted Pendulum Board for NI ELVIS.

5.1 T-S Fuzzy Identification and Control Laws

Following the procedure described in section 3, the first step is to obtain a first affine linear model (8), this model is gotten from a linear identification on equilibrium point of the system (discrete-time linear model), which is taken from [20].

$$x(k+1) = \begin{bmatrix} 1.0000 & 0.0099 & -0.0097 & -0.000 \\ 0.0005 & 0.9839 & -1.8011 & -0.008 \\ 0.0000 & 0.0001 & 1.0099 & 0.010 \\ -0.005 & 0.0102 & 1.8841 & 1.884 \end{bmatrix} x(k) + \\ + \begin{bmatrix} 0.0023 \\ 0.4623 \\ -0.0015 \\ -0.3001 \end{bmatrix} u(k)$$

The T-S fuzzy identification methodology requires of input-output data set. Thus, these data set have been generated from MatLab-Simulink simulations using a dynamic nonlinear model described in (2) as Fig. 4 shows.

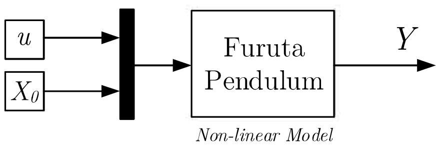

Figure 4: Input-output data generation from Matlab simulations

where, u is input of system, X_0 is the vector of initial conditions, and Y is the outputs vector, in this work it is considered that all states system are measurable. Thus, for an ith MatLab-Simulink simulation an u_i and X_{0_i} are selected randomly with an uniform distribution function, and the output Y_i is taking after a sampling time ($T_s = 10[ms]$) has elapsed. In this work 2000 simulations are used. In table 2 the range of X_0 and u variables are presented.

Table 2 Range of variables for T-S identification

Variable	Range	Unit
θ_0	$[-\pi/3, \pi/3]$	$[rad]$
$\dot{\theta}$	$[-1, 1]$	$[rad/s]$
α	$[-\pi/12, \pi/12]$	$[rad]$
$\dot{\alpha}$	$[-1, 1]$	$[rad/s]$
θ	$[-20, -20]$	$[V]$

In order to estimate the T-S fuzzy model, triangular membership fuzzy set with three subsets for pendulum angular position α and arm speed $\dot{\theta}$ have been chosen as Fig. 5 shows. Thus, nine fuzzy IF-ELSE rules are obtained. For each rule an system $Sys_j^{(i_1 \ldots i_n)}$ is represented by a discrete-time state-space linear model (5), which has been obtained from input-output data using fuzzy identification methodology described in section 3. It is used weighting diagonal matrix of $\gamma = 1$, therefore matrices of each rule is given by:

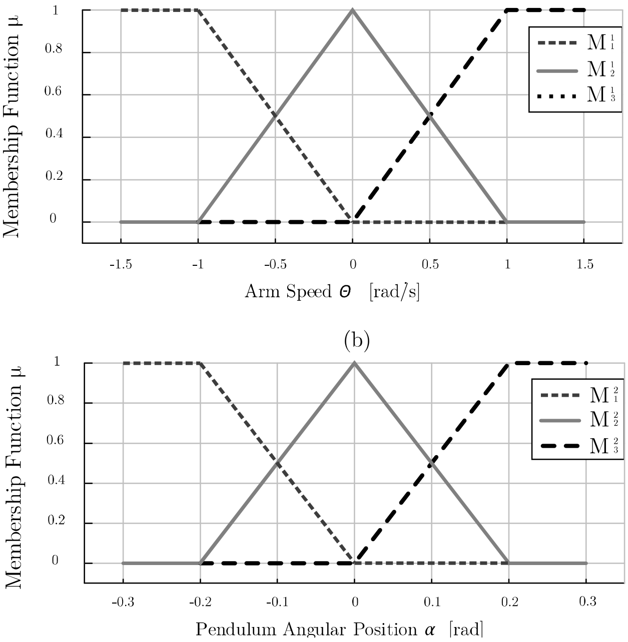

Figure 5: (a) Arm speed fuzzy subsets, (b) Pendulum angular position fuzzy subsets

$Sys^{11} : IF\ x_2(k)\ is\ M_1^1\ and\ x_3(k)\ is\ M_2^1\ THEN :$

$$a_0^{11} = \begin{bmatrix} 0.000 \\ 0.001 \\ 0.000 \\ 0.000 \end{bmatrix}, \quad B^{11} = \begin{bmatrix} 0.000 \\ 0.159 \\ 0.000 \\ -0.102 \end{bmatrix}$$

$$A^{11} = \begin{bmatrix} 1.000 & 0.010 & -0.003 & 0.000 \\ -0.002 & 0.975 & -0.610 & -0.014 \\ 0.000 & 0.000 & 1.005 & 0.010 \\ 0.003 & 0.018 & 1.172 & 1.014 \end{bmatrix}$$

$Sys^{12} : IF\ x_2(k)\ is\ M_1^1\ and\ x_3(k)\ is\ M_2^2\ THEN :$

$$a_0^{12} = \begin{bmatrix} 0.000 \\ -0.016 \\ 0.000 \\ 0.013 \end{bmatrix}, \quad B^{12} = \begin{bmatrix} 0.000 \\ 0.156 \\ 0.000 \\ -0.102 \end{bmatrix}$$

$$A^{12} = \begin{bmatrix} 1.000 & 0.010 & -0.003 & 0.000 \\ -0.028 & 0.972 & -0.636 & 0.006 \\ 0.000 & 0.000 & 1.005 & 0.010 \\ -0.022 & 0.020 & 1.172 & 0.997 \end{bmatrix}$$

$Sys^{13} : IF\ x_2(k)\ is\ M_1^1\ and\ x_3(k)\ is\ M_2^3\ THEN :$

$$a_0^{13} = \begin{bmatrix} 0.000 \\ -0.001 \\ 0.000 \\ 0.001 \end{bmatrix}, \quad B^{13} = \begin{bmatrix} 0.000 \\ 0.159 \\ 0.000 \\ -0.102 \end{bmatrix}$$

$$A^{13} = \begin{bmatrix} 0.999 & 0.010 & -0.003 & 0.000 \\ -0.019 & 0.976 & -0.622 & 0.000 \\ 0.000 & 0.000 & 1.005 & 0.010 \\ 0.015 & 0.016 & 1.165 & 1.002 \end{bmatrix}$$

$Sys^{21} : IF\ x_2(k)\ is\ M_1^2\ and\ x_3(k)\ is\ M_2^1\ THEN :$

$$a_0^{21} = \begin{bmatrix} 0.000 \\ -0.006 \\ 0.000 \\ 0.004 \end{bmatrix}, \quad B^{21} = \begin{bmatrix} 0.000 \\ 0.159 \\ 0.000 \\ -0.102 \end{bmatrix}$$

$$A^{21} = \begin{bmatrix} 0.999 & 0.009 & -0.003 & 0.000 \\ -0.011 & 0.976 & -0.622 & 0.000 \\ 0.000 & 0.000 & 1.005 & 0.010 \\ 0.008 & 0.016 & 1.165 & 1.002 \end{bmatrix}$$

$Sys^{22} : IF\ x_2(k)\ is\ M_1^2\ and\ x_3(k)\ is\ M_2^2\ THEN :$

$$a_0^{22} = \begin{bmatrix} 0.000 \\ 0.004 \\ 0.000 \\ -0.003 \end{bmatrix}, \quad B^{22} = \begin{bmatrix} 0.000 \\ 0.159 \\ 0.000 \\ -0.102 \end{bmatrix}$$

$$A^{22} = \begin{bmatrix} 0.999 & 0.010 & -0.003 & 0.000 \\ -0.044 & 1.003 & -0.635 & -0.029 \\ 0.000 & 0.000 & 1.005 & 0.010 \\ 0.035 & -0.004 & 1.171 & 1.026 \end{bmatrix}$$

$Sys^{23} : IF\ x_2(k)\ is\ M_1^2\ and\ x_3(k)\ is\ M_2^3\ THEN :$

$$a_0^{23} = \begin{bmatrix} 0.000 \\ 0.011 \\ 0.000 \\ -0.008 \end{bmatrix}, \quad B^{23} = \begin{bmatrix} 0.000 \\ 0.159 \\ 0.000 \\ -0.102 \end{bmatrix}$$

$$A^{23} = \begin{bmatrix} 1.000 & 0.009 & -0.003 & 0.000 \\ 0.009 & 0.998 & -0.687 & 0.000 \\ 0.000 & 0.000 & 1.006 & 0.010 \\ -0.007 & 0.007 & 1.215 & 1.003 \end{bmatrix}$$

$Sys^{31} : IF\ x_2(k)\ is\ M_1^3\ and\ x_3(k)\ is\ M_2^1\ THEN :$

$$a_0^{31} = \begin{bmatrix} 0.000 \\ -0.008 \\ 0.000 \\ 0.008 \end{bmatrix}, \quad B^{31} = \begin{bmatrix} 0.000 \\ 0.159 \\ 0.000 \\ -0.102 \end{bmatrix}$$

$$A^{31} = \begin{bmatrix} 1.000 & 0.009 & -0.003 & 0.000 \\ 0.009 & 0.983 & -0.622 & 0.000 \\ 0.000 & 0.000 & 1.005 & 0.010 \\ -0.008 & 0.013 & 1.165 & 1.003 \end{bmatrix}$$

$Sys^{32} : IF\ x_2(k)\ is\ M_1^3\ and\ x_3(k)\ is\ M_2^2\ THEN :$

$$a_0^{32} = \begin{bmatrix} 0.000 \\ -0.002 \\ 0.000 \\ 0.002 \end{bmatrix}, \quad B^{32} = \begin{bmatrix} 0.000 \\ 0.159 \\ 0.000 \\ -0.102 \end{bmatrix}$$

$$A^{32} = \begin{bmatrix} 1.000 & 0.010 & -0.003 & 0.000 \\ 0.022 & 0.994 & -0.639 & 0.022 \\ 0.000 & 0.000 & 1.005 & 0.009 \\ -0.018 & 0.029 & 1.174 & 1.003 \end{bmatrix}$$

$Sys^{33} : IF\ x_2(k)\ is\ M_1^3\ and\ x_3(k)\ is\ M_2^3\ THEN :$

$$a_0^{33} = \begin{bmatrix} 0.000 \\ 0.008 \\ 0.000 \\ -0.006 \end{bmatrix}, \quad B^{33} = \begin{bmatrix} 0.000 \\ 0.159 \\ 0.000 \\ -0.102 \end{bmatrix}$$

$$A^{33} = \begin{bmatrix} 1.000 & 0.009 & -0.003 & 0.000 \\ 0.007 & 0.996 & -0.715 & -0.015 \\ 0.000 & 0.000 & 1.006 & 0.010 \\ -0.006 & 0.027 & 1.239 & 1.0119 \end{bmatrix}$$

In order to validate the T-S fuzzy model identification, Mean Squared Error (MSE) is used, where the errors of four system variables are: $\theta_{MSE} = 1.15 \cdot 10^{-7}$, $\alpha_{MSE} = 7.75 \cdot 10^{-8}$ and $\dot{\theta}_{MSE} = 0.0047$, $\dot{\alpha}_{MSE} = 0.0031$, where the errors are very low.

The feedback optimal vector gains K_a is calculated for each rule, where the matrices Q_w, Q_x and R are determined by trial and error:

$$Q_w = \begin{bmatrix} 0.01 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.01 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.01 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.01 \end{bmatrix}$$

$$Q_x = \begin{bmatrix} 25 & 0.0 & 0.0 & 0.0 \\ 0.0 & 25 & 0.0 & 0.0 \\ 0.0 & 0.0 & 4.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 10 \end{bmatrix}, \quad R = 1$$

The fuzzy feedback control vector gains $K_a{}^{(i_1..i_n)} = \left[K_w{}^{(i_1..i_n)} \quad K_x{}^{(i_1..i_n)} \right]$ for each rule are given by:

$C^{11} :\ IF\ x_2(k)\ is\ M_1^1\ and\ x_3(k)\ is\ M_2^1\ THEN :$

$$K_w^{11} = \begin{bmatrix} 0.001 & 0.000 & 0.000 & 0.000 \end{bmatrix}$$
$$K_x^{11} = \begin{bmatrix} -5.4 & -6.9 & -181.7 & -21.1 \end{bmatrix}$$

$C^{12} :\ IF\ x_2(k)\ is\ M_1^1\ and\ x_3(k)\ is\ M_2^2\ THEN :$

$$K_w^{12} = \begin{bmatrix} 0.001 & 0.000 & 0.000 & 0.000 \end{bmatrix}$$
$$K_x^{12} = \begin{bmatrix} -3.8 & -6.9 & -187.7 & -21.3 \end{bmatrix}$$

$C^{13} :\ IF\ x_2(k)\ is\ M_1^1\ and\ x_3(k)\ is\ M_2^3\ THEN :$

$$K_w^{13} = \begin{bmatrix} 0.001 & 0.000 & 0.000 & 0.000 \end{bmatrix}$$
$$K_x^{13} = \begin{bmatrix} -5.7 & -6.72 & -181.5 & -20.7 \end{bmatrix}$$

$C^{21} :\ IF\ x_2(k)\ is\ M_1^2\ and\ x_3(k)\ is\ M_2^1\ THEN :$

$$K_w^{21} = \begin{bmatrix} 0.001 & 0.000 & 0.000 & 0.000 \end{bmatrix}$$
$$K_x^{21} = \begin{bmatrix} -5.4 & -6.1 & -175.1 & -20.0 \end{bmatrix}$$

$C^{22} =\ IF\ x_2(k)\ is\ M_1^2\ and\ x_3(k)\ is\ M_2^2\ THEN :$

$$K_w^{22} = \begin{bmatrix} 0.001 & 0.000 & 0.000 & 0.000 \end{bmatrix}$$
$$K_x^{22} = \begin{bmatrix} -6.7 & -5.7 & -165.8 & -19.7 \end{bmatrix}$$

$C^{23} =\ IF\ x_2(k)\ is\ M_1^2\ and\ x_3(k)\ is\ M_2^3\ THEN :$

$$K_w^{23} = \begin{bmatrix} 0.001 & 0.000 & 0.000 & 0.000 \end{bmatrix}$$
$$K_x^{23} = \begin{bmatrix} -4.6 & -6.1 & -176.4 & -20.1 \end{bmatrix}$$

$C^{31} =\ IF\ x_2(k)\ is\ M_1^3\ and\ x_3(k)\ is\ M_2^1\ THEN :$

$$K_w^{31} = \begin{bmatrix} 0.001 & 0.000 & 0.000 & 0.000 \end{bmatrix}$$
$$K_x^{31} = \begin{bmatrix} -4.4 & -6.8 & -182.9 & -21.0 \end{bmatrix}$$

$C^{32} =\ IF\ x_2(k)\ is\ M_1^3\ and\ x_3(k)\ is\ M_2^2\ THEN :$

$$K_w^{32} = \begin{bmatrix} 0.001 & 0.000 & 0.000 & 0.000 \end{bmatrix}$$
$$K_x^{32} = \begin{bmatrix} -3.9 & -5.9 & -175.3 & -19.5 \end{bmatrix}$$

$C^{33} :\ IF\ x_2(k)\ is\ M_1^3\ and\ x_3(k)\ is\ M_2^3\ THEN :$

$$K_w^{33} = \begin{bmatrix} 1.000 & 0.009 & -0.003 & 0.000 \end{bmatrix}$$
$$K_x^{33} = \begin{bmatrix} -4.6 & -7.4 & -190.6 & -21.7 \end{bmatrix}$$

5.2 Experimental Results

In this section experimental results are presented, where the dynamic behavior of a classical LQR controller and the proposed controller FLC are compared.

In Fig. 6 the QNET Rotary Inverted Pendulum Trainer (h) for NI ELVIS (g), is presented. Which is composed of: (a) DC Motor of $24[V]$, $286[rpm/V]$ and $I_{max} = 3[A]$, (b) Arm angle encoder of $0.25[dec/count]$ (in quadrature mode), (c) rotary arm, (d) pendulum encoder of $0.0879[dec/count]$ (in quadrature mode), (e) pendulum link and (f) pendulum weight of mass $0.019[kg]$. For this work was used a PC Intel(R), Core(TM) i7-4900MQ CPU @ 2.8GHZ. We used LabVIEW 2013 software with ODE $Runge - kutta1(Euler)$ solver method.

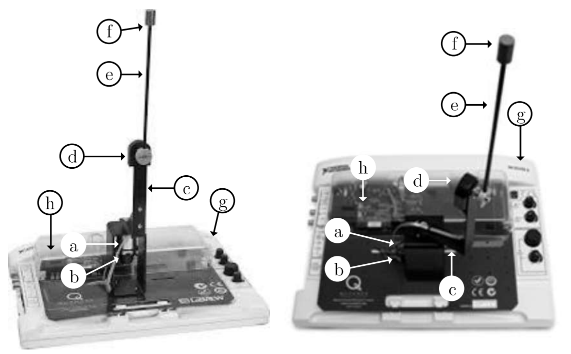

Figure 6: QNET Rotary Inverted Pendulum Trainer for NI ELVIS, [32]

For the purpose of validating the proposed controller three experimental tests are carried out. The first is a reference change test on the arm angular position θ as the Fig. 7 shows, the second test is performed in the presence of an external disturbance. Finally, a pendulum mass change (variation parameter) is made for the third test. For all tests the FP starts in equilibrium point ($\alpha = 0[rad], \theta = 0[rad]$). In order to analyze the performance of the proposed controller, the Integral Square Error (ISE) index is used.

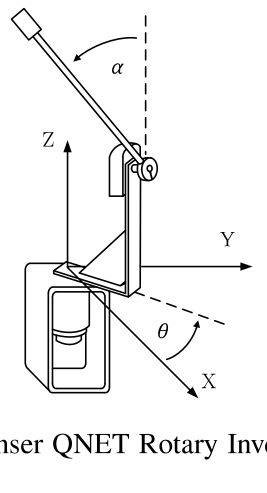

Figure 7: Quanser QNET Rotary Inverted Pendulum scheme

5.2.1 Step Change Reference Test

In this test, the pendulum begins in the upright position (equilibrium point), then a step change of reference on the arm angular position of θ is applied. In Fig. 8 the time-response of arm angular position, in Fig. 9 the time-response of pendulum angular position α and, in Fig. 10 the time-response of input u for a step change reference of $\theta = 0.872[rad]$ are shown.

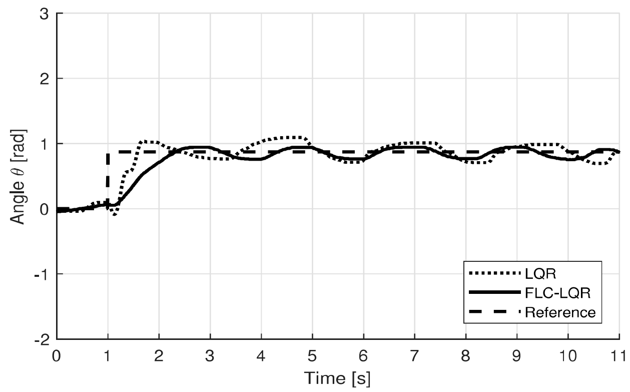

Figure 8: Arm angular position θ time-response for step change reference ($\theta = 0.872[rad]$)

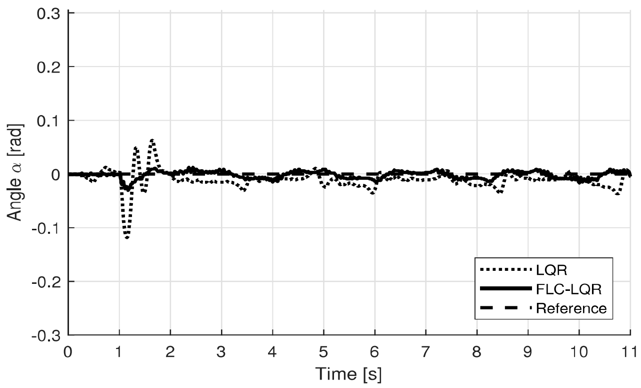

Figure 9: Pendulum angular position α time-response for step change reference ($\theta = 0.872[rad]$)

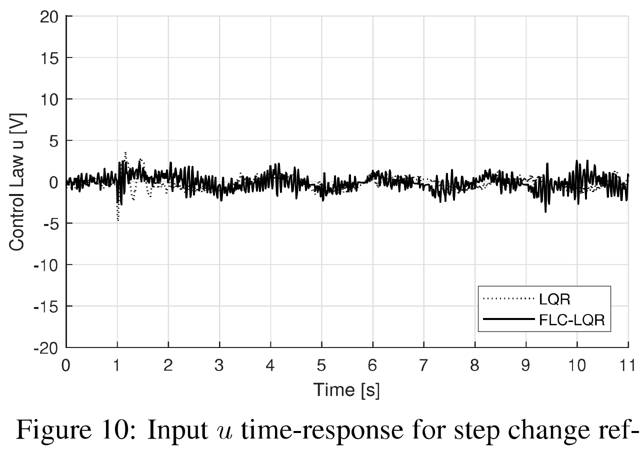

Figure 10: Input u time-response for step change reference ($\theta = 0.872[rad]$)

In Figs. 8 and 9 show that the LQR and FLC-LQR controllers are able to maintain balance (equilibrium point) for a reference change of ($\theta = 0.872[rad]$) with the control action u within the operating range. In Figure 11 presents the real frame sequences of the Furuta's pendulum for a step change reference of ($\theta = 3.577[rad]$) using FLC-LQR.

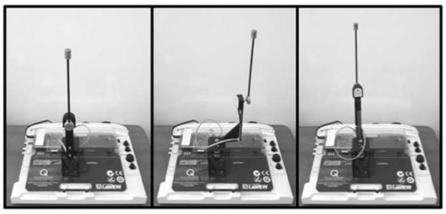

Figure 11: Sequence frame of the Furuta's pendulum for step chance reference ($\theta = 0.872[rad]$) with FLC

In Fig. 12 the time-response of arm angular position, in Fig. 13 the time-response of pendulum angular position (α) and in Fig. 14 the time-response of input u for step change reference ($\theta = 3.577[rad]$) are shown.

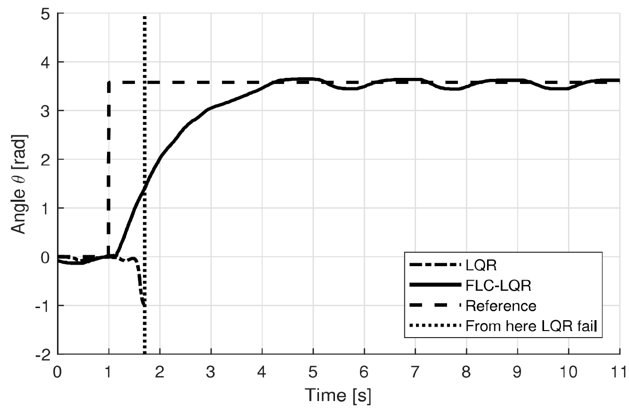

Figure 12: Arm angular position θ evolution for step change reference ($\theta = 3.577[rad]$)

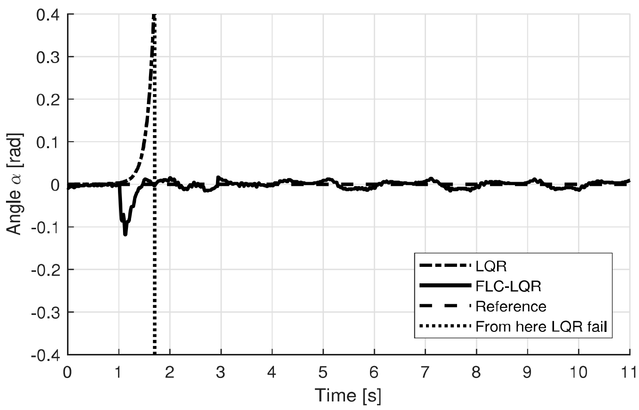

Figure 13: Pendulum angular position α time-response for step change reference ($\theta = 3.577[rad]$)

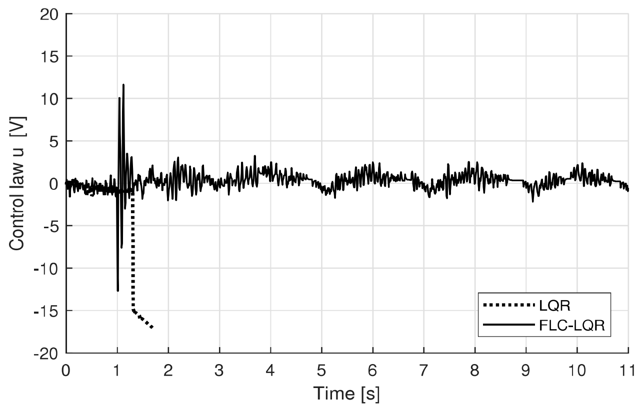

Figure 14: Input u time-response for step change reference ($\theta = 3.577[rad]$)

Figures 12 and 13 indicate that only the FLC controllers are able to maintain balance (equilibrium point) for a reference change of ($\theta = 3.577[rad]$) and the control action u within the operating range. Table 3 shows ISE performance index for both controllers from step change reference tests of ($\theta = 0.872[rad]$) and ($\theta = 3.577[rad]$), where the FLC-LQR controller has less ISE than LQR controller.

Table 3 ISE Step change reference tests

Step Change	**Var.**	**ISE LQR LQR**	**ISE FLC-LQR**
$\theta = 0.872$	α	0.0046	0.00057
	θ	7.9686	5.7195
$\theta = 3.577$	α	$---$	0.0023
	θ	$---$	107.34

5.2.2 External Disturbance Test:

In this test, an external disturbance is applied on Furuta's pendulum as is shown in Fig. 15

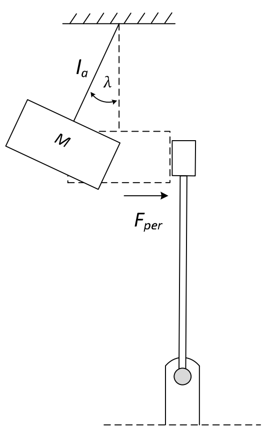

Figure 15: External disturbance test scheme

In this test a mass $M = 0.1[kg]$ is suspended by a cord of distance $l_a = 0.11[m]$ which forms an angle λ with respect to the vertical axis is released towards the rotary inverted pendulum, the perpendicular force F_{per} applied is determined as [37]:

$$F_{per} = M \cdot g \cdot sin(\lambda) \tag{20}$$

where g is the gravitational constant.

Figure 16 presents the real sequence of the FP for a external disturbance test using FLC-LQR.

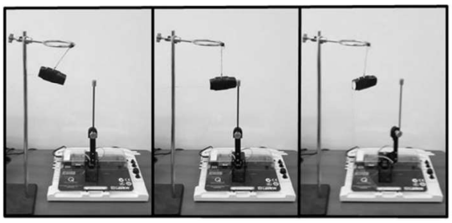

Figure 16: Sequence of the FP for external disturbance test using FLC-LQR

For an angle of $\lambda = 0.6[rad]$, the perpendicular force is $F_{per} = 0.443[N]$. In Fig. 17 the time-response of pendulum angular position, in Fig. 18 the time-response of arm angular position (θ) and in Fig. 19 the time-response of control law u for external disturbance test are shown, where around of $1[s]$ the F_{per} is applied.

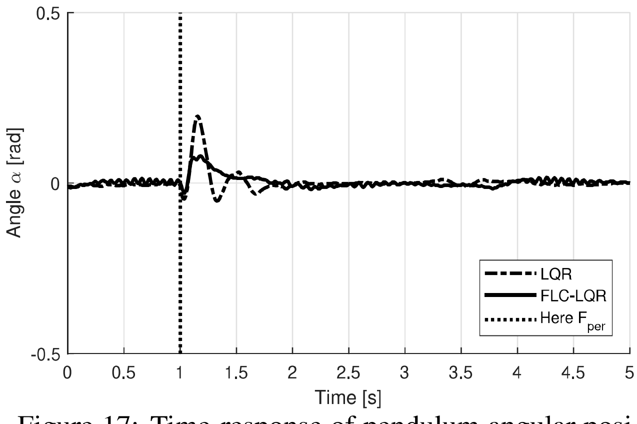

Figure 17: Time-response of pendulum angular position α for a external disturbance test

Taking into consideration Fig. 17 and 18, the LQR and FCL-LQR controllers are able to maintain balance (equilibrium point) for a external disturbance test and the control action u within the operating range.

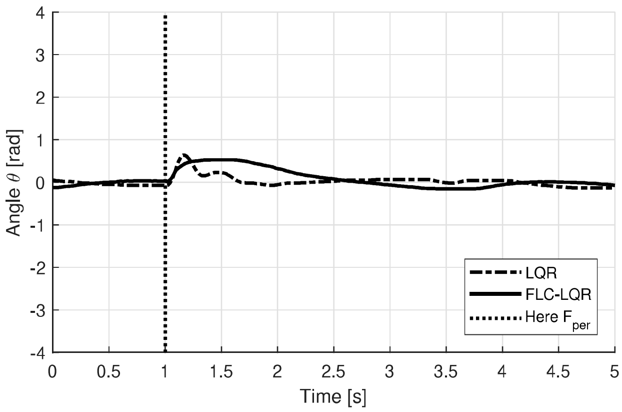

Figure 18: Time-response of arm angular position θ for a external disturbance test

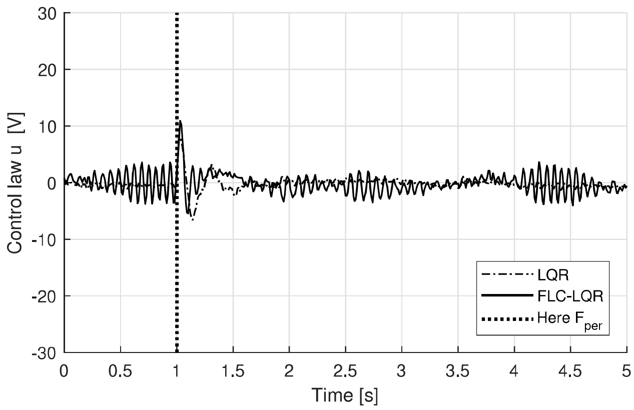

Figure 19: Input u time-response for a external disturbance test

Table 4 shows ISE performance index for both controllers from a external disturbance test, where the FLC-LQR controller has less ISE than LQR controller.

Table 4 ISE Step external disturbance test

Forcer	**Var.**	**ISE LQR**	**ISE FLC-LQR**
$F_{per} = 0.443[N]$	α	0.0058	0.00036
	θ	0.7525	0.6858

5.2.3 Parameter Variation Test:

In this test, FP dynamic model is modified, where a metallic piece of mass $m_p = 138[gr]$ is attached on pendulum as Fig. 20 shows.In Fig. 21 the time-response of pendulum angular position α, in Fig. 22 the time-response of arm angular position (θ) and in Fig. 23 the time-response of input u for parameter variation test are shown. According to Fig. 21 and 22, only the FLC-LQR controller is able to maintain balance (equilibrium point) for parameter variation test.

Figure 20: Parameter variation test on Furuta's pendulum

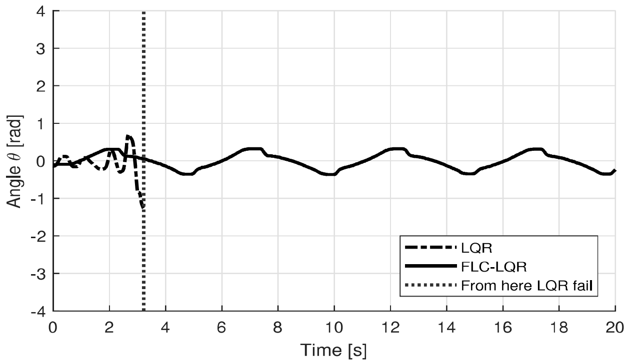

Figure 21: Time-response of pendulum angular position α for parameter variation test

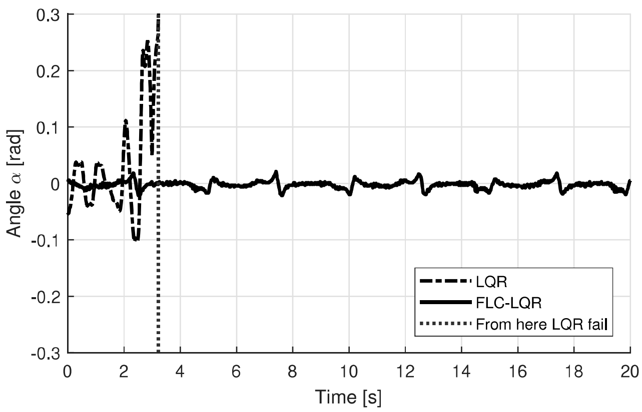

Figure 22: Time-response of arm angular position θ for parameter variation test

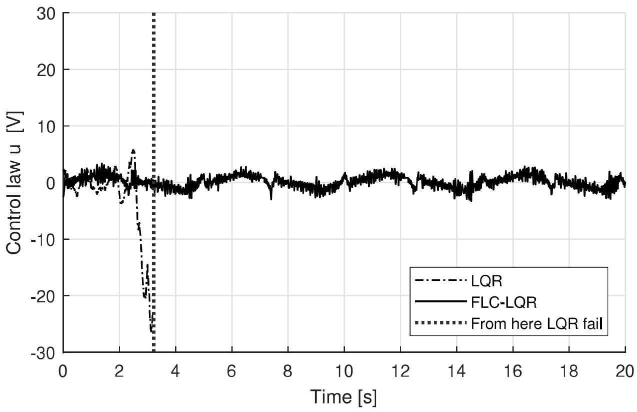

Figure 23: Time-response of input for parameter variation test

6 Conclusion

In this work a fuzzy optimal (FLC-LQR) control has been successful designed and executed for a Furuta's Pendulum. The FLC-LQR was obtained blending of Takagi-Sugeno (T-S) fuzzy modeling and the Linear Quadratic Regulator (LQR). Three experimental tests: arm angular position reference change, external disturbance and finally a test of parameter variation were implemented. Tracking reference and disturbance rejection controllers comparison were done and the performance measured. The FLC-LQR compared against a classic LQR controller presented lower ISE for all experimental test.

References:

[1] Y. Liu and H. Yu, A survey of underactuated mechanical systems, *IET Control Theory Applications*, vol.10, no.7 , pp. 921–935, 2013.

[2] M. Ramírez-Neria, H. Sira-Ramírez, R. Garrido-Moctezuma and A. Luviano-Juárez, Linear active disturbance rejection control of underactuated systems: The case of the Furuta pendulum, *ISA Transactions*, vol.53, no.53, pp.920–928, 2014.

[3] Ye Li, C. Wei, Q. Wu, P. Chen, Y. Jiang and Y. Li, Study of 3 dimension trajectory tracking of underactuated autonomous underwater vehicle. *Ocean Engineering*, vol.10, pp.270-274, 2015.

[4] B. d´Andréa-Novel, S. Thorel, Control of non holonomic or under-actuated mechanical systems: The examples of the unicycle robot and the slider.*ESAIM: Control, Optimisation and*

Calculus of Variations, vol.22, no.4, pp.983–1016, 2016.

[5] J. Á. Acosta and M. I. Sánchez and A. Ollero, CRobust control of underactuated Aerial Manipulators via IDA-PBC, *53rd IEEE Conference on Decision and Control*, IEEE, pp.673–678, 2014.

[6] Y. Xu, M. Iwase and K. Furuta,Time optimal swing-up control of single pendulum, *Journal of Dynamic Systems, Measurement, and Control*, vol.123, no.3, pp.518–527, 2001.

[7] B. S. Cazzolato and Z. Prime, On the dynamics of the Furuta pendulum, *Journal of Control Science and Engineering*, 2011.

[8] V. Sukontanakarn and M. Parnichkun, Real-Time Optimal Control for Rotary Inverted Pendulum , *IAmerican Journal of Applied Sciences*, vol.6, no.6, pp.1106-'1115, 2009.

[9] G. Rigatos, P. Siano, M. Abbaszadeh, S. Ademi and A. Melkikh, Nonlinear H-infinity control for underactuated systems: the Furuta pendulum example, *International Journal of Dynamics and Control*, vol.6, no.2, pp.835–847, 2018.

[10] J. Wen, Y. Shi and X. Lu, Stabilizing a rotary inverted pendulum based on Lyapunov stability theorem, *29th Chinese Control And Decision Conference (CCDC)*, IEEE, pp.5652–5657, 2017.

[11] K. J. Åström and K. Furuta, Swinging up a pendulum by energy control, *Automatica*, vol.36, no.2, pp.287–295, 2000.

[12] N. Muskinja and B. Tovornik, Swinging up and stabilization of a real inverted pendulum, *IEEE Transactions on Industrial Electronics*, vol.53, no.2, pp.631–639, 2006.

[13] C. Aguilar-Avelar and J. Moreno-Valenzuela, A composite controller for trajectory tracking applied to the Furuta pendulum, *ISA Transactions*, vol.57, pp.286–294, 2015.

[14] C. Aguilar-Avelar and J. Moreno-Valenzuela, New Feedback Linearization-Based Control for Arm Trajectory Tracking of the Furuta Pendulum, *IEEE/ASME Transactions on Mechatronics*, vol.21, no.2, pp.638–648, 2016.

[15] C. Aguilar and J. Moreno-Valenzuela, A feedback linearization controller for trajectory tracking of the Furuta pendulum, *American Control Conference*, IEEE, pp.4543–4548, 2014.

[16] J. Moreno-Valenzuela and C. Aguilar-Avelar and S. A. Puga-Guzmán and V. Santibáñez, Adaptive Neural Network Control for the Trajectory Tracking of the Furuta Pendulum, *IEEE Transactions on Cybernetics*, vol.46, no.12, pp.3439–3452, 2016.

[17] I. Hussein and A. Optimal Control of Underactuated Nonholonomic Mechanical Systems, *IEEE Transactions on Automatic Control*, vol.53, no.3, pp.668–682, 2008.

[18] X., Yongcai, M. Iwase, and K. Furuta. Time optimal swing-up control of single pendulum, *Journal of Dynamic Systems, Measurement, and Control*, vol.123, no.3, pp.518-527, 2001.

[19] M. Park, Y. J. Kim and J. J. Lee, Swing-up and LQR stabilization of a rotary inverted pendulum, *Artificial Life and Robotics*,vol.16, pp.94–97, 2011.

[20] I. Paredes and M. Sarzosa and M. Herrera and P. Leica and O. Camacho, Optimal-robust controller for furuta pendulum based on linear model,*Second Ecuador Technical Chapters Meeting (ETCM)*, IEEE, pp. 1-6. DOI: 10.1109/ETCM.2017.8247510, 2017.

[21] J. Akhil, A. Clint, M. Shinu Mohanan and C. Keerthi, Performance study of PID controller and LQR technique for inverted pendulum, *World Journal of Engineering and Technology*, vol.3, no.2, pp.76–81, 2015.

[22] T. Takagi, M. Sugeno, Fuzzy Identification of Systems and Its Applications to Modeling and Control, *IEEE Transactions on Systems, Man, and Cybernetics* , vol.15, pp.116–132, 1985.

[23] B. Jiang, Z. Gao, P. Shi and Y. Xu, Adaptive Fault-Tolerant Tracking Control of Near-Space Vehicle Using Takagi-Sugeno Fuzzy Models, *IEEE Transactions on Fuzzy Systems*, vol.18, no.5, pp.1000–1007, 2010.

[24] J. A. Meda-Campana, J. C. Gomez-Mancilla and B. Castillo-Toledo, Exact Output Regulation for Nonlinear Systems Described by Takagi–Sugeno Fuzzy Models, *IEEE Transactions*

on Fuzzy Systems, vol.20, no.2, pp.235–247, 2012.

[25] M. Klug, E. B. Castelan, V. Leite and L. Silva, Fuzzy dynamic output feedback control through nonlinear Takagi-Sugeno models, *Fuzzy Sets and Systems*, vol.236, pp.92–111, 2015.

[26] J. Wang, Y. Gao, J. Qiu and C. K. Ahn, Choon Ki, Sliding mode control for non-linear systems by Takagi-Sugeno fuzzy model and delta operator approaches, *IET Control Theory & Applications* , vol.11, no.8, pp.1205–1213, 2016.

[27] M. Adánez, B. M. l-Hadithi, A. Jiménez and F. Matía, Optimal Control of a Ball and Beam Nonlinear Model Based on Takagi-Sugeno Fuzzy Model, *Advances in Fuzzy Logic and Technology*, 2017.

[28] A. Jiménez and B. M. Al-Hadithi and L. Alonso and J. Pérez-Oria, Linear quadratic regulator based Takagi-Sugeno model for multivariable nonlinear processes,*17th International Conference on System Theory, Control and Computing (ICSTCC)*, IEEE, pp.285–290, 2013.

[29] M. H. Khooban, N. Vafamand, T. Niknam, T. Dragicevic, and F. Blaabjerg, Model-predictive control based on Takagi-Sugeno fuzzy model for electrical vehicles delayed model, *IET Electric Power Applications*, vol.11, no.5, pp.918–934, 2017.

[30] Q. V. Dang, B. Allouche, L. Vermeiren, A. Dequidt and M. Dambrine, Design and implementation of a robust fuzzy controller for a rotary inverted pendulum using the Takagi-Sugeno descriptor representation, *Symposium on Computational Intelligence in Control and Automation (CICA)*, IEEE, 2014.

[31] U. Farooq, J. Gu, M. E. El-Hawary, V. Balas and M. U. Asad, Experimental study of optimal Takagi Sugeno fuzzy controller for rotary inverted pendulum, *International Conference on Fuzzy Systems*, IEEE, 2015.

[32] Inc. Quanser, User Manual - QNET Rotary Inverted Pendulum Trainer for NI ELVIS, pp.1–26, 2011.

[33] B. M. Al-Hadithi, A. Jiménez and R. Galán, Fuzzy optimal control using generalized Takagi-Sugeno model for multivariable nonlinear systems, *Applied Soft Computing*, vol.30, no.2, pp.205–213, 2015.

[34] B. M. Al-Hadithi, A. Jiménez and Fernando Matía, A new approach to fuzzy estimation of Takagi-Sugeno model and its applications to optimal control for nonlinear systems, *Applied Soft Computing*, vol.12, pp.280–290, 2012.

[35] A. Jiménez, B. M. Al-Hadithi, and Fernando Matía, An Optimal T-S Model for the Estimation and Identification of Nonlinear Functions, *WSEAS Trans. Sys. Ctrl.*, vol.3, no.10, pp.897–906, 2008.

[36] A. Jiménez, B. M. Al-Hadithi, Fernando Matía and R. Haber-Haber, Improvement of Takagi-Sugeno fuzzy model for the estimation of nonlinear functions, *Asian Journal of Control*, vol.14, no.2, pp.320–334, 2012.

[37] J. Villacrés, M. Viscaíno, M. Herrera and O. Camacho, Real-time implementation of different controllers for a two-wheeled inverted pendulum, *International Journal of Circuits, Systems and Signal Processing*, vol.10, pp.281–290, 2016.