Random Cutouts of the d-Dimensional Balls with i.i.d. Centers

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Abstract: Consider the random open balls $B_n(\omega)$ with their centers ω_n independently and uniformly distributed over the d-dimensional unit cube $[0,1]^d$ and with their radii r_n decreasing to zero. In this paper, we show that with probability one Hausdorff dimension of the random cut-out set $[0,1]^d - \bigcup_{n=1}^{\infty} B_n(\omega)$ is at most $d - \frac{\beta(d)c^d}{p}$ and frequently equals $d - \frac{\beta(d)c^d}{p}$ when $r_n = \frac{c}{n^p}$ for some $0 < c < \sqrt[d]{\beta(d)}$ and $pd = 1$.

Key–Words: Fractal, Random fractal, Random measure, Random cut-out set

1 Introduction

The term fractal was first introduced by Mandelbrot in 1975 usually refers to sets which, in some sense, have a self-similar structure. The Cantor ternary set is one of the best known and most easily constructed fractals. Mandelbrot and others have modelled a great deal of real objects by using such fractals [1, 2]. But real phenomenons are more complex, such fractals often are not satisfactory and should be replaced by so called random fractals as pointed out in [2]. Indeed, some form of self-similarity is common in random fractals, in particular those arising from stochastic processes.

Random fractals occured as sets derived from the realizations of non-differentiable stochastic processes and fields, such as the zeros of Brownian motion and the zeros of other recurrent processes with independent stable increments. Studying random fractal aspects is an important topic of modern stochastic geometry. Within stochastic geometry the requirement arises to possess the "pure geometric" constructive examples of random fractals, generated without the aid of random fields, which are easily to handle. In this direction, the earliest investigation occurred in [3]. In [3], Mandelbrot constructed a random set in real line by "cutting out " a sequence of random intervals with decreasing length, the so-called random cut-out set. His dimension calculations were based on a birth and death process. Subsequently, Zähle in [4] considered the generalization of Mandelbrot's cutout model to higher dimension, in which the intervals are cut out are replaced by essentially more arbitrary random open sets. He gave many generalizations in terms of Hausdorff dimension. For other pure geometric constructive examples, we refer to [5-13] and the references therein. The present paper is devoted to the study on the special cases of Zähle cutout model.

Given a probability space $(\Omega, \mathcal{F}, P)$ and let $\{r_n\}_{n\geq 1}$ be a sequence of positive real numbers which is decreasing to zero. Let

$$B_n(\omega) := B(\omega_n, r_n) := \{x \in [0,1]^d : |\omega_n - x| < r_n\}$$

be a random open ball, where $|\cdot|$ denote the Euclidean distance in $\mathbb{R}^d$ and $\{\omega_n\}_{n\geq 1}$ is a sequence of i.i.d. random variables which are defined on $(\Omega, \mathcal{F}, P)$ and take values in the unit cube $[0,1]^d$. For $\omega \in \Omega$, we define $K_0 = K_0(\omega) = [0,1]^d$, and recurrently, $K_n(\omega) = K_{n-1}(\omega) - B_n(\omega)$ for $n \geq 1$. Then $\{K_n(\omega)\}_{n\geq 1}$ be a sequence of random compact sets and $K_{n+1}(\omega) \subset K_n(\omega)$. We shall call

$$K(\omega) = \bigcap_{n=0}^{\infty} K_n(\omega) = [0,1]^d - \bigcup_{n=1}^{\infty} B_n(\omega)$$

a random cutout set. Note that this construction differs from cutout set of [14] in that open balls removed may overlap.

In this paper we are interested in the case that all ω_n are uniformly distributed on $[0,1]^d$ and $\sum_{n=1}^{\infty} r_n^d = \infty$ with $r_n < \frac{1}{2}$. As we shall show in Proposition 6 of Section 3, in this case the Lebesgue measure of set $K(\omega)$ is almost surely (a.s. for short) zero. In [10], Falconer proved that with probability one the Hausdorff dimension of $K(\omega)$ is at most $1-t$ and frequently equals $1-t$ when $r_n = \frac{c}{n}$ with $0 < c < 1$ and $d = 1$, in which every open ball is a open interval. We shall determine the Hausdorff dimension of $K(\omega)$ in the case $r_n = \frac{c}{n^p}$ with $0 < c < \sqrt[d]{\beta(d)}$ and

$pd = 1$, where $\beta(d) = \Gamma(\frac{1}{2})^d/\Gamma(\frac{d}{2}+1)$ is the volume of an d dimensional ball of diameter 1. Our method is pure measure-geometric is different from the study via stochastic process in [3]. The main result of this paper is the following assertion.

Theorem 1. *Suppose $K(\omega)$ is a random cut-out set defined as above. Then*

$$P\{\dim_H K(\omega) \le \overline{\dim}_B K(\omega) \le d - \frac{\beta(d)c^d}{p}\} = 1,$$

and $P\{\dim_H K(\omega) = d - \frac{\beta(d)c^d}{p}\} > 0$, where $\dim_H$ denotes the Hausdorff dimension.

Another interesting motivation of studying random cut-out sets is that it is helpful to the study of Dvoretzky's problem and related topics, see [3, 15]. Dvoretzky's problem [16] was posed in 1956. Subsequently, it attracted the attention of Levy, Kahane, Erdos, Billard, Mandelbrot, et al. In 1972, L. Shepp [17, 18] gave a complete solution to this problem. For further information on Dvoretzky's problem, please refer to [19]. One can see [20-23] and the references therein for more recent developments.

The rest of this paper is organized as follows. Section 2 is the preliminaries. In Section 2.1, we recall the potential theoretic method. In Section 2.2, we recall the definition of Martingale and the related convergence theorems. In Section 2.3, we introduce the notion so called "random martingale measure". The proof of Theorem 1 is given in Section 3.

2 Preliminary

2.1 Potential Theoretic Method

We recall a technique for calculating Hausdorff dimensions that is widely used both in theory and in practice. Let $s \ge 0$, we call

$$I_s(\mu) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{1}{|x-y|^s}\, d\mu(x)\, d\mu(y)$$

s-energy of a measure μ on $\mathbb{R}^d$. Denoting the s-dimension Hausdorff measure by $\mathcal{H}^s$. The following Lemma can be seen in many literatures, here we refer reader to see [10, 11] for more details.

Lemma 2. *Suppose $E \subseteq \mathbb{R}^d$. (a)If there exists a finite measure μ on E with $I_s(\mu) < \infty$ and $\mu(E) > 0$, then $\mathcal{H}^s(E) = \infty$ and $\dim_H E \ge s$. (b)If E is a Borel set with $\mathcal{H}^s(E) > 0$, then for any $t \le s$ there exists a finite measure μ on E such that $I_t(\mu) < \infty$.*

2.2 Martingale and Convergence Theorems

We recall that the definition of Martingale and the related convergence theorems [24]. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots$ an increasing sequence of sub $\sigma-$fields of $\mathcal{F}$. Assume that for each k, X_k is an integrable random variable on $(\Omega, \mathcal{F}_n, P)$. We say that $\{X_k\}_{k\ge0}$, or more precisely, $\{(\mathcal{F}_k, X_k)\}_{k\ge0}$ a martingale if for all $k = 0, 1, 2, \cdots$, $\mathbb{E}(X_{k+1}|\mathcal{F}_k) = X_k$ a.s., a submartingale if $\mathbb{E}(X_{k+1}|\mathcal{F}_k) \ge X_k$ a.s., a supermartingale if $\mathbb{E}(X_{k+1}|\mathcal{F}_k) \le X_k$ a.s.. We say that $\{X_k\}_{k\ge0}$ is a nonnegative supermartingale if it is a supermartingale with $X_k \ge 0$ for all k, a L^2-bounded martingale if it is a martingale with $\sup_{0\le k<\infty} \mathbb{E}(X_k^2) < \infty$.

Lemma 3. *[24] Suppose $\{X_k\}_{k\ge0}$ is a nonnegative supermartingale described as above. Then there exists a non-negative random variable X on $(\Omega, \mathcal{F}, P)$ such that X_k converges to X a.s..*

Lemma 4. *[24] Suppose $\{X_k\}_{k\ge0}$ is a L^2-bounded martingale described as above. Then there exists a nonnegative random variable X on $(\Omega, \mathcal{F}, P)$ such that X_k converges to X a.s., and $\mathbb{E}(X) = \mathbb{E}(X_k)$ for any $k \ge 0$.*

In this paper it is enough to think of $\mathbb{E}(X_{k+1}|\mathcal{F}_k)$ as the mean of X_{k+1} calculated as though $X_0, X_1, \ldots, X_n$ are already known. In this sense, $\mathbb{E}(X_{k+1}|\mathcal{F}_k) = X_k$ means that whatever happens in the first k steps, the expectation of X_{k+1} nevertheless equals X_k.

2.3 Random Martingale Measure

Let $\mathscr{B}(\mathbb{R}^d)$ denote the family of all Borel sets of $\mathbb{R}^d$. We say that μ is a random measure with respect to probability space $(\Omega, \mathcal{F}, P)$ if μ is $\mathcal{F}$-measurable, that is, μ is a function which associates with each $\omega \in \Omega$ a measure μ_ω on $\mathbb{R}^d$ such that, for all $A \in \mathscr{B}(\mathbb{R}^d)$, the function

$$\omega \to \mu_\omega(A)$$

from Ω to $[0, +\infty)$ is $\mathcal{F}$-measurable.

Denoting the Lebesgue measure by $\mathcal{L}$ on $\mathbb{R}^d$. Let $\{\mu_n\}_{n\ge0}$ is a sequence of random measures with respect to probability space $(\Omega, \mathcal{F}, P)$. We say that $\{\mu_n\}_{n\ge0}$ is a random martingale measure, if it satisfying:

(1) μ_0 is a finite, deterministic measure with bounded support;

(2) μ_n is absolutely continuous with respect to $\mathcal{L}$ a.s. for all n;

(3) there exists an increasing sequence of sub $\sigma-$fields $\mathcal{F}_n$ of $\mathcal{F}$ such that μ_n is $\mathcal{F}_n$-measurable. Moreover, for all $A \in \mathscr{B}(\mathbb{R}^d)$,

$$\mathbb{E}(\mu_{n+1}(A)|\mathcal{F}_n) = \mu_n(A);$$

(4) there exists a constant $C > 0$ such that

$$\mu_{n+1} \leq C\mu_n \quad \text{a.s.}$$

for all n.

By Lemma 3, we have

Lemma 5. *If $\{\mu_n\}_{n\geq 0}$ is a random martingale measure, then almost surely, the sequence μ_n is weakly convergent. Denote the limit by μ.*

3 Proof of Theorem 1

Recall that $(\Omega, \mathcal{F}, P)$ is a given probability space and

$$K(\omega) = [0,1]^d - \bigcup_{n=1}^{\infty} B_n(\omega)$$

is a random cutout set, where $B_n(\omega) := B(\omega_n, r_n)$ is a random open ball with radius r_n and center ω_n, $\{r_n\}_{n\geq 1}$ is a sequence of positive real numbers which decreasing to zero and $\{\omega_n\}_{n\geq 1}$ is a sequence of i.i.d. random variables on $(\Omega, \mathcal{F}, P)$ with values in $[0,1]^d$. In the rest of paper, we will always assume that all ω_n are uniformly distributed on $[0,1]^d$. Let $\mathbb{E}(X)$ denote the expectation of a random variable X on $(\Omega, \mathcal{F}, P)$. Firstly, we have

Proposition 6. *(a) If $\sum_{n=1}^{\infty} r_n^d < \infty$, then $P(\mathcal{L}(K(\omega)) > 0) > 0$; (b)If $\sum_{n=1}^{\infty} r_n^d = \infty$, then $P(\mathcal{L}(K(\omega)) = 0) = 1$.*

Proof. Denote by χ_A the characteristic function of a set A. By Fubini Theorem, we have

$$\begin{aligned}
&\mathbb{E}(\mathcal{L}(K(\omega))) \\
&= \int \mathcal{L}(\bigcap_{n=0}^{\infty}([0,1]^d - B_n(\omega_n(\omega), r_n)))dP(\omega) \\
&= \lim_{m\to\infty} \int \mathcal{L}(\bigcap_{n=0}^{m}([0,1]^d - B_n(\omega_n(\omega), r_n))) \\
&\quad dP(\omega) \\
&= \lim_{m\to\infty} \int\int \prod_{n=0}^{m} \chi_{[0,1]^d - B_n(\omega_n(\omega), r_n))}(y) \\
&\quad dydP(\omega) \\
&= \lim_{m\to\infty} \int\int \prod_{n=0}^{m} \chi_{[0,1]^d - B_n(\omega_n(\omega), r_n))}(y) \\
&\quad dP(\omega)dy \\
&= \lim_{m\to\infty} \prod_{n=0}^{m}(1 - \beta(d)r_n^d) \\
&\leq \lim_{m\to\infty} e^{-\sum_{n=0}^{m}\beta(d)r_n^d}.
\end{aligned}$$

Since if $\sum_{k=1}^{\infty} r_n^d < \infty$ then $0 < E(\mathcal{L}^d(K(\omega))) < \infty$, if $\sum_{k=1}^{\infty} r_n^d = \infty$ then $E(\mathcal{L}^d(K(\omega))) = 0$, the desired result follows. This completes this proof. $\square$

In this paper it is convenient to identify $[0,1]^d$ with the d-dimensional torus. For example, we identify the corresponding edge of unit square $[0,1]^2$ on plane, that is, if $\omega_n = x = (x_1, x_2)(0 \leq x_1, x_2 < r_n)$ is a center of open disc $B_n(\omega)$ showed as Figure 1(a), then $B_n(\omega)$ is taken to consist of four fields that are showed by grey($B_n(\omega) \cap [0,1]^2$), red($\big((B_n(\omega) + (1,1)\big) \cap [0,1]^2$), blue($\big(B_n(\omega) + (0,1)\big) \cap [0,1]^2$) and green($\big(B_n(\omega) + (1,0)\big) \cap [0,1]^2$) in Figure 1(b) respectively. For $x = (x_1, x_2, \cdots, x_d), y =$

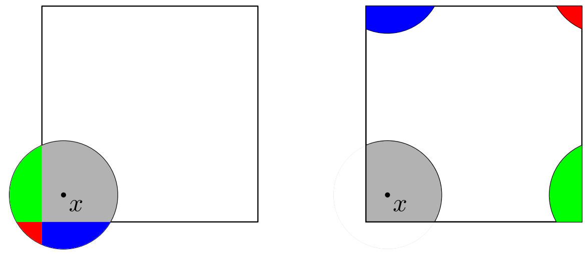

Figure 1: $d = 2$

$(y_1, y_2, \cdots, y_d) \in [0,1]^d$, we write

$$d(x,y) = \sqrt{\sum_{i=1}^{d}(\min(|x_i - y_i|, 1 - |x_i - y_i|))^2},$$

that is the distance between x and y when identifying $[0,1]^d$ with the d-dimensional torus. In particular, for $d = 1$, $d(x,y) = \min\{|x-y|, 1-|x-y|\}$, that is the distance between x and y with 0 and 1 identified. Next, we focus our intention on our research which is presented in the introduction: $r_n = \frac{c}{n^p}$ with $0 < c < \sqrt[d]{\beta(d)}$ and $pd = 1$. In this case, Proposition 6 says that $\mathcal{L}(K(\omega)) = 0$ a.s.. We first estimate the probabilities that a given point, and a given pair of points, are in $K_n(\omega)$.

Lemma 7. *(a) For all $x \in [0,1]^d$ and $n = 1, 2, \cdots$,*

$$P(x \in K_n(\omega)) = \prod_{i=1}^{n}(1 - \beta(d)r_i^d).$$

(b) Given $\varepsilon > 0$ there exists a constant $L > 0$ such that

$$\frac{P(x \in K_n(\omega), y \in K_n(\omega))}{p_n^2} \leq Ld(x,y)^{-\frac{\beta(d)c^d}{p}(1+\varepsilon)}. \tag{1}$$

for all $x, y \in [0,1]^d$ and $n = 1, 2, \cdots$, where $p_n = \prod_{i=1}^{n}(1 - \beta(d)r_i^d)$.

Proof. (a) Since each ω_i is uniformly distributed on $[0,1]^d$, it follows that for all $x \in [0,1]^d$ and $i = 1,2,\cdots$,

$$P(x \in B_i(\omega)) = \beta(d) r_i^d.$$

Note that $x \in K_n(\omega)$ if and only if $x \notin B_i(\omega)$ for all $i = 1,2,\ldots,n$. But the events $(x \notin B_i(\omega))_{i=1}^n$ are independent, so

$$\begin{aligned} P(x \in K_n(\omega)) &= \prod_{i=1}^n P(x \notin B_i(\omega)) \\ &= \prod_{i=1}^n (1-\beta(d) r_i^d). \end{aligned}$$

(b) It follows from the O'Stolz Theorem [25] and $\sum_{i=1}^\infty r_i^d = \infty$ that

$$\begin{aligned} \log p_n &= \sum_{i=1}^n \log(1-\frac{\beta(d)c^d}{i^{pd}}) \\ &\sim -\sum_{i=1}^n \frac{\beta(d)c^d}{i^{pd}} \\ &\sim -\beta(d)c^d \log n. \end{aligned} \quad (2)$$

For any $\varepsilon > 0$, we have from (2) that there exists a constant $L_1 > 0$ such that for all $n = 1,2,\cdots$,

$$\prod_{i=1}^n (1-\frac{\beta(d)c^d}{i^{pd}}) = p_n \geq L_1 n^{-\beta(d)c^d(1+\epsilon)}. \quad (3)$$

For any $x,y \in [0,1]^d$, we consider the positions of ω_i when $B_i(\omega)$ excludes both x and y. Since ω_i is uniformly distribution on $[0,1]^d$, we have

$$P(x \notin B_i(\omega), y \notin B_i(\omega)) \leq$$

$$\begin{cases} 1-\frac{\beta(d)c^d}{i^{pd}} & d(x,y) \leq \frac{c}{i^p}, \\ (1-\frac{\beta(d)c^d}{i^{pd}})^2 & d(x,y) > \frac{c}{i^p}. \end{cases}$$

Thus

$$\frac{P(x \notin B_i(\omega), y \notin B_i(\omega))}{(1-\frac{\beta(d)c^d}{i^{pd}})^2} \leq$$

$$\begin{cases} (1-\frac{\beta(d)c^d}{i^{pd}})^{-1} & d(x,y) \leq \frac{c}{i^p}, \\ 1 & d(x,y) > \frac{c}{i^p}. \end{cases}$$

By (3) and the independence of random open balls are moved, we have

$$\begin{aligned} \frac{P(x \in K_n(\omega), y \in K_n(\omega))}{p_n^2} &= \\ &= \prod_{i=1}^n \frac{P(x \notin B_i(\omega), y \notin B_i(\omega))}{(1-\frac{\beta(d)c^d}{i^{pd}})^2} \\ &\leq \prod_{i: d(x,y) \leq \frac{c}{i^p}} (1-\frac{\beta(d)c^d}{i^{pd}})^{-1} \\ &\leq (p_{i(d(x,y))})^{-1} \\ &\leq L_1^{-1}(i(d(x,y)))^{\beta(d)c^d(1+\varepsilon)}, \end{aligned}$$

where $i(d(x,y))$ is largest positive integer i such that $d(x,y) \leq \frac{c}{i^p}$. From $\frac{c}{(k+1)^p}/\frac{c}{k^p} \to 1 (k \to \infty)$, we have $\frac{c}{i(d(x,y))^p} \sim d(x,y)$. So there exists a suitable constant $L > 0$ such that

$$\frac{P(x \in K_n(w), y \in K_n(w))}{p_n^2} \leq L d(x,y)^{-\frac{\beta(d)c^d}{p}(1+\varepsilon)}$$

□

For any $A \subset \mathbb{R}^d$, we define the sequence of random measures

$$\mu_n(A) = p_n^{-1} \mathcal{L}(A \cap K_n(\omega)), \ \ n = 0,1,\cdots. \quad (4)$$

Then we have

Lemma 8. *$\{\mu_n\}_{n \geq 1}$ is a random martingale measure.*

Proof. It is clear that μ_0 is a finite, deterministic measure with bounded support and μ_n is almost surely absolutely continuous with respect to $\mathcal{L}$ for all n. Let $\mathcal{F}_n$ denote the σ-field underlying the random positions of the centers of $B_1(\omega), B_2(\omega), \ldots, B_n(\omega)$. Then $\{\mathcal{F}_n\}_{n \geq 1}$ is an increasing sequence of sub σ-field of $\mathcal{F}$ and μ_n is $\mathcal{F}_n$-measurable. For each Borel set A, we have by independence of random open balls are moved that

$$\begin{aligned} \mathbb{E}(\mu_{n+1}(A)|\mathcal{F}_n) &= \mathbb{E}(p_{n+1}^{-1}\mathcal{L}(A \cap K_n(w) \\ &\quad \cap([0,1]^d - B_{n+1}(w))|\mathcal{F}_n) \\ &= p_{n+1}^{-1}\mathcal{L}([0,1]^d - B_{n+1}(w)) \\ &\quad \mathcal{L}(A \cap K_n(w)) \\ &= p_{n+1}^{-1}(1-\frac{\beta(d)c^d}{(n+1)^{pd}}) p_n \mu_n(A) \\ &= \mu_n(A). \end{aligned}$$

Thus $\{\mu_n(A)\}_{n \geq 0}$ is a non-negative martingale for each Borel set $A \subset \mathbb{R}^d$. Furthermore, we have $\mu_{n+1}(A) \leq \mu_n(A)$ a.s. for all $A \subset \mathbb{R}^d$ and n since $K_{n+1}(\omega) \subset K_n(\omega)$ for all ω. By the definition of random martingale measure, the desired conclusion follows. □

By Lemma 5, the sequence μ_n in Lemma 8 is weakly convergent a.s.. Denoting the limit by μ. Then

Lemma 9. *$\mu(K(w)) > 0$ is of positive probability.*

Proof. Note that the inequality (1) implies that

$$\begin{aligned}
&\frac{\mathbb{E}(\chi_{K_n(w)\times K_n(w)}(x,y))}{p_n^2} \\
&\quad = \frac{P(x\in K_n(w), y\in K_n(w))}{p_n^2} \\
&\quad \leq Ld(x,y)^{-\frac{\beta(d)c^d}{p}(1+\epsilon)} \qquad . \qquad (5)
\end{aligned}$$

Choose ε such that $\frac{\beta(d)c^d}{p}(1+\varepsilon) < 1$. It follows from (4) and (5) that

$$\begin{aligned}
&\mathbb{E}((\mu_n([0,1]^d))^2) = p_n^{-2}\mathbb{E}((\mathcal{L}(K_n(\omega)))^2) \\
= \;& p_n^{-2}\mathbb{E}(\iint \chi_{K_n(\omega)}(x)\times\chi_{K_n(\omega)}(y)dxdy) \\
= \;& p_n^{-2}\mathbb{E}(\iint \chi_{K_n(\omega)\times K_n(w)}(x,y)dxdy) \\
= \;& p_n^{-2}\mathbb{E}(\mathbb{E}(\chi_{K_n(\omega)\times K_n(\omega)}(x,y))) \\
\leq \;& L\int_{[0,1]^d}\int_{[0,1]^d} d(x,y)^{-\frac{\beta(d)c^d}{p}(1+\varepsilon)}dxdy < \infty.
\end{aligned}$$

Thus $\{\mu_n([0,1]^d)\}_{n\geq 0}$ is a bounded martingale. By Lemma 4, the desired result follows. □

The proof of Theorem 1. Given $\delta > 0$ and let $K(\omega)_\delta$ denote the δ-neighborhood of $K(\omega)$, that is,

$$K(\omega)_\delta = \{x\in\mathbb{R}^d : |x-y| \leq \delta \text{ for some } y\in K(\omega)\}.$$

Denoting the largest positive integer with $r_k > \delta$ by $k(\delta)$. Let $\widetilde{B_j}(\omega)$ be an open ball with the same center as $B_j(\omega)$ and radius $r_j - \delta$ for $j \leq k(\delta)$. Then if $x\in K(\omega)_\delta$ and $j \leq k(\delta)$, then $x \notin \widetilde{B_j}(\omega)$. By the independence of the random open balls are removed, we have that for any $x\in[0,1]^d$,

$$\begin{aligned}
P(x\in K(\omega)_\delta) \;&\leq\; P(x\notin \bigcap_{j=1}^{k(\delta)} \widetilde{B_j}(\omega)) \\
&=\; \prod_{j=1}^{k(\delta)} P(x\notin\widetilde{B_j}(\omega)) \\
&\leq\; \prod_{j=1}^{k(\delta)}(1-\beta(d)(\frac{c}{j^p}-\delta)^d). \quad (6)
\end{aligned}$$

By the maximality of $k(\delta)$, we have $\frac{c}{k(\delta)^p} \sim \delta$. Thus

$$\begin{aligned}
&\log \prod_{j=1}^{k(\delta)}(1-\beta(d)(\frac{c}{j^p}-\delta)^d) \\
= \;& \sum_{j=1}^{k(\delta)}\log(1-\beta(d)(\frac{c}{j^p}-\delta)^d) \\
\leq \;& -\sum_{j=1}^{k(\delta)}\beta(d)(\frac{c}{j^p}-\delta)^d = -\beta(d)\sum_{j=1}^{k(\delta)}(\frac{c}{j^p}-\delta)^d \\
\sim \;& -\beta(d)c^d\sum_{j=1}^{k(\delta)}\frac{1}{j} \sim -\beta(d)c^d\log k(\delta) \\
\sim \;& \frac{\beta(d)c^d}{p}\log\delta. \qquad (7)
\end{aligned}$$

Given $\varepsilon > 0$, it follows from (6) and (7) that there exists a constant M such that for any $\delta \leq 1$,

$$\mathbb{E}(\mathcal{L}(K(\omega)_\delta)) = P(x\in K(\omega)_\delta) \leq M\delta^{\frac{\beta(d)c^d}{p}-\varepsilon}.$$

Thus

$$\mathbb{E}\Bigg(\sum_{\delta=2^{-k}:k=1,2,\cdots}\mathcal{L}(K(\omega)_\delta)\delta^{-\frac{\beta(d)c^d}{p}+2\varepsilon}\Bigg)$$

$$\leq M\sum_{\delta=2^{-k}:k=1,2,\cdots}\delta^\varepsilon < \infty.$$

This implies that

$$P\Bigg(\sum_{\delta=2^{-k}:k=1,2,\cdots}\mathcal{L}^d(K(w)_\delta)\delta^{-\frac{\beta(d)c^d}{p}+2\varepsilon} < \infty\Bigg) = 1$$

and with probability one there exists a constant $M' > 0$ such that for any positive integer k, we have $\mathcal{L}(K(w)_{2^{-k}})(2^{-k})^{-\frac{\beta(d)c^d}{p}+2\varepsilon} \leq M'$, and thus $\mathcal{L}(K(\omega)_\delta)\delta^{-\frac{\beta(d)c^d}{p}+2\varepsilon}$ is bounded for any $0<\delta<1$. It follows from the (2.5) of [10] that

$$P(\overline{\dim}_B K(\omega) \leq d - \frac{\beta(d)c^d}{p} + 2\varepsilon) = 1.$$

Since ε is arbitrary, we conclude that

$$P(\overline{\dim}_B K(\omega) \leq d - \frac{\beta(d)c^d}{p}) = 1.$$

It remains to determine the lower bound. Let $\varepsilon >$ 0 and μ_n and μ be the random measures on $K_n(w)$

and $K(w)$ introduced as before. By Lemma 5 and Fatou's Lemma [24], and using (5), we have

$$\mathbb{E}\Big(\iint |x-y|^{-s} d\mu(x)\mu(y) \Big)$$

$$\begin{aligned}
&= E\Big(\lim_{k\to\infty} \iint |x-y|^{-s} d\mu_n(x)\mu_n(y) \Big) \\
&= \liminf_{n\to\infty} \mathbb{E}\Big(\iint |x-y|^{-s} d\mu_n(x)\mu_n(y) \Big) \\
&= \liminf_{n\to\infty} p_n^{-2} \mathbb{E}\Big(\iint |x-y|^{-s} \\
&\quad \chi_{K_n(w)\times K_n(w)}(x,y) dxdy \Big) \\
&\leq L \int_{[0,1]^d} \int_{[0,1]^d} d(x.y)^{-s} d(x,y)^{-\frac{\beta(d)c^d}{p}(1+\epsilon)} dxdy \\
&= L \int_{[0,1]^d} \int_{[0,1]^d} d(x,y)^{-(s+\frac{\beta(d)c^d}{p}(1+\varepsilon))} dxdy \\
&< \infty
\end{aligned}$$

provided that $s < d - \frac{\beta(d)c^d}{p}(1+\varepsilon)$. This implies that for any $s < d - \frac{\beta(d)c^d}{p}$,

$$P\Big(\int\int |x-y|^{-s} d\mu(x)\mu(y) < \infty \Big) = 1.$$

By Lemma 2 and Lemma 9, $P(\dim_H K(\omega) \geq d - \frac{\beta(d)c^d}{p}) > 0$. This completes the proof.

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