Competitive numerical method for an avian influenza model

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Abstract: A competitive implicit finite-di ference method for the numerical solution of an avian influenz model is constructed. The proposed numerical schemes have two fi ed points which are identical to the critical points of the continuous model and it is shown that they have the same stability properties. It is shown further that the solution sequence is attracted from any set of initial conditions to the correct (stable) fi ed point for an arbitrarily large time step. Numerical Simulations are confirme and compared with well-known numerical methods.

Key–Words: Implicit finite-di ference, Avian influenz

1 Introduction

Bird fl or avian influenza caused by H5N1 virus, is a new emerging infectious disease. The world has been experiencing a relentless spread of bird fl due to importation of chicken birds as they move around the world to seasonal breeding and feeding grounds, infecting domestic flock around the world. More than 150 million birds, mostly chickens, have died or has been culled. Sixty-three out of 124 infected humans have died since December 2003 [28]. Many methods such that hunting and isolation have been taken to control the spread of avian influenza However, it is observed that poultry without any symptom can excrete much highly pathogenic virus, which makes it more difficul to inhibit the H5N1 type virus from spreading.

Many problems in mathematical epidemiology are modelled by autonomous systems of nonlinear ordinary differential equations, which implies the assumption that the parameters of the model are independent of time. These models describe the behavior and relationship between the different subpopulations: susceptible, infective and recovered, which together constitute the total population of a certain region or environment. Generally, the exact solutions of these models are unavailable being necessary to obtain accurate numerical approximations to the solutions in order to understand the dynamics of the systems. A number of deterministic models have been reported in the literature for avian influenz transmission dynamics (see, for instance, [2, 3, 6, 5, 16, 17, 23, 29]) while Iwami S, Takeuchi Y, Liu X. [16] investigated a mathematical model to explain the spread of avian influenz and mutant avian influenza Derouich and Boutayeb [5] presented a mathematical model that deals with the dynamics of human infection by avian influenz both in birds and in humans. Ye and Li [29] developed an avian fl model based on the standard SEIQ model including constant immigration of latent class and an additional property of the avian influenza namely that the asymptomatic individuals in the latent period have infectious force. The stochastic models were proposed to model and predict the worldwide spread of pandemic influenz [3]. Rao et al. [23] developed a groundbreaking methodology based on computer simulations to analyze the spread of H5N1 using stochastic interactions between waterfowl, poultry, and humans. These studies, however, gave no details on the numerical method(s) used to solve the resulting nonlinear initial-value problems(IVPS). Therefore one important task of the mathematical modelling is to obtain accurate numerical solutions.

It is worth mentioning that discretizing the ordinary differential equations (ODEs) of the model by traditional schemes like Euler and Runge-Kutta methods can result in contrived chaos and oscillations for certain values of the discretization parameters [8, 9, 11, 12, 21, 22], moreover some methods, despite using adaptative step size, still fail (see [19]). Although such scheme-dependent numerical instabilities can often be avoided by using small time-steps, the extra computing cost incurred when examining the long-term behaviour of a dynamical system may be substantial. It is, therefore, essential to use a numerical method which allows the largest possible time steps that are consistent with stability and accuracy. In order to circumvent contrived chaos, whilst retaining accuracy and numerical stability, it may be necessary to forego the ease-of-implementation of inexpensive explicit numerical methods in favour of implicit methods (which are known to be more competitive in terms of numerical stability).

The purpose of the current study is to construct a competitive implicit method for solving a simple mathematical model of avian influenz transmission proposed in [5]. The paper is organized as follows. The governing continuous-time model for the evolution of avian influenz disease in human population is given in Section 2. The stability of this model is analyzed in Section 2.1. The construction of proposed numerical schemes is carried out in Section 3 and its fi ed point analyzed in section 4. Numerical results in several situations are reported in section 5.

2 Mathematical model

The model presented in [5] for the transmission of avian influenz using SIRS model for human population and SI model for bird population are expressed as the nonlinear system of differential equations of the form

Human population

$$\begin{aligned} S_h'(t) &= \Lambda - \left(\mu + \frac{\beta I_v}{N_v}\right) S_h + \delta R_h, \\ I_h'(t) &= \frac{\beta I_v S_h}{N_v} - (\mu + \gamma + \alpha) I_h, \\ R_h'(t) &= \gamma I_h - (\mu + \delta) R_h, \end{aligned} \tag{1}$$

Bird population

$$\begin{aligned} S_v'(t) &= \mu_0 N_v - \left(\mu_0 + \frac{\beta_0 I_v}{N_v}\right) S_v, \\ I_v'(t) &= \frac{\beta_0 I_v S_v}{N_v} - \mu_0 I_v. \end{aligned} \tag{2}$$

In (1)-(2), the variables S_h, I_h, R_h, S_v, and I_v denote the populations of susceptible humans, infectious humans, recovered humans, infectious birds and recovered birds at time t, so that the total population of human and bird at time t are given by $N_h = S_h + I_h + R_h$ and $N_v = S_v + I_v$, respectively. Furthermore, Λ is the recruitment of humans into the population (assumed susceptible), β is the infection rate of susceptible humans (which results following effective contact with infectious birds) and μ is the natural death rate of humans. Infectious humans recover (and move into the R_h class) at a rate γ and suffer disease-induced death at a rate α_h. The recovered humans population loss their immunity (so that they acquire avian influenz infection again) and move to the S_h class at a rate σ.

The susceptible birds population is generated by birth at a rate μ_0. This population is reduced by infection, following effective contact with infectious birds, at the rate β_0 and natural death at a rate μ_0. Since the above model monitors human and bird populations, all the associated parameters and state variables are non-negative.

For simplicity, set the new variables

$$s_h = \frac{S_h}{\Lambda/\mu}, i_h = \frac{I_h}{\Lambda/\mu}, r_h = \frac{R_h}{\Lambda/\mu}, s_v = \frac{S_v}{N_v}, i_v = \frac{I_v}{N_v},$$

the total human and bird populations are normalized to unity so that $s_h + i_h + r_h = n_h$ and $s_v = 1 - i_v$. The system (1)-(2), thus, is reduced to the non-linear IVP system

$$\begin{aligned} \frac{ds_h}{dt} &\equiv f_1 = \mu - (\mu + \beta i_v) s_h + \delta r_h,\ s_h(0) = s_h^0 \\ \frac{di_h}{dt} &\equiv f_2 = \beta i_v s_h - (\mu + \gamma + \alpha) i_h,\ i_h(0) = i_h^0 \\ \frac{dr_h}{dt} &\equiv f_3 = \gamma i_h - (\mu + \delta) r_h,\ r_h(0) = r_h^0 \\ \frac{di_v}{dt} &\equiv f_4 = \beta_0 i_v (1 - i_v) - \mu_0 i_v,\ i_v(0) = i_v^0. \end{aligned} \tag{3}$$

2.1 Stability Analysis

The steady-states of the IVP (3) are determined when the time derivatives vanish giving: the trivial critical point (no infected populations), $E_1 = (1, 0, 0, 0)$ and the non-trivial critical point, $E_2(\bar{s}_h, \bar{i}_h, \bar{r}_h, \bar{i}_v)$, with

$$\left.\begin{aligned} \bar{s}_h &= \frac{(\mu + \gamma + \alpha)\bar{i}_h}{\beta \bar{i}_v}, & \bar{i}_h &= \frac{\mu\beta(\mu + \delta)\bar{i}_v}{\beta\omega_1 \bar{i}_v + \omega_2}, \\ \bar{r}_h &= \frac{\gamma \bar{i}_h}{\mu + \delta}, & \bar{i}_v &= \frac{\mu_0}{\beta_0}(\tilde{R} - 1), \end{aligned}\right\} \tag{4}$$

where $\omega_1 = \mu(\mu + \gamma + \alpha) + \delta(\mu + \alpha)$, $\omega_2 = \mu(\mu + \delta)(\mu + \alpha + \gamma)$ and $\tilde{R} = \beta_0/\mu_0$. It follows from (4) that the system (3) has a unique positive solution when $\tilde{R} > 1$.

A critical point is said to be stable if the eigenvalues of the Jacobian evaluated at the critical point, are real and negative or are complex with negative real parts. It is easy to show that the Jacobian associated with f_1, f_2, f_3 and f_4 given in (3) is the matrix

$$J = \begin{bmatrix} \beta i_v - \mu & 0 & \delta & -\beta s_h \\ \beta i_v & -k_2 & 0 & \beta s_h \\ 0 & \gamma & -k_1 & 0 \\ 0 & 0 & 0 & a_{33} \end{bmatrix},$$

where $k_1 = \mu + \delta$, $k_2 = \mu + \gamma + \alpha$ and $a_{33} = \beta_0(1-i_v) - \beta_0 i_v - \mu_0$, the determinant of which vanishes when $\tilde{R} = \beta_0/\mu_0 = 1$. This unique value will be regarded as a bifurcation parameter of the model equations.

At the trivial critical point $s_h = 1,\ ih = r_h = i_v = 0$, the eigenvalues $\lambda_i, i = 1,2,3,4$ of the associated Jacobian are

$$\lambda_1 = -\mu_0(1-\tilde{R}),\ \lambda_2 = -\mu,\ \lambda_3 = -k_1,\ \lambda_4 = -k_2,$$

which it follows that all four eigenvalues are real and negative whenever $\tilde{R} < 1$. On the other hand, one is positive if $\tilde{R} > 1$. It may be concluded, therefore, that the trivial critical point is stable whenever $\tilde{R} < 1$ and unstable whenever $\tilde{R} > 1$.

At the non-trivial critical point E_2, it may be shown that $\lambda_1 = -\mu_0(\tilde{R}-1)$ and the other eigenvalues of the associated Jacobian are the roots of the characteristic equation

$$\lambda^3 + p\lambda^2 + q\lambda + r = 0, \tag{5}$$

where

$$\begin{aligned} p &= \beta\bar{i}_v + k_1 + k_2 + \mu, \\ q &= (\beta\bar{i}_v + \mu)(k_1 + k_2) + \mu k_1 k_2 \\ r &= \beta\bar{i}_v(k_1\alpha + k_2\mu) + \mu k_1 k_2. \end{aligned}$$

It is clear that p, q, r are always positive whenever $\tilde{R} > 1$. Using Routh-Hurwitz criterion, the equation (5) has all negative real part when $\tilde{R} > 1$ and has exactly one positive real root when $\tilde{R} < 1$. It may be concluded, therefore, that the non-trivial critical point is stable whenever $\tilde{R} > 1$ and unstable whenever $\tilde{R} < 1$. Moreover, the equation (5) has a zero eigenvalue when $\tilde{R} = 1$ and then the non-trivial critical point coincides with the trivial critical point which is neutrally stable.

3 Development of the numerical method

The time variable $t \geq 0$ will be discretized at the points $t_n = n\ell(n = 0,1,2,\ldots)$ where $\ell > 0$ is a constant time step. The solutions of the model (3) at the point t_n are $s_h(t_n), i_h(t_n), r_h(t_n)$ and $i_v(t_n)$. The solutions of the numerical method at the same point t_n will be denoted by s_h^n, i_h^n, r_h^n and i_v^n, respectively. The development of the numerical method will be based on the first-orde approximations

$$\frac{dX}{dt} = \frac{X(t+\ell) - X(t)}{\ell} + O(\ell) \text{ as } \ell \to 0, \tag{6}$$

in which $t = t_n$. Approximating the derivatives in (3) by (6), and evaluating the variables on the right-hand sides of (3) as follows: for $n = 0,1,2,\ldots$,

$$\left.\begin{aligned} \frac{s_h^{n+1} - s_h^n}{\ell} &= \mu - (\mu + \beta i_v^n)s_h^{n+1} + \delta r_h^n, \\ \frac{i_h^{n+1} - i_h^n}{\ell} &= \beta i_v^n s_h^n - k_2 i_h^{n+1}, \\ \frac{r_h^{n+1} - r_h^n}{\ell} &= \gamma i_h^n - k_1 r_h^{n+1}, \\ \frac{i_v^{n+1} - i_v^n}{\ell} &= \beta_0 i_v^n - (\beta_0 i_v^n + \mu_0) i_v^{n+1}, \end{aligned}\right\} \tag{7}$$

gives, after re-arranging,

$$\left.\begin{aligned} s_h^{n+1} &\equiv g_1 = \frac{s_h^n + \ell(\mu + \delta r_h^n)}{1 + \ell(\mu + \beta i_v^n)}, \\ i_h^{n+1} &\equiv g_2 = \frac{i_h^n + \ell\beta s_h^n i_v^n}{1 + \ell k_2}, \\ r_h^{n+1} &\equiv g_3 = \frac{r_h^n + \ell\gamma i_h^n}{1 + \ell k_1}, \\ i_v^{n+1} &\equiv g_4 = \frac{i_v^n + \ell\beta_0 i_v^n}{1 + \ell(\beta_0 i_v^n + \mu_0)}, \end{aligned}\right\} \tag{8}$$

This method (8) is denoted as method NFD.

The local truncation errors $\mathcal{L}_X \equiv \mathcal{L}_X[s_h(t), i_h(t), r_h(t), i_v(t); \ell]$ of (8), in which $t = t_n$, may be derived from (7), and are given by

$$\left.\begin{aligned} \mathcal{L}_{s_h} = \ & s_h(t+\ell) - s_h(t) - \ell\mu - \ell\delta r_h(t) \\ & + \ell(\mu + \beta i_v(t))s_h(t+\ell) \\ \mathcal{L}_{i_h} = \ & i_h(t+\ell) - i_h(t) - \ell\beta i_v(t)s_h(t) \\ & + \ell k_2 i_h(t+\ell), \\ \mathcal{L}_{r_h} = \ & r_h(t+\ell) - r_h(t) - \ell\gamma i_h(t) \\ & + \ell k_1 r_h(t+\ell), \\ \mathcal{L}_{i_v} = \ & i_v(t+\ell) - i_v(t) - \ell\beta_0 i_v(t) \\ & + \ell(\beta_0 i_v(t) + \mu_0) i_v(t+\ell), \end{aligned}\right\} \tag{9}$$

Expanding $s_h(t+\ell), i_h(t+\ell), r_h(t+\ell)$ and $i_v(t+\ell)$ as Taylor series about t in (9) lead to

$$\begin{aligned} \mathcal{L}_{s_h} = \ & \left(\tfrac{1}{2}s_h''(t) + \ell(\mu + \beta i_v(t))s_h'(t)\right)\ell^2 \\ & + O(\ell^3) \text{ as } \ell \to 0, \\ \mathcal{L}_{i_h} = \ & \left(\tfrac{1}{2}i_h''(t) + \ell k_2 i_h'(t)\right)\ell^2 \\ & + O(\ell^3) \text{ as } \ell \to 0, \\ \mathcal{L}_{r_h} = \ & \left(\tfrac{1}{2}r_h''(t) + \ell k_1 r_h'(t)\right)\ell^2 \\ & + O(\ell^3) \text{ as } \ell \to 0, \\ \mathcal{L}_{i_v} = \ & \left(\tfrac{1}{2}i_v''(t) + \ell(\beta_0 i_v(t) + \mu_0)i_v'(t)\right)\ell^2 \\ & + O(\ell^3) \text{ as } \ell \to 0, \end{aligned}$$

indicating that the method NFD (8) are first-orde accurate (see, for instance, Lambert [12], pp. 56-57). It should be noted that, although the first-orde method NFD is implicit by construction, it enables the four populations, s_h, i_h, r_h and i_v to be computed explicitly at every time step.

4 Fixed-point analysis

The aim here is to check whether the method FMD consisting of (8), has the same stability property as the original model (3). To do this, we firs consider the associated equations

$$\begin{array}{ll} s_h = g_1(s_h, i_h, r_h, i_v), & i_h = g_2(s_h, i_h, r_h, i_v), \\ r_h = g_3(s_h, i_h, r_h, i_v), & i_v = g_4(s_h, i_h, r_h, i_v), \end{array} \quad (10)$$

associated with the method NFD (8). It is easy to show that the fi ed points of the method NFD are the equilibria of the ODE system (3), namely E_1 and E_2, and it remains to establish the conditions under which the method will converge to one of the fi ed/critical points from the initial conditions s_h^0, i_h^0, r_h^0 and i_v^0.

It is well known (see, for instance, Smith [25], pp. 268-269) that a system of the form (8) converges to a fi ed point if and only if the spectral radius, $\rho(J_1)$, of the Jacobian

$$J_1(E) = \begin{bmatrix} \frac{\partial g_1}{\partial s_h} & \frac{\partial g_1}{\partial i_h} & \frac{\partial g_1}{\partial r_h} & \frac{\partial g_1}{\partial i_v} \\ \frac{\partial g_2}{\partial s_h} & \frac{\partial g_2}{\partial i_h} & \frac{\partial g_2}{\partial r_h} & \frac{\partial g_2}{\partial i_v} \\ \frac{\partial g_3}{\partial s_h} & \frac{\partial g_3}{\partial i_h} & \frac{\partial g_3}{\partial r_h} & \frac{\partial g_3}{\partial i_v} \\ \frac{\partial g_4}{\partial s_h} & \frac{\partial g_4}{\partial i_h} & \frac{\partial g_4}{\partial r_h} & \frac{\partial g_4}{\partial i_v} \end{bmatrix}, \quad (11)$$

evaluated at the fi ed point satisfie the condition $\rho(J_1) < 1$. The fi ed point is stable or attracting if $\rho(J_1) < 1$. The convergence properties of the method NFD with $\ell > 0$ are established in the following theorem.

Theorem 1 *If $\tilde{R} < 1$, then the method* NFD *will converge to the fixed point E_1 for every $\ell > 0$.*

Proof: The eigenvalues of the jacobian matrix J_1 evaluated at the point $E_1(1, 0, 0, 0)$ are given by

$$\lambda_1 = \frac{1}{1+\mu\ell}, \quad \lambda_2 = \frac{1+\ell\beta_0}{1+k_2\ell},$$
$$\lambda_3 = \frac{1}{1+k_1\ell}, \quad \lambda_4 = \frac{1+\beta_0\ell}{1+\mu_0\ell}.$$

It is seen that $|\lambda_i| < 1$ $(i = 1, 2, 3)$ and $|\lambda_4| < 1$ if $\tilde{R} < 1$ i.e. $\beta_0 < \mu_0$. This implies that the stability of the fi ed point E_1 does not depend on the step-size, ℓ (that is, the method NFD is unconditionally convergent to E_1 for all ℓ provided $\tilde{R} < 1$).

Theorem 2 *If $\tilde{R} > 1$, the method* NFD *will converge to the non–trivial fixed point E_2 for every $\ell > 0$.*

Proof: The characteristic equation of the jacobian matrix J_1 evaluated at the point E_2 is given by

$$\left(\lambda - \frac{1+\mu_0\ell}{1+\beta_0\ell}\right)\det(J_2 - \lambda I) = 0 \quad (12)$$

where $e_1 = \beta_0(1+\mu\ell) + \mu_0\beta\ell(\tilde{R}-1)$, $e_2 = 1 + k_2\ell$, $e_3 = 1 + k_1\ell$ and

$$J_2 = \begin{bmatrix} \frac{\beta_0}{e_1} & 0 & \frac{\delta\ell\beta_0}{e_1} \\ \frac{\mu_0\beta\ell(\tilde{R}-1)}{\beta_0 e_2} & \frac{1}{e_2} & 0 \\ 0 & \frac{\gamma\ell}{e_3} & \frac{1}{e_3} \end{bmatrix},$$

Clearly, from (12), one of the eigenvalues of $J_1(E_2)$ is $\lambda_1 = \dfrac{1+\mu_0\ell}{1+\beta_0\ell}$ which is verifie that $|\lambda_1| < 1$ if $\beta_0 > \mu_0$ or $\tilde{R} > 1$. For the other eigenvalues we set up

$$J_2 = \frac{1}{e_1 e_3} Q$$

where $e_1 = \beta_0 + \mu\ell\beta_0 + \beta\ell\mu_0(\tilde{R}-1)$, $e_2 = 1 + (\mu + \gamma + \alpha)\ell$, $e_3 = 1 + (\mu + \delta)\ell$ and Q is the matrix

$$Q = \begin{bmatrix} \beta_0 e_3 & 0 & \delta\ell\beta_0 e_3 \\ \frac{\mu_0\beta\ell(\tilde{R}-1)e_1 e_3}{\beta_0 e_2} & \frac{e_1 e_3}{e_2} & 0 \\ 0 & \gamma\ell e_1 & e_1 \end{bmatrix}.$$

Next, we will show that if the eigenvalues $\lambda_2, \lambda_3, \lambda_4$ of the matrix Q satisfie condition

$$|\lambda_i| < e_1 e_3, \quad i = 2, 3, 4 \quad (13)$$

then the eigenvalues $\lambda_2, \lambda_3, \lambda_4$ of J satisfie $\rho(J) < 1$. A lot of algebraic manipulation revels that the characteristic equation of Q takes the form

$$\psi(\lambda) = -\lambda^3 + u\lambda^2 - v\lambda + w = 0 \quad (14)$$

where $k_3 = \mu\beta_0 + \mu_0\beta(\tilde{R}-1)$,

$$\begin{aligned} u &= \frac{1}{e_2}\Bigg(\beta_0 k_1 k_2 + (k_1+k_2)k_3\Bigg)\ell^2 \\ &\quad + \frac{1}{e_2}\Bigg(2\left((k_1+k_2)\beta_0 + k_3\right)\ell + 3\beta_0\Bigg), \\ v &= \frac{e_1 e_3}{e_2}\Bigg(2\left((k_1+k_2)\beta_0 + k_3\right)\ell + 3\beta_0\Bigg), \\ w &= \frac{e_1^2 e_3^2}{e_2}\left(\gamma\beta\delta\mu_0\ell^3(\tilde{R}-1) + \beta_0\right). \end{aligned}$$

Clearly, $u > 0$, $v > 0$ and $w > 0$ whenever $\tilde{R} > 1$. It is easy to see that $\phi(0) = w > 0$ and that $\phi = \phi(\lambda)$ is a strictly increasing function as $\lambda \to -\infty$ so that $\phi(0) = 0$ has no negative roots. Further algebraic manipulation show that, when $\lambda = e_1 e_3$,

$$\begin{aligned} \psi &= -\frac{e_1^2 e_3^2 \ell^3 \mu_0 \beta(\tilde{R}-1)((\mu+\alpha)\delta + \mu k_1)}{e_2} \\ &\quad -\frac{e_1^2 e_3^2 \ell^3 \mu \beta_0 k_1 k_2}{e_2}; \end{aligned}$$

that is $\phi(\lambda) < 0$ at the point $\lambda = e_1 e_3$ whenever $\tilde{R} > 1$, so that $\phi(\lambda)$ has at least one real root between 0 and $e_1 e_3$ whenever $\tilde{R} > 1$.

The function $\psi(\lambda)$ has a minimum value when $\lambda = \lambda_- = \dfrac{u - (u^2 - 3v)^{1/2}}{3}$ and a maximum value when $\lambda = \lambda_+ = \dfrac{u + (u^2 - 3v)^{1/2}}{3}$. Clearly, $0 < \lambda_- < \lambda_+$ and the condition $\lambda_+ < e_1 e_3$ is satisfie whenever $y > 0$, where

$$y = 9(e_1 e_3)^2 - 6u e_1 e_3 + 3v. \tag{15}$$

Substituting for u and v in (15) gives,

$$\begin{aligned} y &= \frac{3e_1 e_3 \ell^2 k_3(3k_1 k_2 \ell + \mu + \alpha + k_2)}{e_2} \\ &\quad + \frac{3e_1 e_3 \ell^2 \beta_0(\alpha k_1 + \mu k_2 + \delta\gamma)}{e_2} \end{aligned}$$

so that $y > 0$ whenever $\tilde{R} > 1$. Thus, $0 < \lambda_- < \lambda_+ < e_1 e_3$ and $\phi(\lambda) = 0$ has no roots outside the interval bounded by 0 and $e_1 e_3$. It follows that no eigenvalues of J_2 exceed unity in modulus whenever $\tilde{R} > 1$, irrespective of the size of the time step ℓ. It is implied that the numerical method (NFD) will converge unconditionally from any starting values $s_h(0), i_h(0), r_h(0), i_v(0)$ to the non-trivial fi ed point E_2^* whenever $\tilde{R} > 1$. Therefore, we have established the following theorem.

5 Numerical Experiments

5.1 Experiment 1: effect of time-step, ℓ

To verify the convergence properties of the method NFD (8), simulations are carried out with parameter values: $\mu = 0.00004$, $\beta = 0.01$, $\beta_0 = 0.035$, $\gamma = 0.25$, $\alpha = 0.002$, $\delta = 0.1$ and vary the value μ_0. The initial values are chosen for simulation purposes: $s_h(0) = 0.5$, $i_h(0) = 0.2$, $r_h(0) = 0.06$ and $i_v(0) = 0.1$. The results are compared with those obtained using the standard forth–order Runge-Kutta (RK4) method.

The effect of time-step on the two methods is monitored by using various values of time-step ℓ and the threshold quantities are given the values $\tilde{R} = 0.8750$ $(\mu_0 = 0.04)$ and $\tilde{R} = 76.6500$ $(\mu_0 = 0.0004)$, respectively, in the simulations. The results are tabulated in Tables 1-2. It is found that the method NFD (see (8)) has a much better stability property than the RK4 method which failed when $\ell \geq 11.1$ in case $\tilde{R} < 1$ and $\ell \geq 11.3$ in case $\tilde{R} > 1$, respectively.

Table 1: Convergence properties of RK4 and NFD methods using various time-steps with $\mu_0 = 0.04$ $(R_0 = 0.8750 < 1)$

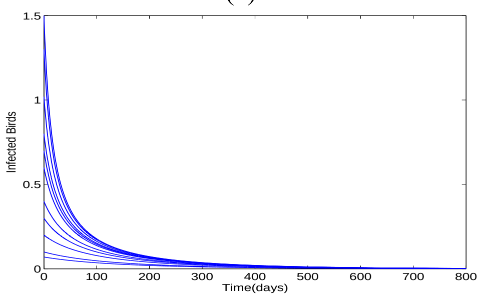
Time step	Numerical Methods	
	RK4	NFD
1	converge	converge
3	converge	converge
8.2	converge	converge
11.1	diverge(method failed)	converge
100	diverge	converge
1000	diverge	converge

Table 2: Convergence properties of RK4 and NFD methods using various time-steps with $\mu_0 = 0.0004(R_0 = 76.6500 > 1)$

Time step	Numerical Methods	
	RK4	NFD
1	converge	converge
3	converge	converge
7	converge	converge
8.1	converge	converge
11.3	diverge(method failed)	converge
100	diverge	converge
1000	diverge	converge

The stability and convergence properties of the method NFD and the RK4 method are investigated. It is found that the method NFD does not give chaotic results and seems to always give numerical results that converge to the correct steady-state solutions, E_1 and E_2 regardless of the size of the step-size used in the simulations, see Figs. 1-2 and Fig. 3(b). On the other hand, the RK4 behaves well for small step-sizes (Table 2) but exhibits scheme-dependent instabilities (divergent) when relatively large step-sizes are used (see Fig. 3(b)).

(a)



(b)

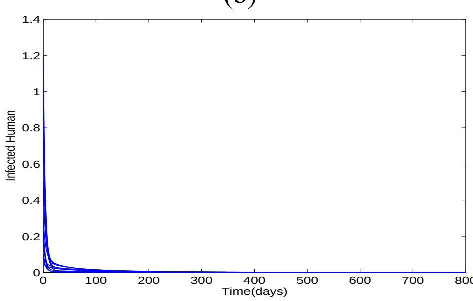
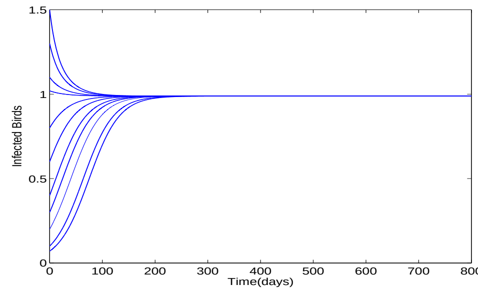


Figure 1: Simulations of the model (8) using NFD method with various initial conditions and $\tilde{R} = 0.8750 < 1$ ($\mu_0 = 0.04$): (a) The number of infected birds; (b) The number of infected humans.

6 Conclusions

In this paper, we propose a numerical scheme to solve a SIR–SI model for the transmission of avian influenz which is interesting to understand the evolution of this disease as well as other diseases of similar characteristics. Proposed method is analyzed and tested in several numerical simulations. It is found that the NFD method gives results that converged (monotonically) to the true steady-states for any time-step used unlike the RK4 method which fails when certain time-step is used. The huge advantage which the method NFD has over the RK4 method is that it may be used with an arbitrarily large value of the time

(a)



(b)

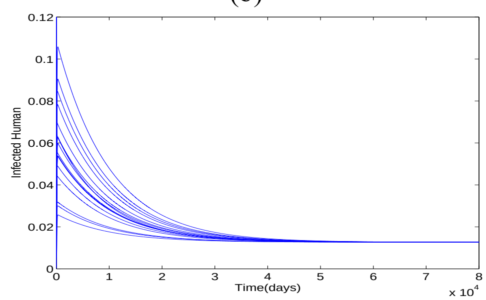
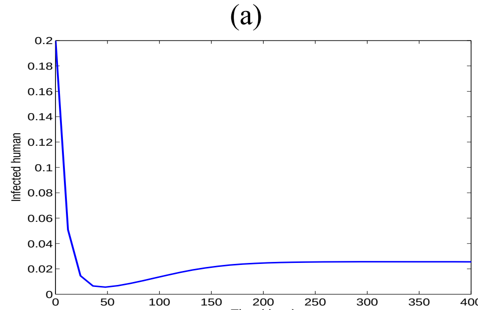


Figure 2: Simulations of the model (8) using NFD method with various initial conditions and $\tilde{R} = 76.6500 > 1$ ($\mu_0 = 0.0004$): (a) The number of infected birds; (b) The number of infected humans.

(a)



(b)

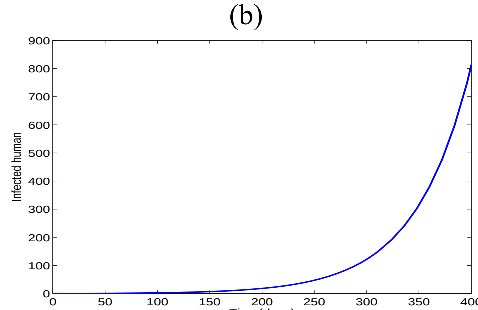


Figure 3: The profil of infected human using the methods NFD and RK4 with $\ell = 12$ and $\mu_0 = 0.04$ ($\tilde{R} > 1$): (a) The method NFD, (b) The RK4 method.

step, thus making it more economical to use when integrating over long time periods to reach steady states.

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