Nonlinear Observer Using Mean Value Theorem And Lsqnonlin Matlab Algorithm

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Abstract: In this article, we propose a new nonlinear observer concept. The basic idea for our observer’s design is to use mean value theorem (MVT) and lsqnonlin algorithm to determine the estimation error ($e = x - \hat{x}$) and MVT parameters $\beta_i$ (between 0 and 1). The stability study is carried out thanks to the quadratic function of Lyapunov. Two numerical examples are provided to show the performance of the proposed approach. The first studies a chaotic system with a linear term ($\dot{x} = f(x, u)$) and the second deals with a linear system ($\dot{x} = Ax + f(x, u)$).

Key–Words: lsqnonlin algorithm; Nonlinear observer; Nonlinear system; mean value theorem; State estimation

1 Introduction

State observation for non-linear has been an active area of research. The goal of this work is to design observers able to reconstruct the non-measurable state of the nonlinear system. Several types of observers have been designed to solve this problem, and many outstanding results have been obtained. Despite significant progress, the main objective remains unresolved, who consist to find generalized observer for all nonlinear systems. We mention just a few: The Nonlinear Luenberger observer approach [1-3], sliding mode observers [4-6] and adaptive observers [7-9].

Several other researchers have worked on observers for nonlinear Lipschitz systems using different approaches. However, a major limitation for this system type is that most of them only work for small values of Lipschitz constant. The major problem is that if this constant is greater than an appropriate value, the design methods cannot be applied. In some recent works [10-12], The mean value theorem is used to write the state estimation error as a linear system with uncertain parameters. The use of this theorem provides a solution even for a large Lipschitz constant.

In this paper, we propose a new nonlinear observer design without unknown inputs. The basic idea is the determination of the estimation error to introduce it into observer structure, based on an error feedback mechanism. The design of this process is done using mean value theorem and lsqnonlin algorithm. The stability analysis is investigate by using a quadratic Lyapunov function. Two numerical examples are provided to show the performance of the proposed approach. The first studies a chaotic system without a linear term ($\dot{x} = f(x, u)$) and the second deals with a nonlinear system with a linear term ($\dot{x} = Ax + f(x, u)$).

2 Preliminaries

In this section, we present some tools that are important for the next section: The mean value theorem and lsqnonlin algorithm.

2.1 Mean-Value Theorem

Lemma 1: Mean Value Theorem for a Vector Function [13-14]

Let $f : R^n \rightarrow R$ is differentiable at each point of the line segment $[a \ b]$, then there exists on that line segment a point $c = b + \beta(a - b)$ between $[a \ b]$ and $\beta \in [0 \ 1]$ such that

$$f(a) - f(b) = \nabla f(c)(a - b)$$

(1)

Note that $\beta$ is a variable that changes continuously with the values of $a$ and $b$. To use the Mean-Value Theorem, it is necessary to determine at each iteration the value of $\beta$. In this paper, we have used lsqnonlin algorithm to solve this problem.
3 Problem formulation and main results

The nonlinear system can be described as following:

\[
\begin{align*}
\dot{x}(t) &= f(x(t), u(t)) \\
y(t) &= Cx(t)
\end{align*}
\]  

(2)

where \( u(t) \in \mathbb{R}^k \) and \( y(t) \in \mathbb{R}^m \) are the input and output vectors. \( f(x(t); u(t)) \) is supposed to be continuously differentiable, and \( C \in \mathbb{R}^{m \times n} = [I_m \ 0] \) are known constant matrices of appropriate dimensions and \( I_m \) is an identity matrix. We assume that \( \text{rank}(C) = m \).

3.1 Proposition of Nonlinear Observer

In this section, we will first present the structure and necessary and sufficient condition for existence of the proposed nonlinear observer. This observer has the following structure:

\[
\begin{align*}
\dot{x}(t) &= f(\hat{x}, u) + G_{\hat{x},u,e} e(t) \\
\dot{e}(t) &= (D_x(f_{\hat{x},u,e})(c_i) - \overline{G}_{\hat{x},u,e}) e(t)
\end{align*}
\]  

(3a)

(3b)

Now we can give a sufficient condition under which the observer given by (3) is indeed a nonlinear observer.

Theorem 1

If there exist positive constants \( \eta \geq 1 \) such that:

\[
D_x(f_{\hat{x},u,e})(c_i) - \overline{G}_{\hat{x},u,e} < 0
\]  

(4)

\( \overline{G}_{\hat{x},u,e} \in \mathbb{R}^{m \times n} \) is matrix which have to be designed such that \( \hat{x} \) asymptotically converges to \( x \), \( c_i = \hat{x} + \beta_i e \) with \( \beta_i \in \{ 0 \ 1 \} \). \( \overline{G}_{\hat{x},u,e} \) is a diagonal matrix, which can be written as follows

\[
\overline{G}_{\hat{x},u,e} = \eta \begin{bmatrix}
\alpha_1(\hat{x}, u, e) & 0 & \cdots & 0 \\
0 & \alpha_2(\hat{x}, u, e) & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \alpha_n(\hat{x}, u, e)
\end{bmatrix}
\]

with

\[
\alpha_k(\hat{x}, u, e) = \sum_{j=1}^{n} |g_{kj}| + \sum_{i=1}^{n} |g_{ik}|
\]  

(5)

With \( k = \{ 1, 2, ..., n \} \), \( g_{ij} \) represent the coefficients of the matrix \( D_x(f_{\hat{x},u,e})(c_i) \). \( \eta \) is positive constant \( \eta \geq 1 \), \( e(t) \in \mathbb{R}^n \) is the vector of estimation error determined by observer, \( y(t) - \hat{y}(t) \in \mathbb{R}^m \) is the measurable estimation error and \( e_1(t) \in \mathbb{R}^n \) is vector that includes measurable estimation errors and the errors determined by observer such that \( e_1(t) = [y_1(t) - \hat{y}_1(t), \ldots, y_m(t) - \hat{y}_m(t), e_{m+1}(t), \ldots, e_n(t)]^T \).

3.2 Proof of condition (3):

In order to investigate the stability and the convergence of the observer the state estimation error dynamics is analyzed:

\[
\dot{e}(t) = \dot{x}(t) - \dot{\hat{x}}(t) = f(x, u) - (f_{\hat{x},u} + G_{\hat{x},u,e} e_1(t))
\]  

(7)

We consider that \( f(x, u) \) is expressed according to the known terms, such that \( e(t) \) and \( \dot{\hat{x}}(t) \) will be determined by observer (3), \( c_i \) using lsqnonlin optimization algorithm (11-12) and the input system \( u(t) \):

\[
f_{\hat{x},u} = f_{\hat{x},u} + D_x(f_{\hat{x},u,e})(c_i) e
\]  

(8)

With \( c_i = \hat{x} + \beta_i (x - \hat{x}) = \hat{x} + \beta_i e \) such that \( \beta \in [0 \ 1] \) and \( i = \{1, 2, ..., n\} \). \( D_x \) is the differential operator defined by

\[
D_x(f_{\hat{x},u,e})(c_i) = \frac{\partial f_{\hat{x},u,e}(c_i)}{\partial x} |_{x = c_i}
\]  

(9)

Using the mean value theorem and dynamic error equation becomes:

\[
\dot{e}(t) = (D_x(f_{\hat{x},u,e})(c_i) - \overline{G}_{\hat{x},u,e}) e(t)
\]  

(10)

3.2.1 Determination of the parameters \( \beta_i \)

For the determination of the \( \beta_i \) parameters, we use lsqnonlin algorithm and Simulink block ”Interpreted MATLAB Function” (figure 2). Based on mean value theorem (1), we consider the following system of equations:

\[
X(\beta_i) = f_{\hat{x}+e,u} - f_{\hat{x},u} - D_x(f_{\hat{x},u,e})(\hat{x} + \beta_i e) e = 0
\]  

(11)

\[
X(\beta_i) = [X_1, X_2, \ldots, X_n]^T, \quad f_{\hat{x}+e,u} = [f_{\hat{x}+e,u}, f_{\hat{x}+e,u}, \ldots, f_{\hat{x}+e,u}]^T, \quad f_{\hat{x},u} = [f_{\hat{x},u}, f_{\hat{x},u}, \ldots, f_{\hat{x},u}]^T, \quad D_x(f_{\hat{x},u,e})(\hat{x} + \beta_i e) = [\nabla f_{\hat{x},u,e}(\hat{x} + \beta_i e), \nabla f_{\hat{x},u,e}(\hat{x} + \beta_i e), \ldots, \nabla f_{\hat{x},u,e} + \beta_i e]^T
\]  

and \( \beta_i = [\beta_1, \beta_2, \ldots, \beta_n]^T \).

The parameter estimation \( \beta_i \) is transformed into an optimization problem. A performance objective function is defined for the minimization; \( f_{\text{obj}} \) is used as objective function and is given by:

\[
\min_{\beta_i \in [0 \ 1]} f_{\text{obj}}(\beta_i) = \min_{\beta_i \in [0 \ 1]} \left( \sum_{j=1}^{n} (X_j(\beta_i))^2 \right)
\]  

(12)
We assume that $\beta_i$ is a diagonal matrix, which can be written as follows:

\[ \beta = \begin{bmatrix} \alpha_1(\hat{x}, u, e) & 0 & \ldots & 0 \\ 0 & \alpha_2(\hat{x}, u, e) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \alpha_n(\hat{x}, u, e) \end{bmatrix} \]

where

\[ \alpha_k(\hat{x}, u, e) = \sum_{j=1}^{n} |g_{kj}| + \sum_{i=1}^{n} |g_{ik}| \] (20)

Such that $k = \{1, 2, \ldots, n\}$.

We obtain the following expression:

\[ \dot{V}(e) = e^T (D_x(f_{\hat{x}, u})(c_i) - \mathcal{G}_{\hat{x}, u, e}) e \] (21)

The previous expression becomes negative if $\eta \geq 1$.

### 4 Illustrative example

- **Example 1:** Nonlinear system without linear term

The numerical simulation example provided to verify the effectiveness of the proposed approach is represented by ordinary differential equations. We consider the chaotic system without linear term [18]:

\[ \dot{x}(t) = \begin{pmatrix} \ln(0.1 + \exp(x_2 - x_1)) \\ 0.2 - x_1 x_2 \\ x_1 x_3 \end{pmatrix}, \ y(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} x(t) \] (22)

with $u(t) = 0$. We consider the initial conditions given by:

\[ x_0 = [-2, -1, -1]^T, \hat{x}_0 = [-1, -0.5, -1.5]^T \] and \[ e_0 = x_0 - \hat{x}_0 = [-1, -0.5, 0.5]^T. \]

Two case studies are considered:

In the first, we assume that the proposed observer has the initial estimation error of (3b) equal at $e_0 = \mathcal{N}(\beta_1) \simeq 0$ (figure 4) at each moment of the simulation for the $\beta_i$ (figure 3) parameters that evolve between 0 and 1, such that $i = 1, 2, 3$. 

#### 4.0.1 Determination of $\beta_i$ parameters

Based on the mean value theorem, Isqnonlin algorithm and equation (12). We note that the functions $\mathcal{N}(\beta_i) \simeq 0$ (figure 4) at each moment of the simulation for the $\beta_i$ (figure 3) parameters that evolve between 0 and 1, such that $i = 1, 2, 3$. 

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**Figure 1:** Determination of $\beta_i$ parameters
4.0.2 Comparison between observer’s error for $e_0$, $e_{01}$ and real error

As shown fig.5, we notice that both observer’s errors for $e_0$ and real errors are confused, the observer’s error for $e_{01}$ converge to real errors. Which validates the proposed development in the paper.

4.0.3 Comparison between two case $e_0$ and $e_{01}$ for the proposed observer

Figure 6 show satisfactory of proposed observer performance in dealing with a nonlinear system without linear term.

Example 2 Nonlinear system with linear part

In this example, we will compare the proposed observer with two recent nonlinear observers design [17] and [16]. Consider a single-link flexible robotic [19]:

$$f(x,u) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 19.5 & 0 & -19.5 & 3.33 \\ 0 & 21.6 & 0 & 0 \\ 0 & 0 & -3.33 \sin(x_3) & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 19.5 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \ u(t) = \sin(t).$$

We consider the initial conditions given by:

$$x_0 = [3 \ 3 \ 3 \ 3]; \ \hat{x}_0 = [-1 \ -2 \ -0.5 \ -1]$$

and $e_0 = x_0 - \hat{x}_0$.

- Proposed observer

First case

We chosen that $\eta = 1$ and $e_0 = x_0 - \hat{x}_0 = [2 \ 1 \ 2.5 \ 2]^T$.

Second case

We chosen that $\eta = 1$ and $e_0 = e_{01} = [2 \ 1 \ 2.5 \ 2.5]^T$.

- [17] observer

We keep the same condition considered in this paper, for more details see [17].

- [16] observer

We consider the following matrices:

$$L = \begin{bmatrix} 0.5409 & -0.6315 & 0.2392 \\ -10.3506 & 21.2355 & 22.8782 \\ 1.8370 & 3.4444 & 10.6638 \\ 15.5282 & 8.4967 & 62.7550 \end{bmatrix}.$$  

For more details see [16].
4.0.4 Determination of $\beta_i$ parameters

We note that the functions $X(\beta_i) \simeq 0$ (figure 8) at each moment of the simulation for the $\beta_i$ (figure 7) parameters that evolve between 0 and 1, such that $i = 1, 2, 3, 4$.

4.0.5 Comparison between evolution observer’s error and real error

As shown fig.9, we notice that both observer’s error (for $e_0$ and $e_{01}$) follows the real errors.

5 Conclusion

A full order nonlinear observer was proposed for a large class of nonlinear systems with unknown inputs. Mean value theorem and lsqonlin algorithm are the tools to use for the design of this observer. Numerical example is provided to show high performances of the proposed approach and the large class of nonlinear dynamical systems that are concerned.

References:


