A Controller Design based on a Functional H_∞ Filter for Delayed Singular Systems : The Time and Frequency Domain Cases

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Abstract: This paper deals with the time and frequency domain of a controller design based on a function reduced order filter for linear multi-variable delayed singular systems where measurements are affected by bounded disturbances. The control gain is designed using H_∞ techniques. The time procedure design is based on the unbiasedness of the estimation error using Sylvester equation and on Lyapunov-Krasovskii stability theory. Then a new method to avoid the time derivative of the disturbance in filtering error is proposed and the problem is solved by means of Linear Matrix Inequalities (LMIs). Both cases where the H_∞ technique is dependent or independent from the delay are dervied separetely. The frequency domain approach is derived from the time domain one by applying the factorization approach. A numerical example is given to illustrate the proposed approach.

Key–Words: Controller, singular systems, H_∞ functional filter, delay, disturbance.

1 Introduction

Singular models or descriptor models have been a topic of recurring interests of many researches. In fact, these generalized mathematical representations better describe physical systems than regular ones [23]. So a great deal of work based on the theory of state-space systems has been extended to the descriptor models.

Furthermore, delay modeling has been extensively studied as it influences the stability robustness and other performances of the systems [22]. This situation becomes obvious when dealing with communication networks, economic systems and chemical processes [1, 22, 2].

On the other hand, a great deal of work has been devoted to the design of the filter-based controller for delayed singular systems [5, 16, 8, 9]. This controller is of major importance, mainly, when the states of the systems is partially measurable. This kind of controller is getting more and more interesting especially that a great part of control designs are developed with the assumption that the state components of the system are available for the feedback [11]. However, only a few part of the state can be measured. Motivated by these facts, a recurring interests of researches are focused on the development of filtering techniques to estimate a functional of state which can be used, also, on control laws based on the state feedback principle [6]. In addition, these controllers are handled by H_∞ filters able to minimize the perturbation effect on the estimation error.

However, in the frequency domain, few results has been developed to the controller design based on a functional H_∞ filter for singular delayed systems [10, 11].

In this framework, a new time and frequency domain design procedure of filter-based controller for singular delayed systems is proposed. The time domain approach is obtained into two steps. Firstly, we give conditions ensuring the admissibility of the H_∞ problem. Secondly, we propose a functional H_∞ filter design essential to derive the control law. The estimation problem is extended to a singular one in order to avoid the time derivative of the disturbance ($\dot{w}(t)$) on the estimation error dynamic. This filter, based on unbiasedness conditions, estimate a functional of state according to a H_∞ criteria and is proposed by means of LMIs conditions. These conditions are of two kinds. One satisfies the H_∞ criteria independently from the delay and the second respects the same criteria but dependently on the state delay. Furthermore and based on the time domain results, a frequency domain approach is set to the design of the functional H_∞ filter based controller using MFDs. The main reason of formulating the results of the time domain in the frequency one is the advantages that it presents for the filter-based control. In fact, the compensator is driven by the input and the output of the system. So, only the input-output behavior of the compensator (characterized by its transfer function) influences the properties of the closed-loop system.

The outlines of the paper are as follows. Section 2 gives assumptions used through this paper and presents the functional filter-based controller problem that we propose to solve. Section 3, presents the first contribution of the paper by giving the time domain design of the controller. This contribution is presented in two parts. First, the state feedback gain is designed with respect to the H_∞ performance criteria and satisfying the problem admissibility. Second, we propose to design the filter-based controller using the unbiasedness condition, dependently and independently from the state delay. The problem is transformed into a matrix inequalities to solve. A LMI approach is then applied to optimize the gain implemented in the filter. The fourth section presents the second result of the paper by giving a frequency domain description of the controller using polynomial MFDs. A summary of the filter based controller is presented in the fifth part of the article. Section 6 gives numerical examples to illustrate our approaches and section 7 concludes the paper.

2 Problem Formulation

Let's consider the following continuous-time linear time-delay singular system described by :

$$
\begin{aligned}
E\dot{x}(t) &= Ax(t) + A_d x(t-d) + Bu(t) \\
&\quad + B_d u(t-d) + D_1 w(t) && \text{(1a)} \\
z(t) &= F_1 x(t) && \text{(1b)} \\
y(t) &= Cx(t) + D_2 w(t) && \text{(1c)} \\
x(t_0) &= \phi_0 && \text{(1d)}
\end{aligned}
$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^q$ is the output vector, $u(t) \in \mathbb{R}^p$ is the input vector, $w(t) \in \mathbb{R}^m$ is the bounded disturbance and $z(t) \in \mathbb{R}^{m_z}$ is the functional state.

E, A, A_d, B, C, B_d, D_1, F_1 and D_2 are known matrices of appropriate dimensions. ϕ_0 is the value of the initial state, $d \in \mathbb{R}_+$ is the considered delay.

In the sequel, we suppose that :

Hypothesis 1. *[5]*

1. $rank(E) = r_1 \leq n$
2. $rank(\begin{bmatrix} E \\ C \end{bmatrix}) = n$

Purpose:

The main objective of this paper is to design in the time and the frequency domain a controller based on a functional H_∞ filter for delayed singular systems. The same delay is assumed in the state and the input vectors.

3 Time domain design of the filter-based controller

We propose to solve the observer-based controller problem into two steps. First, we propose to design the control gain K_c satisfying the admissibility of the subsystem {(1a)-(1b)}. Second, we aim to design a filter-based controller in order to reconstruct the control law by estimating only the state functional essential to the controller design.

Under hypothesis 1, there exists a non singular matrix S where $S = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$ with $a_0 \in \mathbb{R}^{n\times n}$, $b_0 \in \mathbb{R}^{n\times q}$, $c_0 \in \mathbb{R}^{q\times n}$ and $d_0 \in \mathbb{R}^{q\times q}$ such that :

$$
\begin{aligned}
a_0 E + b_0 C &= I_n && \text{(2a)} \\
c_0 E + d_0 C &= 0_{q\times n} && \text{(2b)}
\end{aligned}
$$

where E and C are given in (1).

The purpose of the paper is to design a functional filter-based controller for system (1) of the form :

$$
\begin{aligned}
\dot{\chi}(t) &= F\chi(t) + F_d \chi(t-d) + H_d u(t-d) \\
&\quad + Hu(t) + L_1 y(t) + L_2 y(t-d) && \text{(3a)} \\
u(t) &= \chi(t) + My(t) && \text{(3b)}
\end{aligned}
$$

where $M = K_c b_0 + K d_0$.

Matrices F, F_d, H, H_d, L_1, L_2, K_c and K are to be designed.

Problem:

Our main objective is to build a functional filter and a control law following $(3a)$ and $(3b)$ in time domain and its equivalent descriptor in the frequency domain such that :

a) $lim_{t\to+\infty}u - K_c x = 0$, if $w = 0$.

b) The system state vector and the filtering error are asymptotically stable and satisfy the H_∞ performance.

The H_∞ criteria is given by:

$$0 < \|H_{\epsilon w}\|_\infty = sup_{w\neq 0}\frac{\|\epsilon\|_2}{\|w\|_2} < \gamma \qquad (4)$$

with $H_{\epsilon w}(s) = \frac{\epsilon(s)}{w(s)}$ is a transfer matrix, γ is a positive scalar and $\epsilon = u - K_c x$ is the estimation error.

3.1 State feedback synthesis

Let us consider the subsystem given by $(1a) - (1b)$:

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + A_d x(t-d) + Bu(t) \\ &\quad + B_d u(t-d) + D_1 w(t) && \text{(5a)} \\ z(t) &= F_1 x(t) && \text{(5b)} \end{aligned}$$

Lemma 1. *The system (5) is admissible and satisfies the H_∞ performance requirement given by (4) if and only if there exist a matrix X and a symmetric definite positive matrix Z such that :*

$$X^T E^T = EX \geq 0 \qquad (6)$$

$$V < 0 \qquad (7)$$

where

$$V = \begin{pmatrix} \alpha & A_d X & X^T F_1^T \\ * & -Z & 0 \\ * & * & -I_n \end{pmatrix} \qquad (8)$$

▲ with

$$\alpha = AX + X^T A^T + Z + \gamma^{-2} D_1 D_1^T \qquad (9)$$

By replacing u by $K_c x$ in system (5), we have:

$$\begin{aligned} E\dot{x}(t) &= (A + BK_c)x(t) + (A_d + B_d K_c)x(t-d) \\ &\quad + D_1 w(t) && \text{(10a)} \\ z(t) &= F_1 x(t) && \text{(10b)} \end{aligned}$$

So results given by Lemma 1 can be applied on system (10) in order to design the state feedback K_c according to the next proposed theorem.

Theorem 2. *The system (10) is admissible and satisfies the H_∞ criteria if and only if there exist matrices : X_c, Y_c and $Z_c = Z_c^T > 0$ such that :*

$$X_c^T E^T = EX_c \geq 0 \qquad (11)$$

$$V_c < 0 \qquad (12)$$

where

$$V_c = \begin{pmatrix} \alpha_c & \beta_x & X_c^T F_1^T \\ * & -Z_c & 0 \\ * & * & -I_n \end{pmatrix} \qquad (13)$$

with

$$\begin{aligned} \alpha_c &= AX_c + BY_c + (AX_c + BY_c)^T \\ &\quad + Z_c + \gamma^{-2} D_1 D_1^T && \text{(14)} \end{aligned}$$

and

$$\beta_x = A_d X_c + B_d Y_c \qquad (15)$$

So the state feedback gain is given by :

$$K_c = Y_c X_c^{-1} \qquad (16)$$

▲

Proof 1. *It's obtained by applying the results of [25] on system (10) and using a transformation of the main result according to the Schur lemma [3].*
In fact, the system considered on [25] is a delayed singular system with uncertain parameters. When setting the uncertainty components to zero, the admissibility conditions of the H_∞ problem is given by:

$$X_c^T E^T = EX_c \geq 0 \qquad (17)$$

and

$$M_c = \begin{pmatrix} \alpha_c & \beta_x & X_c^T F_1^T & 0 \\ * & -Z_c & 0 & 0 \\ * & * & -I_n & 0 \\ * & * & * & -\epsilon_c I \end{pmatrix} < 0 \qquad (18)$$

with ϵ_c is a positive scalar.
By realizing that M_c is a bloc-diagonal matrix and $-\epsilon_c I$ is a negative matrix, then equation (13) is the condition for the H_∞ problem admissibility. ■

3.2 Filter-based controller synthesis

3.2.1 The unbiasedness conditions of the filter-based controller

Let $\epsilon(t)$ be the estimation error: Considering (2) and $(3b)$, $\epsilon(t)$ is given by :

$$\begin{aligned} \epsilon(t) &= u(t) - K_c x(t) && \text{(19a)} \\ &= \chi(t) - (K_c a_0 + Kc_0)Ex(t) \\ &\quad + (K_c b_0 + Kd_0)D_2 w(t) && \text{(19b)} \\ &= \chi(t) - \Psi_1 Ex(t) + \Psi_2 w(t) && \text{(19c)} \end{aligned}$$

with

$$\begin{aligned}\Psi_1 &= K_c a_0 + K c_0 &(20)\\ \Psi_2 &= (K_c b_0 + K d_0) D_2 &(21)\end{aligned}$$

Given the singular system (1) and the functional filter-based controller (3), we aim to design the filter matrices F, F_d, H, H_d, L_1, L_2 and M which verify the unbiasedness estimation error conditions (if $w(t) = 0$) and the attenuation of the disturbance effect given by (4) (if $w(t) \neq 0$)

The unbiasedness of the estimation error dynamics is verified according to the following proposed theorem :

Theorem 3. *The unbiasedness of the estimation error given by (19) relative to system (1) and filter (3) is verified such that:*

$$\begin{aligned}\dot{\epsilon}(t) &= F\epsilon(t) + F_d\epsilon(t-d)\\ &\quad +\alpha w(t) + \beta w(t-d) - \zeta \dot{w}(t) \quad (22)\end{aligned}$$

if and only if the following equations are satisfied :

i) $L_1 C + F\Psi_1 E - \Psi_1 A = 0$

ii) $L_2 C + F_d \Psi_1 E - \Psi_1 A_d = 0$

iii) $H = \Psi_1 B$

iv) $H_d = \Psi_1 B_d$

with

$$\begin{aligned}\alpha &= L_1 D_2 - F\Psi_2 - \Psi_1 D_1 &(23)\\ \beta &= L_2 D_2 - F_d \Psi_2 &(24)\end{aligned}$$

and

$$\zeta = -\Psi_2 \quad (25)$$

▲

Proof 2. *The derivative of the estimation error is given as follows :*

$$\dot{\epsilon}(t) = \dot{\chi}(t) - \Psi_1 E\dot{x}(t) + \Psi_2 \dot{w}(t) \quad (26)$$

By replacing in (26) $E\dot{x}(t)$ and $\dot{\chi}(t)$ by their expressions given by (1) and (3) respectively, we have:

$$\begin{aligned}\dot{\epsilon}(t) &= F\epsilon(t) + F_d\epsilon(t-d)\\ &\quad -(\Psi_1 B - H)u(t)\\ &\quad -(\Psi_1 B_d - H_d)u(t-d)\\ &\quad +(L_2 D_2 - F_d\Psi_2)w(t-d)\\ &\quad +(L_1 D_2 - F\Psi_2 - \Psi_1 D_1)w(t)\\ &\quad +(L_2 C + F_d\Psi_1 E - \Psi_1 A_d)x(t-d)\\ &\quad +(L_1 C + F\Psi_1 E - \Psi_1 A)x(t)\\ &\quad +\Psi_2 \dot{w}(t) \quad (27)\end{aligned}$$

with the initial condition $\epsilon_0 = u_0 - K_c x_0$ ■

Purpose 1. *At this stage, we propose to design the filter-based controller dependently and independently from the state delay using the obtained state feedback gain (16).*

By replacing, in condition i) of theorem 1, Ψ_1 and Ψ_2 by their expressions in (20) and (21) respectively, we have :

$$FK_c a_0 E + JC - Kc_0 A = K_c a_0 A \quad (28)$$

with :

$$J = L_1 - FKd_0 \quad (29)$$

Similarly for condition ii) of theorem 1, we obtain :

$$F_d K_c a_0 E + J_d C - Kc_0 A_d = K_c a_0 A_d \quad (30)$$

with

$$J_d = L_2 - F_d K d_0 \quad (31)$$

Equations (28)-(31) can be written in the following matrix form :

$$X\Sigma = \Theta \quad (32)$$

where,

$$X = \begin{bmatrix} F & F_d & -K & J & J_d \end{bmatrix} \quad (33)$$

$$\Sigma = \begin{bmatrix} K_c a_0 E & 0 \\ 0 & K_c a_0 E \\ c_0 A & c_0 A_d \\ C & 0 \\ 0 & C \end{bmatrix} \quad (34)$$

$$\Theta = \begin{bmatrix} K_c a_0 A & K_c a_0 A_d \end{bmatrix} \quad (35)$$

Note that a general solution of (32), exists if and only if

$$rank \begin{bmatrix} \Sigma \\ \Theta \end{bmatrix} = rank(\Sigma) \quad (36)$$

In this case, the general solution for (32) is given by:

$$X = \Theta\Sigma^+ - Z(I - \Sigma\Sigma^+) \quad (37)$$

where Σ^+ is the generalized inverse of matrix Σ given by (34) and Z is an arbitrary matrix of appropriate dimensions, that will be determined in the sequel using LMI approach.

The unknown matrix F in (33) can be given by :

$$F = X \begin{pmatrix} I \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (38)$$

By replacing (37) in (38), we obtain :

$$F = \Theta\Sigma^{+}\begin{pmatrix} I \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} - Z(I - \Sigma\Sigma^{+})\begin{pmatrix} I \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (39)$$

Let's consider :

$$F_{11} = \Theta\Sigma^{+}\begin{pmatrix} I \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (40)$$

and

$$F_{22} = (I - \Sigma\Sigma^{+})\begin{pmatrix} I \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (41)$$

Then,

$$F = F_{11} - ZF_{22} \quad (42)$$

Similarly for matrix F_d, we obtain :

$$F_d = F_{d_{11}} - ZF_{d_{22}} \quad (43)$$

where

$$F_{d_{11}} = \Theta\Sigma^{+}\begin{pmatrix} 0 \\ I \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (44)$$

and

$$F_{d_{22}} = (I - \Sigma\Sigma^{+})\begin{pmatrix} 0 \\ I \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (45)$$

We have :

$$J = J_{11} - ZJ_{22} \quad (46)$$

where

$$J_{11} = \Theta\Sigma^{+}\begin{pmatrix} 0 \\ 0 \\ 0 \\ I \\ 0 \end{pmatrix} \quad (47)$$

and

$$J_{22} = (I - \Sigma\Sigma^{+})\begin{pmatrix} 0 \\ 0 \\ 0 \\ I \\ 0 \end{pmatrix} \quad (48)$$

Similarly for matrix J_d, we obtain :

$$J_d = J_{d_{11}} - ZJ_{d_{22}} \quad (49)$$

where

$$J_{d_{11}} = \Theta\Sigma^{+}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I \end{pmatrix} \quad (50)$$

and

$$J_{d_{22}} = (I - \Sigma\Sigma^{+})\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I \end{pmatrix} \quad (51)$$

and,

$$K = -K_{11} + ZK_{22} \quad (52)$$

with

$$K_{11} = \Theta\Sigma^{+}\begin{pmatrix} 0 \\ 0 \\ I \\ 0 \\ 0 \end{pmatrix} \quad (53)$$

$$K_{22} = (I - \Sigma\Sigma^{+})\begin{pmatrix} 0 \\ 0 \\ I \\ 0 \\ 0 \end{pmatrix} \quad (54)$$

By combining (21) and (52), we obtain :

$$\Psi_2 = \Psi_{211} - Z\Psi_{222} \quad (55)$$

with :

$$\begin{aligned} \Psi_{211} &= N_{11}D_2 && (56) \\ \Psi_{222} &= N_{22}D_2 && (57) \end{aligned}$$

and

$$\begin{aligned} N_{11} &= K_c b_0 + K_{11}d_0 && (58) \\ N_{22} &= K_{22}d_0 && (59) \end{aligned}$$

In order to avoid the derivative component of the perturbation $(\dot{w}(t))$ which affect the filtering error dynamic (22), we propose to reformulate the equation (22) on the following singular state-space form:

$$\begin{aligned} \begin{bmatrix} I_n & \zeta \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{\epsilon}(t) \\ \dot{w}(t) \end{bmatrix} &= \begin{bmatrix} F & \alpha \\ 0 & -I_m \end{bmatrix}\begin{bmatrix} \epsilon(t) \\ w(t) \end{bmatrix} \\ &+ \begin{bmatrix} F_d & \beta \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \epsilon(t-d) \\ w(t-d) \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ I_m \end{bmatrix} w(t) \quad (60) \end{aligned}$$

Let us consider $\xi = \begin{bmatrix} \epsilon(t) \\ w(t) \end{bmatrix}$, so we have :

$$\rho\dot{\xi}(t) = \widetilde{F}\xi(t) + \widetilde{F}_d\xi(t-d_1) + \widetilde{B}w(t) \tag{61}$$

where

$$\rho = \begin{bmatrix} I_n & \zeta \\ 0 & 0 \end{bmatrix} \tag{62}$$

$$\widetilde{F} = \begin{bmatrix} F & \alpha \\ 0 & -I_m \end{bmatrix} \tag{63}$$

$$\widetilde{F}_d = \begin{bmatrix} F_d & \beta \\ 0 & 0 \end{bmatrix} \tag{64}$$

and

$$\widetilde{B} = \begin{bmatrix} 0 \\ I_m \end{bmatrix} \tag{65}$$

Since the matrix ζ in (62) depends on the unknown matrix Z (See (25) and (55)). We assume that the gain matrix Z satisfies the following relation :

$$Z\Psi_{22} = Z\zeta_{22} = 0 \tag{66}$$

This assumption enables us to avoid an unknown (to be designed) gain matrix Z in the singular matrix ρ given by (62). So there always exists a matrix Z_1 such that:

$$Z = Z_1(I - \zeta_{22}\ \zeta_{22}^{+}) \tag{67}$$

with

$$\zeta_{22} = -N_{22}D_2 \tag{68}$$

and ζ_{22}^{+} is the pseudo inverse of ζ_{22} such that :

$$\zeta_{22}\zeta_{22}^{+}\zeta_{22} = \zeta_{22} \tag{69}$$

In fact,

$$\begin{aligned} Z\zeta_{22} &= Z_1(I - \zeta_{22}\zeta_{22}^{+})\zeta_{22} && (70)\\ &= Z_1(\zeta_{22} - \zeta_{22}\zeta_{22}^{+}\zeta_{22}) && (71)\\ &= Z_1(\zeta_{22} - \zeta_{22}) && (72)\\ &= 0 && (73) \end{aligned}$$

then, we have :

$$\begin{aligned} \zeta &= \zeta_{11} - Z\zeta_{22} \\ &= \zeta_{11} && (74) \end{aligned}$$

where

$$\zeta_{11} = -N_{11}D_2 \tag{75}$$

3.2.2 Filter-based controller design independent from the delay

The design procedure is based on Lyapunov-Krasovskii stability theory using LMIs approach. The filter-based controller stability conditions are independent from the delay, so the estimated state converges asymptotically to the real one for any constant time delay with the satisfaction of condition (4).

At this stage, and based on theorem 2 and Lyapunov-Krasovskii stability theory, one can get the gain matrix Z which parametrizes the filter matrices, as proposed in the following theorem.

Theorem 4. *The filter-based controller in the form of (3) is a H_∞ controller for system (1) if there exist matrices $P_{1s} = P_{1s}^T$, P_{2s}, $P_{3s} = P_{3s}^T$, $Q_1 = Q_1^T$, $Q_4 = Q_4^T$, Q_2 and Y_s satisfying the following linear matrix inequalities :*

$$Q_s = \begin{pmatrix} Q_1 & Q_2 \\ Q_2^T & Q_4 \end{pmatrix} > 0 \tag{76}$$

$$P_s\rho = \rho^T P_s^T > 0 \tag{77}$$

where

$$P_s = \begin{pmatrix} P_{1s} & {P_{2s}}^T \\ P_{2s} & P_{3s} \end{pmatrix} \tag{78}$$

with

$$P_{2s} = LP_{1s} \tag{79}$$

satisfying

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} \\ * & \alpha_{22} & \alpha_{23} & \alpha_{24} & \alpha_{25} \\ * & * & \alpha_{33} & \alpha_{34} & \alpha_{35} \\ * & * & * & \alpha_{44} & \alpha_{45} \\ * & * & * & * & \alpha_{55} \end{pmatrix} < 0 \tag{80}$$

with ${\alpha_{ij}}_{1\le i,j\le 5}$ are given in appendix A.

and then the gain Z_1 is given by:

$$Z_1 = P_{1s}^{-1}Y_s \tag{81}$$

▲

We note that according to (77), we have:

$$L^T = -N_{11}D_2 \tag{82}$$

In fact when replacing P_{2s} in (78) by its expression in (79), we have :

$$P_s\rho = \begin{pmatrix} P_{1s} & P_{1s}\zeta_{11} \\ LP_{1s} & LP_{1s}\zeta_{11} \end{pmatrix} \tag{83}$$

and

$$\rho^T P_s^T = \begin{pmatrix} P_{1s} & P_{1s}L^T \\ \zeta_{11}^T P_{1s} & \zeta_{11}^T P_{1s}L^T \end{pmatrix} \quad (84)$$

So, according to (77) and by identification we have:

$$L^T = \zeta_{11} \quad (85)$$

and when using (75), equation (82) holds.

Proof 3. *Let $V(\xi, t)$ be the Lyapunov-Krasovskii (See [16]) functional of the form :*

$$V(\xi, t) = \xi^T(t)\ P_s\rho\ \xi(t) + \int_{t-d}^{t} \xi(\mu)^T Q_s \xi(\mu) d\mu \quad (86)$$

where Q_s and P_s verify respectively (76) and (77). In order to establish sufficient conditions for existence of (3) according to (4), we should verify the following inequality:

$$H(\epsilon, w) = \dot{V}(\xi, t) + \epsilon^T(t)\epsilon(t) - \gamma^2 w^T(t)w(t) < 0 \quad (87)$$

Since $w^T w > 0$, we can write:

$$H(\epsilon, w) < H(\epsilon, w) + w^T(t)w(t) \quad (88)$$

Equation (88) can be written as:

$$H(\epsilon, w) < \dot{V}(\xi, t) + \epsilon^T(t)\epsilon(t) - (\gamma^2 - 1)w^T(t)w(t) \quad (89)$$

By considering $\gamma_x^2 = \gamma^2 - 1$ and since

$$\xi^T\xi - \gamma^2 w^T w = \epsilon^T\epsilon - \gamma_x^2 w^T w \quad (90)$$

Equation (89) can be written like:

$$H(\epsilon, w) < H_x(\xi, w) \quad (91)$$

With:

$$H_x(\xi, w) = \dot{V}(\xi, t) + \xi^T(t)\xi(t) - \gamma_x^2 w^T(t)w(t) \quad (92)$$

So it's sufficient to impose that:

$$H_x(\xi, w) < 0 \quad (93)$$

By differentiating $V(\xi, t)$ along the solution (61), we obtain :

$$\begin{aligned} H_x(\xi, w) \;=\; & \xi^T(t)[\widetilde{F}^T P_s^T + P_s\widetilde{F} + Q_s + I]\xi(t) \\ & + \xi^T(t-d)\widetilde{F}_d^T P_s^T \xi(t) \\ & + \xi^T(t) P_s \widetilde{F}_d \xi(t-d) \\ & - \xi^T(t-d) Q_s \xi(t-d) \\ & + w^T(t)\widetilde{B} P_s^T \xi(t) + \xi^T(t) P_s \widetilde{B} w(t) \\ & - \gamma_x^2 w^T(t)w(t) < 0 \end{aligned} \quad (94)$$

and it can be written like :

$$v^T \begin{pmatrix} \alpha_s & P_s\widetilde{F}_d & P_s\widetilde{B} \\ * & -Q_s & 0 \\ * & * & -\gamma_x^2 I_m \end{pmatrix} v < 0 \quad (95)$$

with

$$\alpha_s = \widetilde{F}^T P_s^T + P_s\widetilde{F} + Q_s + I \quad (96)$$

where $v^T = \left[\ \xi^T \quad \xi^T(t-d) \quad w^T(t)\ \right]$
From (95), $H(\xi, w) < 0$ if

$$\begin{pmatrix} \alpha_s & P_s\widetilde{F}_d & P_s\widetilde{B} \\ * & -Q_s & 0 \\ * & * & -\gamma_x^2 I_m \end{pmatrix} < 0 \quad (97)$$

By replacing $\widetilde{F}$, $\widetilde{F}_d$, $\widetilde{B}$, Q_s and P_s by their expressions given, respectively, by (63), (64), (65), (76) and (78) in (97) and according to equations (42), (43), (46), (49) and (52), the matrix in (97) equals that in (80) which prove theorem 3. ■

Once Z_1 is calculated using (81) and Z is calculated using (67), all filter matrices can also be given by equations (42), (43), (46), (49) and (52).

3.2.3 Filter-based controller design dependent on the delay

In this paragraph, we aim to design a filter-based controller dependently on the delay. Based on the Lyapunov-Krasovskii stability theory and with respect to the H_∞ criteria given by (4), one can get the gain matrix Z which parametrizes the filter matrices, as proposed in theorem 4. This type of design is of great importance, especially, when dealing with unknown or variable delay with known bounds such that:

$$0 < {\tau^*}_1 \leq d \leq {\tau^*}_2 \quad (98)$$

where τ_1^* and τ_2^* are scalars.

Theorem 5. *The filter-based controller in the form of (3) is a H_∞ filter for system (1) if there exist matrices $P_1 = P_1^T$, P_2, $P_3 = P_3^T$ and Y satisfying the following linear matrix inequalities :*

$$P\rho \;=\; \rho^T P^T > 0 \quad (99)$$

where

$$P = \begin{pmatrix} P_1 & {P_2}^T \\ P_2 & P_3 \end{pmatrix} \quad (100)$$

with

$$P_2 = LP_1 \quad (101)$$

satisfying

$$\Omega = \begin{pmatrix} \Xi & Q \\ Q^T & U \end{pmatrix} < 0 \quad (102)$$

Where Ξ and U are symmetric matrices of dimension $3m_z + 4m$, $Q \in \mathbb{R}^{(3m_z+4m)\times(3m_z+4m)}$.
Ξ, U and Q are given in Appendix B.

and we have :

$$Z_1 = P_1^{-1} Y \tag{103}$$

▲

We note that according to (99), we have:

$$L^T = -N_{11} D_2 \tag{104}$$

Proof 4. *The chosen Lyapunov functional is (See [14]):*

$$V(t) = V_1(t) + d[V_2(t) + V_3(t)] \tag{105}$$

with

$$V_1(t) = \xi(t)^T P\rho\xi(t) \tag{106}$$

$$V_2(t) = \int_0^d \int_{t-\theta}^t \xi(s)^T \widetilde{F}^T P \widetilde{F} \xi(s) ds d\theta \tag{107}$$

$$V_3(t) = \int_0^d \int_{t-\theta}^t \xi(s)^T \widetilde{F}_d^T P \widetilde{F}_d \xi(s) ds d\theta \tag{108}$$

In order to establish sufficient conditions for existence of (3) according to (4), we should verify the inequality (93).
The derivative of the functional $V(t)$ is :

$$\dot{V}(t) = \dot{V}_1(t) + d\dot{V}_2(t) + d\dot{V}_3(t) \tag{109}$$

According to equations (61) and (99), we have:

$$\begin{aligned} \dot{V}_1(t) = \; & \xi(t)^T [\widetilde{F}^T P + P\widetilde{F}] \xi(t) \\ & + \xi(t-d)^T \widetilde{F}_d^T P \xi(t) \\ & + \xi(t)^T P \widetilde{F}_d \xi(t-d) \\ & + w^T(t) \widetilde{B}^T P \xi(t) \\ & + \xi^T(t) P \widetilde{B} w(t) \end{aligned} \tag{110}$$

Then,

$$\begin{aligned} \dot{V}_2(t) = \int_0^d & [\xi(t)^T \widetilde{F}^T P \widetilde{F} \xi(t) \\ & - \xi(t-\theta)^T \widetilde{F}^T P \widetilde{F} \xi(t-\theta)] d\theta \end{aligned} \tag{111}$$

so,

$$\begin{aligned} \dot{V}_2(t) = \; & d\xi(t)^T \widetilde{F}^T P \widetilde{F} \xi(t) \\ & - \int_0^d \xi(t-\theta)^T \widetilde{F}^T P \widetilde{F} \xi(t-\theta)] d\theta \end{aligned} \tag{112}$$

Let's

$$\Upsilon(t-\theta) = -\widetilde{F}\xi(t-\theta) \in \mathbb{R}^n \tag{113}$$

so we write :

$$\begin{aligned} \dot{V}_2(t) = \; & d\xi(t)^T \widetilde{F}^T P \widetilde{F} \xi(t) \\ & - \int_0^d \Upsilon(t-\theta)^T P \Upsilon(t-\theta)] d\theta \end{aligned} \tag{114}$$

and

$$\begin{aligned} \dot{V}_3(t) = \int_0^d & [\xi(t)^T \widetilde{F}_d^T P \widetilde{F}_d \xi(t) \\ & - \xi(t-\theta)^T \widetilde{F}_d^T P \widetilde{F}_d \xi(t-\theta)] d\theta \end{aligned} \tag{115}$$

Let's :

$$\Upsilon_d(t-\theta) = -\widetilde{F}_d \xi(t-\theta) \in \mathbb{R}^n \tag{116}$$

so,

$$\begin{aligned} \dot{V}_3(t) = \; & d\xi(t)^T \widetilde{F}_d^T P \widetilde{F}_d \xi(t) \\ & - \int_0^d \Upsilon_d(t-\theta)^T P \Upsilon_d(t-\theta) d\theta \end{aligned} \tag{117}$$

Uniform asymptotic stability implies that :

$$\lim_{t\to+\infty} \dot{V}(t) \leq 0 \tag{118}$$

As θ is bounded, the quantities $\Upsilon(t-\theta)$ and $\Upsilon_d(t-\theta)$, respectively, given by (113) and (116) satisfy :

$$\lim_{t\to+\infty} \Upsilon(t-\theta) = \lim_{t\to+\infty} \Upsilon(t) \tag{119}$$

and,

$$\lim_{t\to+\infty} \Upsilon_d(t-\theta) = \lim_{t\to+\infty} \Upsilon_d(t) \tag{120}$$

Consequently,

$$\begin{aligned} & \lim_{t\to+\infty} \left(\int_0^d \Upsilon(t-\theta)^T P \Upsilon(t-\theta) d\theta \right) \\ & \qquad = d \lim_{t\to+\infty} \Upsilon(t)^T P \Upsilon(t) \end{aligned} \tag{121}$$

and,

$$\begin{aligned} & \lim_{t\to+\infty} \left(\int_0^d \Upsilon_d(t-\theta)^T P \Upsilon_d(t-\theta) d\theta \right) \\ & \qquad = d \lim_{t\to+\infty} \Upsilon_d(t)^T P \Upsilon_d(t) \end{aligned} \tag{122}$$

We set the variable's changes:

$$\gamma_v = \lim_{t\to+\infty} \Upsilon(t) \tag{123}$$

$$\nu = \lim_{t\to+\infty} \Upsilon_d(t) \tag{124}$$

The equations (121) and (122) can be written as :

$$\lim_{t\to+\infty}\left(\int_0^d \Upsilon(t-\theta)^T P\Upsilon(t-\theta)d\theta\right) = d\gamma_v^T P\gamma_v \quad (125)$$

and,

$$\lim_{t\to+\infty}\left(\int_d^{2d} \Upsilon_d(t-\theta)^T P\Upsilon_d(t-\theta)d\theta\right) = d\nu^T P\nu \quad (126)$$

We suppose that $\xi = \lim_{t\to+\infty}\xi(t)$, we have :

$$\begin{aligned}\lim_{t\to+\infty}\dot{V}(t) &= [\xi^T[(\widetilde{F}+\widetilde{F}_d)^T P + P(\widetilde{F}+\widetilde{F}_d)]\xi \\ &\quad +d^2\xi^T\widetilde{F}^T P\widetilde{F}\xi \\ &\quad +d^2\xi^T\widetilde{F}_d^T P\widetilde{F}_d\xi] \\ &\quad +[-d^2\gamma_v^T P\gamma_v - d^2\nu^T P\nu] \quad (127)\end{aligned}$$

Therefore, according to equation (98) we have :

$$\begin{aligned}\lim_{t\to+\infty}\dot{V}(t) &\leq [\xi^T[(\widetilde{F}+\widetilde{F}_d)^T P + P(\widetilde{F}+\widetilde{F}_d)]\xi \\ &\quad +{\tau^*}_2^2\xi^T\widetilde{F}^T P\widetilde{F}\xi \\ &\quad +{\tau^*}_2^2\xi^T\widetilde{F}_d^T P\widetilde{F}_d\xi] \\ &\quad +[\xi^T(t)P\widetilde{B}w(t) + w^T(t)\widetilde{B}P\xi(t)] \\ &\quad -{\tau^*}_1^2\gamma_v^T P\gamma_v \\ &\quad -{\tau^*}_1^2\nu^T P\nu] \quad (128)\end{aligned}$$

Then, according to (87) we have :

$$H(\xi,t) < \begin{bmatrix}\xi^T & \gamma_v^T & \nu^T\end{bmatrix}\Psi\begin{bmatrix}\xi\\ \gamma_v\\ \nu\end{bmatrix} < 0 \quad (129)$$

where

$$\Psi = \begin{pmatrix}\beta_c & 0 & 0 & P\widetilde{B}\\ 0 & -{\tau^*}_1^2 P & 0 & 0\\ 0 & 0 & {\tau^*}_1^2 P & 0\\ \widetilde{B}^T P & 0 & 0 & -\gamma_x^2 I_m\end{pmatrix} \quad (130)$$

and

$$\begin{aligned}\beta_c &= \widetilde{F}^T P + P\widetilde{F} + \widetilde{F}_d^T P + P\widetilde{F}_d \\ &\quad +{\tau^*}_2^2\widetilde{F}^T P\widetilde{F} + {\tau^*}_2^2\widetilde{F}_d^T P\widetilde{F}_d + I \quad (131)\end{aligned}$$

Since Ψ is a symmetric matrix then relation (129) is equivalent to :

$$\Psi < 0 \quad (132)$$

In order to avoid the quadratic form present in β_c, we propose to transform the inequality given by (132) in an other form according to the Schur Lemma (See [3]).
In fact, matrix Ψ can be written as :

$$\Psi = U - Q_v^T\Gamma^{-1}Q_v \quad (133)$$

where

$$\Gamma = \Xi^{-1} \quad (134)$$

According to the Schur lemma, $\Psi < 0$ and $\Gamma < 0$ if and only if :

$$\Omega_v = \begin{pmatrix}\Gamma & Q_v\\ {Q_v}^T & U\end{pmatrix} < 0 \quad (135)$$

where

$$Q_v = \begin{pmatrix}{\tau^*}_2\widetilde{F} & 0 & 0 & 0\\ 0 & I & 0 & 0\\ {\tau^*}_2\widetilde{F}_d & 0 & I & 0\\ 0 & 0 & 0 & \gamma_x I\end{pmatrix} \quad (136)$$

Now, we apply a congruence transformation to Ω_v such that:

$$\Pi = T^T\Omega_v T < 0 \quad (137)$$

Where T is a non singular matrix given by :

$$T = \begin{pmatrix}P & 0 & \dots & \dots & \dots & \dots & 0\\ 0 & P & \ddots & & & & \vdots\\ \vdots & \ddots & P & \ddots & & & \vdots\\ \vdots & & \ddots & I & \ddots & & \vdots\\ \vdots & & & \ddots & I & \ddots & \vdots\\ \vdots & & & & \ddots & I & 0\\ 0 & \dots & \dots & \dots & \dots & 0 & I\end{pmatrix} \quad (138)$$

Then, by replacing P, F, F_d, α and β by their expressions given, respectively, by (100), (42), (43), (23) and (24) in (137) and considering equation (103), theorem 4 holds. ■

4 Filter-based controller design in the frequency domain

In this section and based on time domain results, we propose the filter-based controller design procedure that operates in the frequency domain, dependently and independently form the delay, using left co-prime factorization of a transfer matrix [9, 24]. So, the filter transfer function is given by the following theorem :

Theorem 6. *The frequency domain description of the H_∞ functional filter-based controller (3) for the linear singular delayed system (1) is given by:*

$$\begin{aligned} u(s) &= T_1(s) \times y(s) \\ &\quad + T_2(s)e^{-ds} \times y(s) \end{aligned} \tag{139}$$

where

$$\begin{aligned} T_1(s) &= [I - N_1^{-1}(s)M_1(s) - N_2^{-1}(s)M_2(s)] \\ &\quad \times N_3^{-1}(s)M_3(s) \\ &= [I - T_c(s)(H + H_d e^{-ds})]^{-1} \\ &\quad \times [M + T_c(s)L_1] \end{aligned} \tag{140a}$$

$$\begin{aligned} T_2(s) &= [I - N_1^{-1}(s)M_1(s) - N_2^{-1}(s)M_2(s)] \\ &\quad \times N_4^{-1}(s)M_4(s) \\ &= [I - T_c(s)(H + H_d e^{-ds})]^{-1} \\ &\quad \times T_c(s)L_2 \end{aligned} \tag{140b}$$

with

$$T_c(s) = sI_{m_z} - F_x(s) \tag{141}$$

and

$$F_x(s) = F + F_d e^{-d_s} \tag{142}$$

and where, using left coprime factorization [9], all matrices implemented in this design are given by :

$$N_1(s) = -(sI_{mz} - F_x(s) + X_1)^{-1} + I_{m_z} \tag{143}$$
$$M_1(s) = (sI_{mz} - F_x(s) + X_1)^{-1}H \tag{144}$$
$$N_2(s) = (sI_{mz} - F_x(s) - X_2)^{-1}X_2 + I_{m_z} \tag{145}$$
$$M_2(s) = (sI_{mz} - F_x(s) - X_2)^{-1}H_d e^{-ds} \tag{146}$$
$$N_3(s) = -(sI_{mz} - F_x(s) + X_3)^{-1} + I_{m_z} \tag{147}$$
$$M_3(s) = (sI_{mz} - F_x(s) + X_3)^{-1}(L_1 - X_3 M) + M \tag{148}$$
$$N_4(s) = (sI_{mz} - F_x(s) - X_4)^{-1}X_4 + I_{m_z} \tag{149}$$
$$M_4(s) = (sI_{mz} - F_x(s) - X_4)^{-1}L_2 e^{-ds} \tag{150}$$

Note that X_1, X_2, X_3 and X_4 are matrices of appropriate dimensions such that, respectively, $det(sI_{m_z} - F_x(s) + X_1)$, $det(sI_{m_z} - F_x(s) - X_2)$, $det(sI_{m_z} - F_x(s) + X_3)$, and $det(sI_{m_z} - F_x(s) - X_4)$ are Hurwitz. ▲

Proof 5. *By applying the Laplace transform to (3a) and taking into accounts (141), we write :*

$$\begin{aligned} \chi(s) &= (sI_{m_z} - F_x(s))^{-1}Hu(s) \\ &\quad + (sI_{m_z} - F_x(s))^{-1}H_d e^{-ds}u(s) \\ &\quad + (sI_{m_z} - F_x(s))^{-1}L_2 e^{-ds}y(s) \\ &\quad + (sI_{m_z} - F_x(s))^{-1}L_1 y(s) \end{aligned} \tag{151}$$

By replacing in (3b) $\chi(s)$ by its expression in (151), we have:

$$\begin{aligned} u(s) &= [I_{m_z} - T_c(s)(H + H_d e^{-ds})]^{-1} \\ &\quad \times [M + T_c(s)L_1]y(s) \\ &\quad + [I_{m_z} - T_c(s)(H + H_d e^{-ds})]^{-1} \\ &\quad \times T_c(s)L_2 e^{-ds}y(s) \end{aligned} \tag{152}$$

So, the proposed frequency domain description holds.

5 filter-based controller design steps summary

5.1 State Feedback Synthesis

Step 1) Compute matrices X_c, Y_c and Z_c using (11) and (12).
Step 2) Compute matrix K_c using (16).

5.2 Time Domain Functional Filter-Based Controller Design

Step 1) Compute matrix S using (2).
Step 2) Compute matrices Θ and Σ using (34) and (35).
Step 3) Compute matrices F_{11}, F_{22}, $F_{d_{11}}$, $F_{d_{22}}$, J_{11}, J_{22}, $J_{d_{11}}$ and $J_{d_{22}}$ using (40), (41), (44), (45), (47), (48), (50) and (51).

5.2.1 Time Domain Design Independent from the state delay

Step 4) Compute matrix L using (82).
Step 5) Compute matrices P_s, Q_s and Y_s by solving the LMIs given by (76), (77) and (80).
Step 6) Compute matrix gain Z_1 using (81).
Step 7) Compute matrix gain Z using (67).
Step 8) Compute F and F_d using equations (42) and (43).
Step 9) Compute J, J_d and K using respectively equations (46), (49) and (52).
Step 10) Get matrices L_1 and L_2 from (29) and (31).
Step 11) Get matrices H and H_d using, respectively, conditions iii) and iv) from theorem 2.

5.2.2 Time Domain Design Dependent on the state delay

Step 4) Compute matrix L using (104).
Step 5) Compute matrices P, Q and Y by solving the LMIs given by (99), (102) and (169).
Step 6) Compute matrix gain Z_1 using (103).
Step 7) Compute matrix gain Z using (67).
Step 8) Compute F and F_d using equations (42) and (43).

Step 9) Compute J, J_d and K using respectively equations (46), (49) and (52).
Step 10) Get matrices L_1 and L_2 from (29) and (31).
Step 11) Get matrices H and H_d using, respectively, conditions iii) and iv) from theorem 2.

5.3 Functional Filter-Based Controller Design in the Frequency Domain

Step 1) Calculate X_1, X_2, X_3 and X_4 using MFDs such that, $det(sI_{m_z} - F_x(s) + X_1)$, $det(sI_{m_z} - F_x(s) - X_2)$,$det(sI_{m_z} - F_x(s) + X_3)$, $det(sI_{m_z} - F_x(s) - X_4)$ are Hurwitz.
Step 2) Compute $N_1(s)$, $M_1(s)$, $N_2(s)$, $M_2(s)$, $N_3(s)$, $M_3(s)$, $N_4(s)$, $M_4(s)$, using (143)-(150), where $F_x(s)$ is given by (142).
Step 3) Deduce from step 2) the values of $T_1(s)$, $T_2(s)$, $T_3(s)$ and $T_4(s)$.
Step 4) Compute the filter-based controller design given by (139).

6 Numerical Examples

Let's consider system (1), where :

$$E = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix},\ A = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix},\ B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$A_d = \begin{pmatrix} 0.5 & -1 \\ 1 & -1 \end{pmatrix},\ B_d = \begin{pmatrix} 1 \\ -1 \end{pmatrix},\ D_2 = 10^{-4},$$

$$C = \begin{pmatrix} 1 & 1 \end{pmatrix},\ F = \begin{pmatrix} 1 & 2 \end{pmatrix},\ D_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

We have $rank \begin{pmatrix} E \\ C \end{pmatrix} = 2$, so hypothesis 1 is verified.

According to conditions given by equation (2), we have:

$$a_0 = \begin{pmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix},\ b_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$c_0 = \begin{pmatrix} 1 & 1 \end{pmatrix},\ d_0 = 0.$$

Figure 1 illustrates the used bounded disturbance such that $\|w\|_2 = 4109.6\,units$

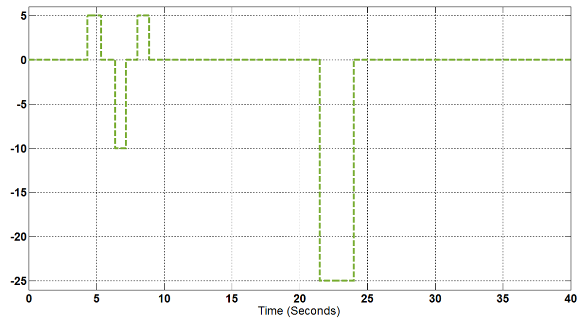


Figure 1: The Disturbance $w(t)$

6.1 The feedback gain synthesis:

The resolution of the LMI system given by equations (11) and (12) gives :

$$Z_c = \begin{pmatrix} 0.586 & 0.917 \\ 0.917 & 1.463 \end{pmatrix},$$

$$X_c = \begin{pmatrix} -0.716 & -1.303 \\ 0.064 & -0.064 \end{pmatrix},$$

$$Y_c = \begin{pmatrix} -0.0244 & -0.0948 \end{pmatrix}$$

The resolution of equation (16) leads to :

$$K_c = \begin{pmatrix} 0.0590 & 0.2772 \end{pmatrix}$$

6.2 Filter-based controller design independent from the delay:

In this paragraph we impose :

$$\gamma = 7.07, \qquad d = 1s.$$

Solving LMI (80), we get :

$$P_s = \begin{pmatrix} 0.0824 & -4.8602 \times 10^{-7} \\ -4.8602 \times 10^{-7} & 38.3992 \end{pmatrix},$$

$$Q_s = \begin{pmatrix} 44.3913 & 0.18 \\ 0.18 & 27.089 \end{pmatrix}$$

$$Y_s = \begin{pmatrix} 44.56 & 31.19 & 1.94 & -5.83 & -2.91 \end{pmatrix},$$

and

$$Z = \begin{pmatrix} Z_1 & Z_2 \end{pmatrix},$$

with

$$Z_1 = \begin{pmatrix} 541.04 & 378.73 \end{pmatrix},$$

and

$$Z_2 = \begin{pmatrix} 23.61 & -70.83 & -35.41 \end{pmatrix},$$

So, the filter-based controller matrices values are given as follows:

$$F = -541.0033,$$

$$F_d = -378.8023,$$

$$L_1 = 70.7845$$

$$L_2 = 35.3923,$$

$$H = 23.5221,$$

$$H_d = 0.2182,$$

$$K = 23.6312$$

Figures 2 and 3 show a comparison between the desired law control defined by $K_c x(t)$ and the output of the filter-based controller given by equation $(3b)$.

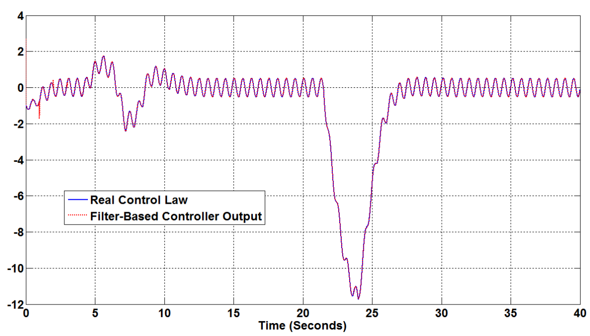
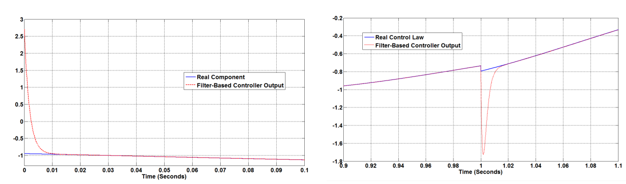


Figure 2: The Control Laws Evolution



(a) Transient Phase of the Control Laws Evolution.

(b) Delay Effect on the Control Laws Evolution (t=1s).

Figure 3: Zoom In of the Control Laws Evolution.

Figures 4 and 5 represent the evolution of the estimation error.
We can remark as it shown in figure $(4a)$ that the response time of the filter based controller is relatively small, so it has a rapid estimation dynamic. Then the disturbance effect on the estimation error (Figure (4b)) is not noticed.

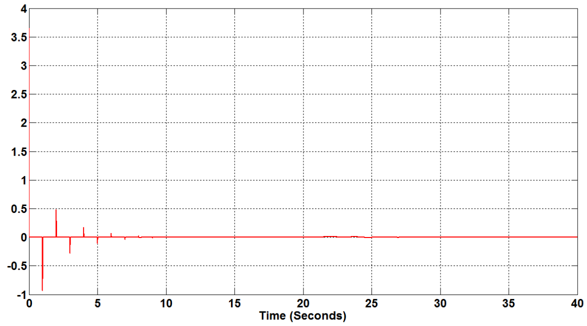
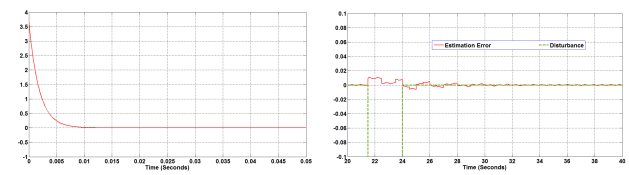


Figure 4: The Estimation Error



(a) Transient Phase of the Estimation Error.

(b) Disturbance Effect on the Estimation Error(Zone :(22s–>24s)).

Figure 5: Zoom In of the Estimation Error.

Then the signal $(y(t))$ to noise $(w(t))$ ratio is evaluated by:

$$SNR = 20Log_{10}(\frac{\|y\|_2}{\|w\|_2}) = 7.3843dB$$

and the norm H_∞ of the the transfer function of the error to the disturbance is evaluated by:

$$\|H_{\epsilon w}\|_\infty = 3.6546 < \gamma.$$

6.3 Filter-based controller design dependent on the delay

Similarly to paragraph (6.2), we suppose that:

$$\gamma = 7.07, \qquad d = 1s.$$

Then, we can consider, as an application of theorem 4 on a constant known delay, according to equation (98) that :

$$\tau^*{}_1 = \tau^*{}_2 = 1s.$$

Using equation (102), we get :

$$P = \begin{pmatrix} 0.5349 & -2.9705 \times 10^{-6} \\ -2.9705 \times 10^{-6} & 41.9921 \end{pmatrix},$$

$$Y = \begin{pmatrix} 0.58 & 0.41 & 0.02 & -0.08 & -0.04 \end{pmatrix},$$

and

$$Z = \begin{pmatrix} 1.10 & 0.77 & 0.05 & -0.15 & -0.07 \end{pmatrix},$$

So, the filter-based controller matrices values are given as follows:

$$F = -1.0599, \; F_d = -0.842,$$

$$L_1 = 0.0993, L_2 = 0.0496,$$

$$H = -0.0428, \; H_d = 0.2277,$$

$$K = 0.071$$

Using the same disturbance function $w(t)$ as used in paragraph (6.2), we draw the estimation error and the evolution of the law control.

Figure 6 shows the disturbance effect on the estimation error where figure 7 represents a comparison between the real law control and the estimated one during the Transient phase, permanent phase and the Transient duration when disturbance is applied.

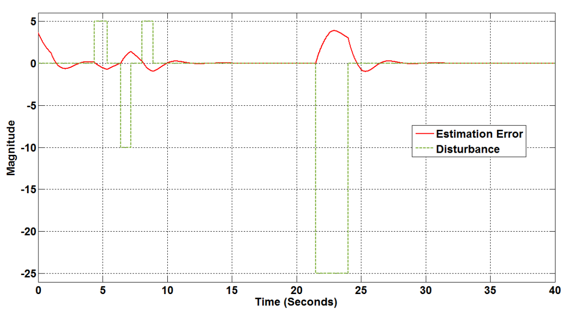


Figure 6: The Estimation Error

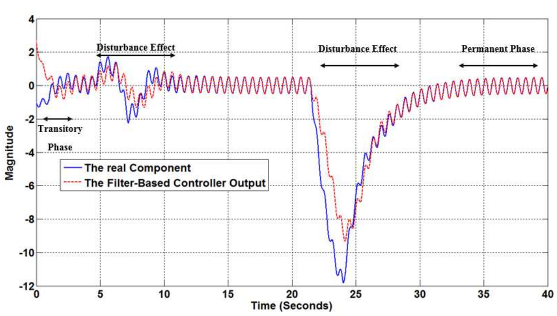


Figure 7: The Control Laws Evolution

The signal ($y(t)$) to noise ($w(t)$) ratio is evaluated by :

$$SNR = 20Log_{10}(\frac{\|y\|_2}{\|w\|_2}) = 7.7223dB$$

and the norm H_∞ of the the transfer function of the error to the disturbance is evaluated by:

$$\|H_{\epsilon w}\|_\infty = 3.8624 < \gamma.$$

Comparison between the design dependent on the delay and the independent from delay technique :

Figures 8 and 9 represent a comparison between the estimation errors using the two mentioned methods. Then, we note a quicker dynamic when using the independent from state delay technique but with a greater magnitude (figure $9a$).
It's obvious in (figure $9b$) that the dependent on delay method leads to a better error magnitude during the permanent phase.

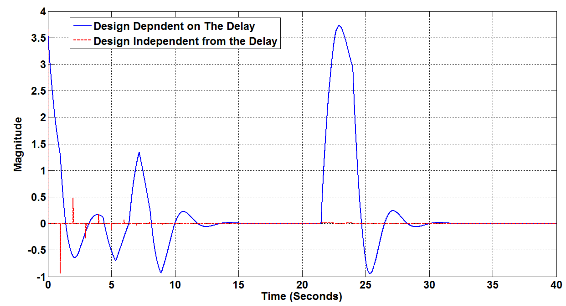
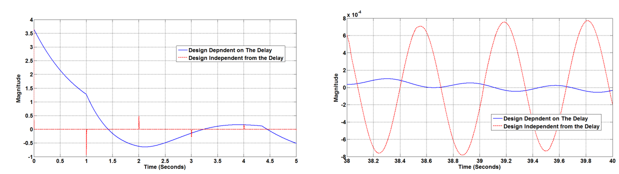


Figure 8: The Estimation Error



(a) Transient Phase of the Evolution of the estimation Error.

(b) Permanent Phase of the evolution of the Estimation Error.

Figure 9: Zoom In of the Estimation Error : Dependent and Independent techniques.

6.4 Filter-based controller design Dependent on the delay: Application on a linear singular system with variable state delay

In this paragraph, we switch the constant state delay to a variable one $d(t)$ such that :

$$d(t) = 0.3sin(t) + 0.7$$

Then, we have :

$$\tau^*{}_1 = 0.4s. \qquad \tau^*{}_2 = 1s.$$

Using equation (102), we get :

$$P = \begin{pmatrix} 0.5312 & -3.1338 \times 10^{-6} \\ -3.1338 \times 10^{-6} & 42.0079 \end{pmatrix},$$

$$Y = \begin{pmatrix} 0.60 & 0.42 & 0.02 & -0.07 & -0.04 \end{pmatrix},$$

and

$$Z = \begin{pmatrix} 1.13 & 0.79 & 0.04 & -0.14 & -0.07 \end{pmatrix},$$

So, the filter-based controller matrices values are given as follows:

$$F = -1.0924, \ F_d = -0.8647,$$

$$L_1 = 0.0994, L_2 = 0.0497,$$

$$H = -0.0396, H_d = 0.2182,$$

$$K = 0.0695$$

When using the same disturbance function $w(t)$ as used in paragraph (6.2), we draw the estimation error as shown in Figure 11 and the evolution of the function $u(t)$ and $K_c x(t)$:

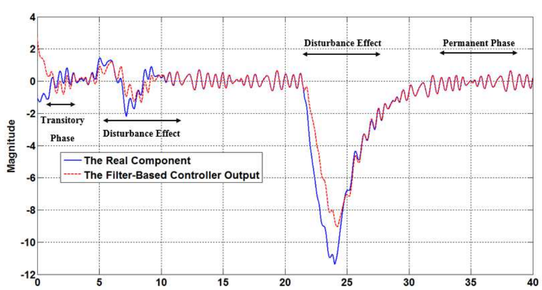


Figure 10: The Control Laws Evolution

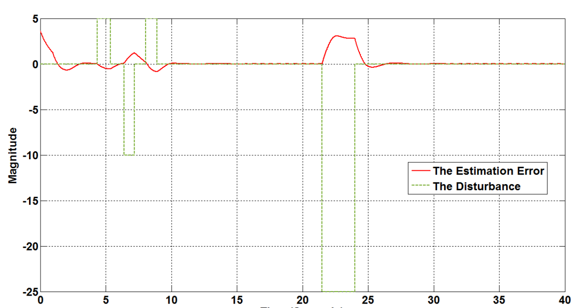


Figure 11: The Estimation Error

The signal $(y(t))$ to noise $(w(t))$ ratio is evaluated by :

$$SNR = 20Log_{10}(\frac{\|y\|_2}{\|w\|_2}) = 7.5782dB$$

and the norm H_∞ of the the transfer function of the error to the disturbance is evaluated by:

$$\|H_{\epsilon w}\|_\infty = 3.6831 < \gamma.$$

6.5 Filter-based controller synthesis in the frequency domain

6.5.1 Filter-based controller design independent from the delay

The considered controller is the same as in paragraph (6.2).

By using the left co-prime factorization, matrices of the frequency domain description of the filter-based controller for singular system (1) are given by :

$$X_1 = X_3 = 0.9575,$$

$$X_2 = X_4 = 0.9649$$

$$N_1(s) = \frac{s + 378.8e^{-s} + 540.9}{s + 378.8e^{-s} + 541.9},$$

$$M_1(s) = \frac{23.52}{s + 378.8e^{-s} + 541.9}$$

Then :

$$N_1{}^{-1}M_1(s) = \frac{23.52}{s + 378.8e^{-s} + 540.9};$$

and we have :

$$N_2(s) = \frac{s + 365.5e^{-s} + 522}{s + 378.8e^{-s} + 540},$$

$$M_2(s) = \frac{0.2e^{-s}}{s + 378.8e^{-s} + 540}$$

Then :

$$N_2{}^{-1}M_2(s) = \frac{0.2e^{-s}}{s + 365.5e^{-s} + 522};$$

Similarly to $N_1(s)$ and $M_1(s)$, we get :

$$N_3(s) = N_1(s)$$

$$M_3(s) = \frac{0.06s + 22.3e^{-s} + 38.8}{s + 378.8e^{-s} + 541.9}$$

Then :

$$N_3{}^{-1}M_3(s) = \frac{0.06s + 22.3e^{-s} + 38.8}{s + 378.8e^{-s} + 540.9};$$

And finally :

$$N_4(s) = N_2(s)$$

$$M_4(s) = \frac{35.4e^{-s}}{s + 378.8e^{-s} + 540.03}$$

Then

$$N_4{}^{-1}M_4(s) \quad = \quad \frac{35.4e^{-s}}{s + 365.5e^{-s} + 522};$$

The singular values plot is given by figure 12.

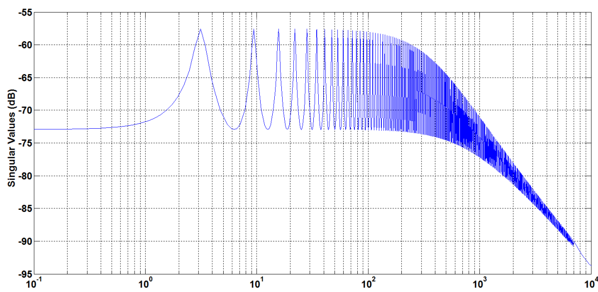


Figure 12: The singular values plot

6.5.2 Filter-based controller design dependent on the delay

The considered controller is the same as in paragraph (6.3).
By using the left co-prime factorization, matrices of the frequency domain description of the filter-based controller for singular system (1) are given by :

$$X_1 = X_3 = 0.1576,$$

$$X_2 = X_4 = 0.9706$$

$$N_1(s) \quad = \quad \frac{s + 0.74e^{-s} + 0.079}{s + 0.74e^{-s} + 1.079},$$

$$M_1(s) \quad = \quad \frac{-0.047}{s + 378.8e^{-s} + 541.9}$$

Then :

$$N_1{}^{-1}M_1(s) \quad = \quad \frac{-0.047}{s + 0.74e^{-s} + 0.079};$$

and we have :

$$N_2(s) \quad = \quad \frac{0.97s + 0.72e^{-s} + 0.95}{s + 0.74e^{-s} - 0.0492},$$

$$M_2(s) \quad = \quad \frac{0.22e^{-s}}{s + 0.74e^{-s} - 0.0492}$$

Then :

$$N_2{}^{-1}M_2(s) \quad = \quad \frac{0.22e^{-s}}{0.97s + 0.72e^{-s} + 0.95};$$

Similarly to $N_1(s)$ and $M_1(s)$, we get :

$$N_3(s) \quad = \quad N_1(s)$$

$$M_3(s) \quad = \quad \frac{0.086s + 0.064e^{-s} + 0.034}{s + 0.74e^{-s} + 1.079}$$

Then :

$$N_3{}^{-1}M_3(s) \quad = \quad \frac{0.086s + 0.064e^{-s} + 0.034}{s + 0.74e^{-s} + 0.079};$$

And finally :

$$N_4(s) \quad = \quad N_2(s)$$

$$M_4(s) \quad = \quad \frac{0.038e^{-s}}{s + 0.74e^{-s} - 0.0492}$$

Then :

$$N_4{}^{-1}M_4(s) \quad = \quad \frac{0.038e^{-s}}{0.97s + 0.72e^{-s} + 0.95};$$

The singular values plot is given by figure 13.

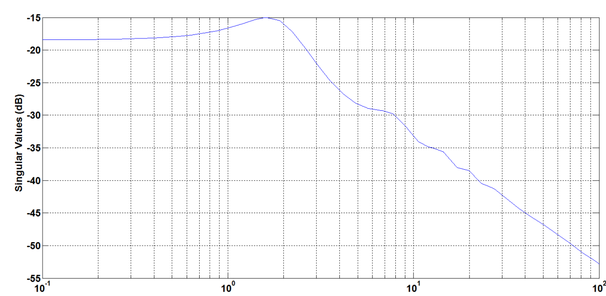


Figure 13: The Singular values plot

In this section, we show the filter-based controller designs effectiveness in numerical examples. So, we highlight the effectiveness of the design techniques independently from the state delay and dependently on the delay with an application on a variable state delay.

7 Conclusion

In this paper, we have studied the problem of controller design based on a functional H_∞ filter for singular systems with delay in both state and input vector. The controller is set in time and frequency domains. The time domain method begin with computing the feedback gain for the control law design with the respect to the admissibility problem and a H_∞ criteria by means of LMIs. Then, a functional filter techniques are used to reconstruct this control law. Note that the filter synthesis verifies a LMI condition dependently and independently from the delay and based on Lyapunov-Krasovskii theory. The frequency domain approach is based on the time domain result. So using some useful MFDs, functional H_∞ filter description is given. The proposed approaches have been applied on a numerical example and they show their effectiveness.

References:

[1] Y. Ariba, Sur la stabilité des systèmes à retards variant dans le temps : Théorie et application au contrôle de congestion d'un routeur. Ph.D. dissertation, Université de Toulouse 3 Paul Sabatier, France, 2009.

[2] E. K. Boukas, and N. F. Al-Muthairi, (2006). *Delay-dependent stabilization of singular linear systems with delays.,International Journal of Innovative Computing, Information and Control*, 283-291.

[3] S. Boyd and L. Vandenberghe, "Convex Optimization", Cambridge University Press, First edition, 2004.

[4] M. Darouach, *Linear Functional Filters for Systems With Delays in State Variables : The Discrete-Time Case.*(2005). *IEEE Transactions on Automatic Control*. Vol. 50 No. 2, pp. 228-233.

[5] M. Darouach, and M. Boutayeb, (1995). *Design of filters for descriptor systems,* 40^{th}*IEEE Transactions on Automatic Control*, no. 7, 1323-1327.

[6] M. Ezzine, M. Darouach, H. Souley Ali and H. Messaoud, (2010a).*A controller design based on function observer for singular linear systems : Time and frequency domains approaches, 18th Mediterranean Conference*, pp. 413-418, Marrakech, Morocco.

[7] M. Ezzine, M. Darouach, H. Souley Ali and H. Messaoud, (2010b). *Time and Frequency domain design of Functional Filters, American Control Conference*, pp. 599-604, Baltimore, MD, USA.

[8] M. Ezzine, M. Darouach, H. Souley Ali and H. Messaoud, (2010c). *Synthése temporelle et fréquentielle de filtres H_∞ d'ordre plein pour les systèmes singuliers [Design of full order H_∞ filters for singular systems in the time and the frequency domain] ,Conférence Internationale Francophone d'Automatique*, Nancy, France.

[9] M. Ezzine, M. Darouach, H. Souley Ali and H. Messaoud, (2011a). *Unknown Inputs Functional Filters Designs for Descriptor Systems with Constant Time Delay,*18^{th} *IFAC*, Milano-Italy.

[10] M. Ezzine, M. Darouach, H. Souley Ali and H. Messaoud, (2011b). *Full order H_∞ filtering for Linear systems in the frequency domain : The time and frequency domain cases,International Journal of Control, Automation, and Systems*, pp. 558-565.

[11] M. Ezzine, M. Darouach, H. Souley Ali and H. Messaoud, (2012). *A controller design based on a functional H_∞ filter for descriptor systems : The time and frequency domain cases,Automatica*, pp. 542-549.

[12] A. Fattouh, O. Sename and J-M. Dion,D. Koenig, D. Jacquet and S. Mammar, *Robust filter design for time-delay systems: a Riccati equation approach,Kybernetika*, Vol. 35, No. 6, pp. 753-764.

[13] J. Hale, and S. Lunel, (1991). *Introduction to functional differential equations,*Applied Mathematical Sciences 99, Springer-Verlag.

[14] M. Khadhraoui, M. Ezzine, H. Messaoud, and M. Darouach, (2014a). *Design of full order unknown input observers for delayed singular systems with tate variable time delay, International Symposium on Communications, Control and Signal Processing (ISCCSP2014)*, pp. 624-627, Athens, Grec.

[15] M. Khadhraoui, M. Ezzine, H. Messaoud, and M. Darouach, (2014b). *Functional Observers design with unknown inputs for delayed singular systems : The discrete time case, Mediterranean Conference of Control and Automation (MED2014c)*, pp. 127-132, Palermo, Italy.

[16] M. Khadhraoui, M. Ezzine, H. Messaoud, and M. Darouach, (2014c). *Design of Full Order H_∞ Filter for Delayed Singular Systems with Unknown Input and Bounded Disturbance, International Conference on Control, Decision and Information Technologies (CODIT2014)*, pp. 429-434, Metz-France.

[17] M. Khadhraoui, M. Ezzine, H. Messaoud, and M. Darouach, (2014d). *Design of Full Order Ob-*

servers with Unknown Inputs for Delayed Singular Systems with Constant Time Delay, International Conference on Control, Decision and Information Technologies (CODIT2014), pp. 423-428, Metz-France.

[18] D. Koenig, D. Jacquet and S. Mammar, (2006). *Delay-dependent H_∞ filter of linear delay descriptor systems,Proc. American Control Conference*, pp.3813-3817, Minneapolis, MN, USA.

[19] D. Koenig and B. Marx, (2004). *Design of Observers for descriptor systems with delayed state and unknown inputs,Proc. American Control Conference*, pp.4806-4810, Boston, MA, USA.

[20] N. Krasovskii, (1963). *Stability of Motion,Stanford University Press*.

[21] I. Masubuchi, Y. Kamitane, A. Ohara and N. Suda, (1997). *H_∞ Control for descriptor systems: A matrix inequalities approach,Automatica*, vol. 33, no. 4, pp. 669-673.

[22] I. S. Niculescu, (2001). *Delay effect on stability, A robust control approach, Springer Verlag.*

[23] Verghese, G., Levy, B. and Kailath, T. (1981). *A generalized state-space for singular systems,IEEE Transactions on Automatic Control*, pp. 811-831.

[24] M. Vidyasagar, (1985). *Control systems Synthesis: A factorization approach,M.I.T, Press*, Cambridge.

[25] S. Xu, B. J. Lam and C. Yang, (2003). *Robust H_∞ control for uncertain singular systems with state delay,International Journal of Robust and Nonlinear Control*, vol. 13, pp. 1213-1223.

Appendix A : Theorem 4 equations:

$$\begin{aligned}
\alpha_{11} &= P_{1s}F_{11} - Y_s\Delta F_{22} + (P_{1s}F_{11} - Y_s\Delta F_{22})^T \\
&\quad +Q_1 + I_n \qquad (153) \\
\alpha_{12} &= -P_{1s}K_c a_0 D_1 - P_{1s}K_{11}c_0 D_1 \\
&\quad -P_{1s}F_{11}K_c b_0 D_2 + Y_s\Delta F_{22}K_c b_0 D_2 \\
&\quad -Y_s\Delta J_{22}D_2 + (LP_{1s}F_{11} - LY_s\Delta F_{22})^T \\
&\quad -P_{1s}L^T + Q_2 + P_{1s}J_{11}D_2 \qquad (154) \\
\alpha_{13} &= P_{1s}F_{d_{11}} - Y_s\Delta F_{d_{22}} \qquad (155) \\
\alpha_{14} &= -P_{1s}F_{d_{11}}K_c b_0 D_2 - Y_s\Delta F_{d_{22}}K_c b_0 D_2 \\
&\quad -Y_s\Delta J_{d_{22}}D_2 + P_{1s}J_{d_{11}}D_2 \qquad (156) \\
\alpha_{15} &= P_{1s}L^T \qquad (157) \\
\alpha_{22} &= -LP_{1s}K_c a_0 D_1 - LP_{1s}K_{11}c_0 D_1 + LP_{1s}J_{11}D_2 \\
&\quad -LP_{1s}F_{11}K_c b_0 D_2 + LY_s\Delta F_{22}K_c b_0 D_2 \\
&\quad -LY_s\Delta J_{22}D_2 + (LP_{1s}a_0 D_1 - LP_{1s}K_{11}c_0 D_1 \\
&\quad -LP_{1s}F_{11}K_c b_0 D_2 + LY_s\Delta F_{22}K_c b_0 D_2 \\
&\quad +LP_{1s}J_{11}D_2 - LY_s\Delta J_{22}D_2 - LP_{1s}L^T)^T \\
&\quad -P_{3s} - P_{3s}^T + Q_4 + I_m \qquad (158) \\
\alpha_{23} &= LP_{1s}F_{d_{11}} - LY_s F_{d_{22}} \qquad (159) \\
\alpha_{24} &= LP_{1s}F_{d_{11}}K_c b_0 D_2 - LY_s\Delta F_{d_{22}}K_c b_0 D_2 \\
&\quad +LY_s\Delta J_{d_{22}}D_2 - LP_{1s}J_{d_{11}}D_2 \qquad (160) \\
\alpha_{25} &= P_{3s} \qquad (161) \\
\alpha_{33} &= -Q_1 \qquad (162) \\
\alpha_{34} &= -Q_2 \qquad (163) \\
\alpha_{35} &= 0_{n\times m} \qquad (164) \\
\alpha_{44} &= -Q_4 \qquad (165) \\
\alpha_{45} &= 0_m \qquad (166) \\
\alpha_{55} &= -(\gamma^2 - 1)I_m \qquad (167)
\end{aligned}$$

and

$$\Delta = (I - \zeta_{22}\zeta_{22}^+) \qquad (168)$$

Appendix B: Theorem 5 equations:

$$\Xi = \begin{pmatrix} -P & 0 & 0 & 0 \\ 0 & -P & 0 & 0 \\ 0 & 0 & -P & 0 \\ 0 & 0 & 0 & -I_m \end{pmatrix} \qquad (169)$$

$$\begin{aligned}
U_{11} &= h_1 + h_1^T + I_{m_z} \qquad (170) \\
U_{22} &= h_2 + h_2^T + I_m \qquad (171) \\
U_{33} &= U_{55} = U_{34} = -(1 + {\tau^*}_1^2)P_1 \qquad (172) \\
U_{44} &= U_{66} = -(1 + {\tau^*}_1^2)P_3 \qquad (173) \\
U_{77} &= -2\gamma_x^2 I_m \qquad (174) \\
U_{56} &= (1 + {\tau^*}_1^2)P_1 N_{11} D_2 \qquad (175) \\
U_{12} &= h_3 + h_4 + h_5 L^T \qquad (176) \\
U_{13} &= U_{35} = 0_{m_z\times m_z} \qquad (177) \\
U_{14} &= U_{36} = U_{37} = U_{57} = 0_{m_z\times m} \qquad (178) \\
U_{15} &= -h_5 \qquad (179)
\end{aligned}$$

$$
\begin{aligned}
U_{16} &= -h_5 L^T && (180)\\
U_{17} &= -P_1 N_{11} D_2 && (181)\\
U_{23} &= U_{45} = 0_{m\times m_z} && (182)\\
U_{24} &= U_{46} = U_{47} = U_{67} = 0_m && (183)\\
U_{25} &= -h_4^T && (184)\\
U_{26} &= -h_4^T L^T && (185)\\
U_{27} &= P_3 && (186)
\end{aligned}
$$

with

$$
\begin{aligned}
h_1 &= P_1F_{11} - Y\Delta F_{22} + P_1F_{d_{11}}\\
&\quad -Y\Delta F_{d_{22}} && (187)\\
h_2 &= -LP_1K_ca_0D_1 - LP_1K_{11}c_0D_1\\
&\quad +LY\Delta F_{22}K_cb_0D_2\\
&\quad -LP_1F_{11}K_cb_0D_2 - LY\Delta J_{22}D_2\\
&\quad +(LP_1a_0D_1 - LP_1K_{11}c_0D_1\\
&\quad -LP_1F_{11}K_cb_0D_2 - LP_1L^T\\
&\quad +LY\Delta F_{22}K_cb_0D_2\\
&\quad +LP_1J_{11}D_2 - LY\Delta J_{22}D_2)^T\\
&\quad +LP_1F_{d_{11}}K_cb_0D_2 + LY\Delta J_{d_{22}}D_2\\
&\quad -LY\Delta F_{d_{22}}K_cb_0D_2 - LP_1J_{d_{11}}D_2\\
&\quad +LP_1J_{11}D_2 - P_3 - P_3^T && (188)\\
h_3 &= -P_1K_ca_0D_1 - P_1K_{11}c_0D_1\\
&\quad -P_1F_{11}K_cb_0D_2 + Y\Delta F_{22}K_cb_0D_2\\
&\quad +Q_2 + (LP_1F_{11} - LY\Delta F_{22})^T\\
&\quad -P_1L^T - Y\Delta J_{22}D_2 + P_1J_{11}D_2 && (189)\\
h_4 &= P_1F_{d_{11}}K_cb_0D_2 - Y\Delta F_{d_{22}}K_cb_0D_2\\
&\quad -Y\Delta J_{d_{22}}D_2 + P_1J_{d_{11}}D_2 && (190)\\
h_5 &= (P_1F_{d_{11}} - YF_{d_{22}})^T && (191)
\end{aligned}
$$

and

$$
\begin{aligned}
Q_{11} &= \tau^*{}_2P_1F_{11} - \tau^*{}_2Y\Delta F_{22} && (192)\\
Q_{12} &= \tau^*{}_2(-P_1K_ca_0D_1 - P_1K_{11}c_0D_1\\
&\quad -P_1F_{11}K_cb_0D_2 + Y\Delta F_{22}K_cb_0D_2\\
&\quad -Y\Delta J_{22}D_2 - P_1L^T + P_1J_{11}D_2) && (193)\\
Q_{21} &= \tau^*{}_2(LP_1F_{11} - LY\Delta F_{22}) && (194)\\
Q_{22} &= \tau^*{}_2(-LP_1K_ca_0D_1 - LP_1K_{11}c_0D_1\\
&\quad -LP_1F_{11}K_cb_0D_2 + LY\Delta F_{22}K_cb_0D_2\\
&\quad +LP_1J_{11}D_2 - LY\Delta J_{22}D_2 - P_3) && (195)\\
Q_{51} &= \tau^*{}_2(P_1F_{d_{11}} - Y\Delta F_{d_{22}}) && (196)\\
Q_{52} &= \tau^*{}_2(-P_1F_{d_{11}}K_cb_0D_2 - Y\Delta F_{d_{22}}K_cb_0D_2\\
&\quad -Y\Delta J_{d_{22}}D_2 + P_1J_{d_{11}}D_2) && (197)\\
Q_{61} &= \tau^*{}_2(LP_1F_{d_{11}} - LYF_{d_{22}}) && (198)\\
Q_{62} &= \tau^*{}_2(LP_1F_{d_{11}}K_cb_0D_2 - LY\Delta F_{d_{22}}K_cb_0D_2\\
&\quad +LY\Delta J_{d_{22}}D_2 - LP_1J_{d_{11}}D_2) && (199)
\end{aligned}
$$

$$
\begin{aligned}
Q_{33} &= Q_{55} = P_1 && (200)\\
Q_{34} &= Q_{56} = -P_1N_{11}D_2 && (201)\\
Q_{44} &= Q_{66} = P_3 && (202)\\
Q_{77} &= I_m && (203)\\
Q_{13} &= Q_{15} = Q_{35} = Q_{31} = Q_{53} = 0_{m_z\times m_z} && (204)\\
Q_{24} &= Q_{26} = Q_{46} = Q_{42} = Q_{64} = Q_{72} = Q_{74}\\
&= Q_{76} = Q_{27} = Q_{47} = Q_{67} = 0_{m\times m} && (205)\\
Q_{14} &= Q_{16} = Q_{32} = Q_{54} = Q_{17}\\
&= Q_{37} = Q_{57} = Q_{36} = 0_{m_z\times m} && (206)\\
Q_{23} &= Q_{25} = Q_{43} = Q_{45} = Q_{41} = Q_{63} = Q_{71}\\
&= Q_{65} = Q_{73} = Q_{75} = 0_{m\times m_z} && (207)
\end{aligned}
$$

and

$$\Delta = (I - \zeta_{22}\zeta_{22}^+) \quad (208)$$