A Controller Design based on a Functional $H_\infty$ Filter for Delayed Singular Systems : The Time and Frequency Domain Cases

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Abstract: This paper deals with the time and frequency domain of a controller design based on a function reduced order filter for linear multi-variable delayed singular systems where measurements are affected by bounded disturbances. The control gain is designed using $H_\infty$ techniques. The time procedure design is based on the unbiasedness of the estimation error using Sylvester equation and on Lyapunov-Krasovskii stability theory. Then a new method to avoid the time derivative of the disturbance in filtering error is proposed and the problem is solved by means of Linear Matrix Inequalities (LMIs). Both cases where the $H_\infty$ technique is dependent or independent from the delay are derived separately. The frequency domain approach is derived from the time domain one by applying the factorization approach. A numerical example is given to illustrate the proposed approach.

Key–Words: Controller, singular systems, $H_\infty$ functional filter, delay, disturbance.

1 Introduction

Singular models or descriptor models have been a topic of recurring interests of many researches. In fact, these generalized mathematical representations better describe physical systems than regular ones [23]. So a great deal of work based on the theory of state-space systems has been extended to the descriptor models.

Furthermore, delay modeling has been extensively studied as it influences the stability robustness and other performances of the systems [22]. This situation becomes obvious when dealing with communication networks, economic systems and chemical processes [1, 22, 2].

On the other hand, a great deal of work has been devoted to the design of the filter-based controller for delayed singular systems [5, 16, 8, 9]. This controller is of major importance, mainly, when the states of the systems is partially measurable. This kind of controller is getting more and more interesting especially that a great part of control designs are developed with the assumption that the state components of the system are available for the feedback [11]. However, only a few part of the state can be measured. Motivated by these facts, a recurring interests of researches are focused on the development of filtering techniques to estimate a functional of state which can be used, also, on control laws based on the state feedback principle [6]. In addition, these controllers are handled by $H_\infty$ filters able to minimize the perturbation effect on the estimation error.

However, in the frequency domain, few results has been developed to the controller design based on a functional $H_\infty$ filter for singular delayed systems [10, 11].

In this framework, a new time and frequency domain design procedure of filter-based controller for singular delayed systems is proposed. The time domain approach is obtained into two steps. Firstly, we give conditions ensuring the admissibility of the $H_\infty$ problem. Secondly, we propose a functional $H_\infty$ filter design essential to derive the control law. The estimation problem is extended to a singular one in order to avoid the time derivative of the disturbance ($\dot{w}(t)$) on the estimation error dynamic. This filter, based on unbiasedness conditions, estimate a functional of state according to a $H_\infty$ criteria and
is proposed by means of LMIs conditions. These conditions are of two kinds. One satisfies the $H_\infty$ criteria independently from the delay and the second respects the same criteria but dependently on the state delay. Furthermore and based on the time domain results, a frequency domain approach is set to the design of the functional $H_\infty$ filter based controller using MFDs. The main reason of formulating the results of the time domain in the frequency one is the advantages that it presents for the filter-based control. In fact, the compensator is driven by the input and the output of the system. So, only the input-output behavior of the compensator (characterized by its transfer function) influences the properties of the closed-loop system.

The outlines of the paper are as follows. Section 2 gives assumptions used through this paper and presents the functional filter-based controller problem that we propose to solve. Section 3, presents the first contribution of the paper by giving the time domain design of the controller. This contribution is presented in two parts. First, the state feedback gain is designed with respect to the $H_\infty$ performance criteria and satisfying the problem admissibility. Second, we propose to design the filter-based controller using the unbiasedness condition, dependently and independently from the state delay. The problem is transformed into a matrix inequalities to solve. A LMI approach is then applied to optimize the gain implemented in the filter. The fourth section presents the second result of the paper by giving a frequency domain description of the controller using polynomial MFDs. A summary of the filter based controller is presented in the fifth part of the article. Section 6 gives numerical examples to illustrate our approaches and section 7 concludes the paper.

## 2 Problem Formulation

Let’s consider the following continuous-time linear time-delay singular system described by:

\[
\begin{align*}
E \dot{x}(t) &= Ax(t) + A_d x(t-d) + Bu(t) + B_d u(t-d) + D_1 w(t) \\
\phi_0 &= x(t_0) = \phi_0
\end{align*}
\]

where $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^q$ is the output vector, $u(t) \in \mathbb{R}^p$ is the input vector, $w(t) \in \mathbb{R}^m$ is the bounded disturbance and $z(t) \in \mathbb{R}^{m_z}$ is the functional state.

$E$, $A$, $A_d$, $B$, $C$, $B_d$, $D_1$, $F_1$ and $D_2$ are known matrices of appropriate dimensions. $\phi_0$ is the value of the initial state, $d \in \mathbb{R}_+$ is the considered delay.

In the sequel, we suppose that:

**Hypothesis 1.** [5]

1. $\text{rank}(E) = r_1 \leq n$
2. $\text{rank}\left(\begin{bmatrix} E \\ C \end{bmatrix}\right) = n$

**Purpose:**

The main objective of this paper is to design in the time and the frequency domain a controller based on a functional $H_\infty$ filter for delayed singular systems. The same delay is assumed in the state and the input vectors.

## 3 Time domain design of the filter-based controller

We propose to solve the observer-based controller problem into two steps. First, we propose to design the control gain $K_c$ satisfying the admissibility of the subsystem $\{(1a)-(1b)\}$. Second, we aim to design a filter-based controller in order to reconstruct the control law by estimating only the state functional essential to the controller design.

Under hypothesis 1, there exists a non singular matrix $S$ where $S = \left(\begin{array}{cc} a_0 & b_0 \\ c_0 & d_0 \end{array}\right)$ with $a_0 \in \mathbb{R}^{n \times n}$, $b_0 \in \mathbb{R}^{n \times q}$, $c_0 \in \mathbb{R}^{q \times n}$ and $d_0 \in \mathbb{R}^{q \times q}$ such that:

\[
\begin{align*}
a_0 E + b_0 C &= I_n \\
c_0 E + d_0 C &= 0_{q \times n}
\end{align*}
\]

where $E$ and $C$ are given in (1).

The purpose of the paper is to design a functional filter-based controller for system (1) of the form:

\[
\begin{align*}
\chi(t) &= F \chi(t) + F_d \chi(t-d) + H_d u(t-d) + H u(t) + L_1 y(t) + L_2 y(t-d) \\
u(t) &= \chi(t) + M y(t)
\end{align*}
\]

where $M = K_c b_0 + K d_0$. Matrices $F$, $F_d$, $H$, $H_d$, $L_1$, $L_2$, $K_c$ and $K$ are to be designed.

**Problem:**
Our main objective is to build a functional filter and a control law following (3a) and (3b) in time domain and its equivalent descriptor in the frequency domain such that:

a) \( \lim_{t \to +\infty} u - K_c x = 0, \) if \( w = 0. \)

b) The system state vector and the filtering error are asymptotically stable and satisfy the \( H_\infty \) performance.

The \( H_\infty \) criteria is given by:

\[
0 < \| H_{ew} \|_\infty = \sup_{w \neq 0} \frac{\| e \|_2}{\| w \|_2} < \gamma \tag{4}
\]

with \( H_{ew}(s) = \frac{\epsilon(s)}{w(s)} \) is a transfer matrix, \( \gamma \) is a positive scalar and \( \epsilon = u - K_c x \) is the estimation error.

### 3.1 State feedback synthesis

Let us consider the subsystem given by (1a) – (1b):

\[
\begin{align*}
E \dot{x}(t) &= Ax(t) + A_d x(t - d) + Bu(t) + B_d u(t - d) + D_1 w(t) \\
\quad z(t) &= F_1 x(t) \tag{5a}
\end{align*}
\]

**Lemma 1.** The system (5) is admissible and satisfies the \( H_\infty \) performance requirement given by (4) if and only if there exist a matrix \( X \) and a symmetric definite positive matrix \( Z \) such that:

\[
X^T E^T = E X \geq 0 \tag{6}
\]

\[
V < 0 \tag{7}
\]

where

\[
V = \begin{pmatrix}
\alpha & A_d X & X^T F_1^T \\
* & -Z & 0 \\
* & * & -I_n
\end{pmatrix} \tag{8}
\]

\( \alpha = AX + X^T A^T + Z + \gamma^{-2} D_1 D_1^T \tag{9} \)

By replacing \( u \) by \( K_c x \) in system (5), we have:

\[
\begin{align*}
E \dot{x}(t) &= (A + BK_c)x(t) + (A_d + B_d K_c)x(t - d) + D_1 w(t) \\
\quad z(t) &= F_1 x(t) \tag{10a}
\end{align*}
\]

So results given by Lemma 1 can be applied on system (10) in order to design the state feedback \( K_c \) according to the next proposed theorem.

**Theorem 2.** The system (10) is admissible and satisfies the \( H_\infty \) criteria if and only if there exist matrices \( X_c, Y_c \) and \( Z_c = Z_c^T > 0 \) such that:

\[
X_c^T E^T = E X_c \geq 0 \quad \tag{11}
\]

\[
V_c < 0 \quad \tag{12}
\]

where

\[
V_c = \begin{pmatrix}
\alpha_c & \beta_x & X_c^T F_1^T \\
* & -Z_c & 0 \\
* & * & -I_n
\end{pmatrix} < 0 \tag{13}
\]

with \( \alpha_c = AX_c + BY_c + (AX_c + BY_c)^T + Z_c + \gamma^{-2} D_1 D_1^T \tag{14} \)

and

\[
\beta_x = A_d X_c + B_d Y_c \tag{15}
\]

So the state feedback gain is given by:

\[
K_c = Y_c X_c^{-1} \tag{16}
\]

**Proof 1.** It’s obtained by applying the results of [25] on system (10) and using a transformation of the main result according to the Schur lemma [3]. In fact, the system considered on [25] is a delayed singular system with uncertain parameters. When setting the uncertainty components to zero, the admissibility conditions of the \( H_\infty \) problem is given by:

\[
X_c^T E^T = E X_c \geq 0 \tag{17}
\]

and

\[
M_c = \begin{pmatrix}
\alpha_c & \beta_x & X_c^T F_1^T \\
* & -Z_c & 0 \\
* & * & -I_n
\end{pmatrix} < 0 \tag{18}
\]

with \( \epsilon_c \) is a positive scalar.

By realizing that \( M_c \) is a bloc-diagonal matrix and \( -\epsilon_c I \) is a negative matrix, then equation (13) is the condition for the \( H_\infty \) problem admissibility.

### 3.2 Filter-based controller synthesis

#### 3.2.1 The unbiasedness conditions of the filter-based controller

Let \( \epsilon(t) \) be the estimation error: Considering (2) and (3b), \( \epsilon(t) \) is given by:

\[
\epsilon(t) = u(t) - K_c x(t) \tag{19a}
\]

\[
= \chi(t) - (K_c a_0 + K c_0) E x(t) + (K_c b_0 + K d_0) D_2 w(t) \tag{19b}
\]

\[
= \chi(t) - \Psi_1 E x(t) + \Psi_2 w(t) \tag{19c}
\]
with
\[
\Psi_1 = K_c a_0 + K_c a_0 \quad (20)
\]
\[
\Psi_2 = (K_c b_0 + K_d b_0) D_2 \quad (21)
\]
Given the singular system (1) and the functional filter-based controller (3), we aim to design the filter matrices \( F, F_d, H, H_d, L_1, L_2 \) and \( M \) which verify the unbiasedness estimation error conditions (if \( w(t) = 0 \)) and the attenuation of the disturbance effect given by (4) (if \( w(t) \neq 0 \)).

The unbiasedness of the estimation error dynamics is verified according to the following proposed theorem:

**Theorem 3.** The unbiasedness of the estimation error given by (19) relative to system (1) and filter (3) is verified such that:
\[
\dot{e}(t) = F e(t) + F_d e(t - d) + \alpha w(t) + \beta w(t - d) - \zeta \hat{w}(t) \quad (22)
\]
if and only if the following equations are satisfied:

i) \( L_1 C + F \Psi_1 E - \Psi_1 A = 0 \)

ii) \( L_2 C + F_d \Psi_1 E - \Psi_1 A_d = 0 \)

iii) \( H = \Psi_1 B \)

iv) \( H_d = - \Psi_1 B_d \)

with
\[
\alpha = L_1 D_2 - F \Psi_2 - \Psi_1 D_1 \quad (23)
\]
\[
\beta = L_2 D_2 - F_d \Psi_2 \quad (24)
\]
and
\[
\zeta = - \Psi_2 \quad (25)
\]

**Proof 2.** The derivative of the estimation error is given as follows:
\[
\dot{e}(t) = \dot{\chi}(t) - \Psi_1 E \dot{\hat{e}}(t) + \Psi_2 \hat{w}(t) \quad (26)
\]
By replacing in (26) \( E \dot{\hat{e}}(t) \) and \( \dot{\chi}(t) \) by their expressions given by (1) and (3) respectively, we have:
\[
\dot{e}(t) = F e(t) + F_d e(t - d) - (\Psi_1 B - H) u(t) - (\Psi_1 B_d - H_d) u(t - d) + (L_2 D_2 - F_d \Psi_2) w(t - d) + (L_1 D_2 - F \Psi_2 - \Psi_1 D_1) w(t)
\]
\[
+ (L_2 C + F_d \Psi_1 E - \Psi_1 A_d) x(t - d) + (L_1 C + F \Psi_1 E - \Psi_1 A) x(t) + \Psi_2 \hat{w}(t) \quad (27)
\]
with the initial condition \( e_0 = u_0 - K_c x_0 \)

**Purpose 1.** At this stage, we propose to design the filter-based controller dependently and independently from the state delay using the obtained state feedback gain (16).

By replacing, in condition i) of theorem 1, \( \Psi_1 \) and \( \Psi_2 \) by their expressions in (20) and (21) respectively, we have:
\[
FK_c a_0 E + JC - K_c a_0 A = K_c a_0 A \quad (28)
\]
with:
\[
J = L_1 - FK_d b_0 \quad (29)
\]
Similarly for condition ii) of theorem 1, we obtain:
\[
F_d K_c a_0 E + J_d C - K_c a_0 A_d = K_c a_0 A_d \quad (30)
\]
with
\[
J_d = L_2 - F_d K_d b_0 \quad (31)
\]
Equations (28)-(31) can be written in the following matrix form:
\[
X \Sigma = \Theta \quad (32)
\]
where,
\[
X = \begin{bmatrix} F & F_d & -K & J & J_d \end{bmatrix} \quad (33)
\]
\[
\Sigma = \begin{bmatrix} K_c a_0 E & 0 \\ 0 & K_c a_0 E \\ c_0 A & c_0 A_d \\ C & 0 \\ 0 & C \end{bmatrix} \quad (34)
\]
\[
\Theta = \begin{bmatrix} K_c a_0 A & K_c a_0 A_d \end{bmatrix} \quad (35)
\]
Note that a general solution of (32), exists if and only if
\[
\text{rank} \begin{bmatrix} \Sigma \\ \Theta \end{bmatrix} = \text{rank}(\Sigma) \quad (36)
\]
In this case, the general solution for (32) is given by:
\[
X = \Theta \Sigma^+ - Z (I - \Sigma \Sigma^+) \quad (37)
\]
where \( \Sigma^+ \) is the generalized inverse of matrix \( \Sigma \) given by (34) and \( Z \) is an arbitrary matrix of appropriate dimensions, that will be determined in the sequel using LMI approach.

The unknown matrix \( F \) in (33) can be given by:
\[
F = X \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (38)
\]
By replacing (37) in (38), we obtain:

\[
F = \Theta \Sigma^+ \begin{pmatrix} I & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} - Z(I - \Sigma \Sigma^+) \begin{pmatrix} I & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}
\]  
(39)

Let’s consider:

\[
F_{11} = \Theta \Sigma^+ \begin{pmatrix} I & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}
\]  
(40)

and

\[
F_{22} = (I - \Sigma \Sigma^+) \begin{pmatrix} I & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}
\]  
(41)

Then,

\[
F = F_{11} - ZF_{22}
\]  
(42)

Similarly for matrix \( F_d \), we obtain:

\[
F_d = F_{d11} - ZF_{d22}
\]  
(43)

where

\[
F_{d11} = \Theta \Sigma^+ \begin{pmatrix} 0 & I \\ 0 & 0 \\ 0 & 0 \end{pmatrix}
\]  
(44)

and

\[
F_{d22} = (I - \Sigma \Sigma^+) \begin{pmatrix} 0 & I \\ 0 & 0 \\ 0 & 0 \end{pmatrix}
\]  
(45)

We have:

\[
J = J_{11} - ZJ_{22}
\]  
(46)

where

\[
J_{11} = \Theta \Sigma^+ \begin{pmatrix} 0 & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix}
\]  
(47)

and

\[
J_{22} = (I - \Sigma \Sigma^+) \begin{pmatrix} 0 & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix}
\]  
(48)

Similarly for matrix \( J_d \), we obtain:

\[
J_d = J_{d11} - ZJ_{d22}
\]  
(49)

where

\[
J_{d11} = \Theta \Sigma^+ \begin{pmatrix} 0 & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix}
\]  
(50)

and

\[
J_{d22} = (I - \Sigma \Sigma^+) \begin{pmatrix} 0 & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix}
\]  
(51)

and,

\[
K = -K_{11} + ZK_{22}
\]  
(52)

with

\[
K_{11} = \Theta \Sigma^+ \begin{pmatrix} 0 & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix}
\]  
(53)

and

\[
K_{22} = (I - \Sigma \Sigma^+) \begin{pmatrix} 0 & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix}
\]  
(54)

By combining (21) and (52), we obtain:

\[
\Psi_2 = \Psi_{211} - Z\Psi_{222}
\]  
(55)

with:

\[
\Psi_{211} = N_{11}D_2
\]  
(56)

\[
\Psi_{222} = N_{22}D_2
\]  
(57)

and

\[
N_{11} = K_c b_0 + K_{11}d_0
\]  
(58)

\[
N_{22} = K_{22}d_0
\]  
(59)

In order to avoid the derivative component of the perturbation \( \dot{w}(t) \), we propose to reformulate the equation (22) on the following singular state-space form:

\[
\begin{bmatrix}
I_n & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{\epsilon}(t) \\
\dot{w}(t)
\end{bmatrix}
= 
\begin{bmatrix}
F & 0 \\
0 & -I_m
\end{bmatrix}
\begin{bmatrix}
\epsilon(t) \\
w(t)
\end{bmatrix}
+ 
\begin{bmatrix}
F_d & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\epsilon(t-d) \\
w(t-d)
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\
I_m
\end{bmatrix}
w(t)
\]  
(60)

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Let us consider $\xi = \begin{bmatrix} \epsilon(t) \\ w(t) \end{bmatrix}$, so we have:

$$\rho \dot{\xi}(t) = \ddot{F}\xi(t) + \ddot{F_d}\xi(t - d_1) + \ddot{B}w(t)$$  \hfill (61)

where

$$\rho = \begin{bmatrix} I \alpha \\ 0 \end{bmatrix}$$  \hfill (62)

$$\ddot{F} = \begin{bmatrix} F \alpha \\ 0 \end{bmatrix}$$  \hfill (63)

$$\ddot{F_d} = \begin{bmatrix} F_d \beta \\ 0 \end{bmatrix}$$  \hfill (64)

and

$$\ddot{B} = \begin{bmatrix} 0 \\ I \end{bmatrix}$$  \hfill (65)

Since the matrix $\zeta$ in (62) depends on the unknown matrix $Z$ (See (25) and (55)). We assume that the gain matrix $Z$ satisfies the following relation:

$$Z\Psi_{22} = Z\zeta_{22} = 0$$  \hfill (66)

This assumption enables us to avoid an unknown (to be designed) gain matrix $Z$ in the singular matrix $\rho$ given by (62). So there always exists a matrix $Z_1$ such that:

$$Z = Z_1(I - \zeta_{22} \zeta_{22}^+ \zeta_{22})$$  \hfill (67)

with

$$\zeta_{22} = -N_{22}D_2$$  \hfill (68)

and $\zeta_{22}^+$ is the pseudo inverse of $\zeta_{22}$ such that:

$$\zeta_{22}\zeta_{22}^+\zeta_{22} = \zeta_{22}$$  \hfill (69)

In fact,

$$Z\zeta_{22} = Z_1(I - \zeta_{22} \zeta_{22}^+ \zeta_{22})\zeta_{22}$$  \hfill (70)

$$= Z_1(\zeta_{22} - \zeta_{22} \zeta_{22}^+ \zeta_{22})$$  \hfill (71)

$$= Z_1(\zeta_{22} - \zeta_{22})$$  \hfill (72)

$$= 0$$  \hfill (73)

then, we have:

$$\zeta = \zeta_{11} - Z\zeta_{22}$$  \hfill (74)

where

$$\zeta_{11} = -N_{11}D_2$$  \hfill (75)

### 3.2.2 Filter-based controller design independent from the delay

The design procedure is based on Lyapunov-Krasovskii stability theory using LMIs approach. The filter-based controller stability conditions are independent from the delay, so the estimated state converges asymptotically to the real one for any constant time delay with the satisfaction of condition (4).

At this stage, and based on theorem 2 and Lyapunov-Krasovskii stability theory, one can get the gain matrix $Z$ which parametrizes the filter matrices, as proposed in the following theorem.

**Theorem 4.** The filter-based controller in the form of (3) is a $H_\infty$ controller for system (1) if there exist matrices $P_{1s} = P_{1s}^T$, $P_{2s}$, $P_{3s} = P_{3s}^T$, $Q_1 = Q_1^T$, $Q_4 = Q_4^T$, $Q_2$ and $Y_s$ satisfying the following linear matrix inequalities:

$$Q_s = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_4 \end{bmatrix} > 0$$  \hfill (76)

$$P_s\rho = \rho^TP_s^T > 0$$  \hfill (77)

where

$$P_s = \begin{bmatrix} P_{1s} \\ P_{2s} \\ P_{3s} \end{bmatrix}$$  \hfill (78)

with

$$P_{2s} = LP_{1s}$$  \hfill (79)

satisfying

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} \\ * & \alpha_{22} & \alpha_{23} & \alpha_{24} & \alpha_{25} \\ * & * & \alpha_{33} & \alpha_{34} & \alpha_{35} \\ * & * & * & \alpha_{44} & \alpha_{45} \\ * & * & * & * & \alpha_{55} \end{pmatrix} < 0$$  \hfill (80)

with $\alpha_{ij1\leq i,j\leq 5}$ are given in appendix A.

and then the gain $Z_1$ is given by:

$$Z_1 = P_{1s}^{-1}Y_s$$  \hfill (81)

We note that according to (77), we have:

$$L^T = -N_{11}D_2$$  \hfill (82)

In fact when replacing $P_{2s}$ in (78) by its expression in (79), we have:

$$P_s\rho = \begin{bmatrix} P_{1s} \\ LP_{1s} \end{bmatrix}$$  \hfill (83)
and
\[
\rho^T P_s^T = \begin{pmatrix} P_1 s & P_1 s L_s \\ \zeta s P_1 s & \zeta s P_1 s L_s \end{pmatrix} \tag{84}
\]

So, according to (77) and by identification we have:
\[
L_s = \zeta s \tag{85}
\]
and when using (75), equation (82) holds.

**Proof 3.** Let \( V(\xi, t) \) be the Lyapunov-Krasovskii (See [16]) functional of the form:
\[
V(\xi, t) = \xi^T(t) P_s \rho \xi(t) + \int_{t-d}^{t} \xi(\mu)^T Q_s \xi(\mu) d\mu \tag{86}
\]
where \( Q_s \) and \( P_s \) verify respectively (76) and (77).

In order to establish sufficient conditions for existence of (3) according to (4), we should verify the following inequality:
\[
H(\epsilon, w) = \dot{V}(\xi, t) + \epsilon^T(t) \epsilon(t) - \gamma^2 w^T(t) w(t) < 0 \tag{87}
\]
Since \( w^T w > 0 \), we can write:
\[
H(\epsilon, w) < H(\epsilon, w) + w^T(t) w(t) \tag{88}
\]
Equation (88) can be written as:
\[
H(\epsilon, w) < \dot{V}(\xi, t) + \epsilon^T(t) \epsilon(t) - (\gamma^2 - 1) w^T(t) w(t) \tag{89}
\]
By considering \( \gamma^2 = \gamma^2 - 1 \) and since
\[
\xi^T(t) - \gamma^2 w^T(t) w = \epsilon^T \epsilon - \gamma^2 w^T(t) w \tag{90}
\]
Equation (89) can be written like:
\[
H(\epsilon, w) < H_x(\xi, w) \tag{91}
\]
With:
\[
H_x(\xi, w) = \dot{V}(\xi, t) + \xi^T(t) \xi(t) - \gamma^2 w^T(t) w(t) \tag{92}
\]
So it's sufficient to impose that:
\[
H_x(\xi, w) < 0 \tag{93}
\]
By differentiating \( V(\xi, t) \) along the solution (61), we obtain:
\[
H_x(\xi, w) = \xi^T(t) [\tilde{F}_d^T P_s + P_s \tilde{F} + Q_s + I] \xi(t) + \xi^T(t-d) \tilde{F}_d^T P_s \xi(t) + \xi^T(t) P_s \tilde{F}_d \xi(t-d) - \xi^T(t-d) Q_s \xi(t-d) + w^T(t) B P_s \xi(t) + \xi^T(t) P_s \tilde{B} w(t) - \gamma^2 w^T(t) w(t) < 0 \tag{94}
\]

and it can be written like:
\[
v^T \begin{pmatrix} \alpha_s P_s \tilde{F}_d & P_s \tilde{B} \\ * & -Q_s & 0 \\ * * & * & -\gamma^2 I_m \end{pmatrix} v < 0 \tag{95}
\]
with
\[
\alpha_s = \tilde{F}_d^T P_s + P_s \tilde{F} + Q_s + I \tag{96}
\]
where \( v^T = [\xi^T \xi^T(t-d) w^T(t) \]

From (95), \( H(\xi, w) < 0 \) if
\[
\begin{pmatrix} \alpha_s P_s \tilde{F}_d & P_s \tilde{B} \\ * & -Q_s & 0 \\ * * & * & -\gamma^2 I_m \end{pmatrix} < 0 \tag{97}
\]

By replacing \( \tilde{F}, \tilde{F}_d, \tilde{B}, Q_s \) and \( P_s \) by their expressions given, respectively, by (63), (64), (65), (76) and (78) in (97) and according to equations (42), (43), (46), (49) and (52), the matrix in (97) equals that in (80) which prove theorem 3.

Once \( Z_1 \) is calculated using (81) and \( Z \) is calculated using (67), all filter matrices can also be given by equations (42), (43), (46), (49) and (52).

### 3.2.3 Filter-based controller design dependent on the delay

In this paragraph, we aim to design a filter-based controller dependently on the delay. Based on the Lyapunov-Krasovskii stability theory and with respect to the \( H_\infty \) criteria given by (4), one can get the gain matrix \( Z \) which parametrizes the filter matrices, as proposed in theorem 4. This type of design is of great importance, especially, when dealing with unknown or variable delay with known bounds such that:
\[
0 < \tau^* \leq d \leq \tau^* \tag{98}
\]
where \( \tau^* \) and \( \tau^* \) are scalars.

**Theorem 5.** The filter-based controller in the form of (3) is a \( H_\infty \) filter for system (1) if there exist matrices \( P_1 = P_1^T, P_2, P_3 = P_3^T \) and \( Y \) satisfying the following linear matrix inequalities:
\[
P \rho = \rho^T P \rho > 0 \tag{99}
\]
where
\[
P = \begin{pmatrix} P_1 & P_2^T \\ P_2 & P_3 \end{pmatrix} \tag{100}
\]
with
\[
P_2 = LP_1 \tag{101}
\]
satisfying
\[
\Omega = \begin{pmatrix} \Xi & Q \\ QT & U \end{pmatrix} < 0 \tag{102}
\]
Where $\Xi$ and $U$ are symmetric matrices of dimension $3m + 4m$, $Q \in \mathbb{R}^{(3m + 4m) \times (3m + 4m)}$. $\Xi$, $U$ and $Q$ are given in Appendix B.

and we have:

$$Z_1 = P_1^{-1} Y$$  \hspace{1cm} (103)

We note that according to (99), we have:

$$L^T = -N_{11} D_2$$ \hspace{1cm} (104)

**Proof 4.** The chosen Lyapunov functional is (See [14]):

$$V(t) = V_1(t) + d[V_2(t) + V_3(t)]$$ \hspace{1cm} (105)

with

$$V_1(t) = \xi(t)^T P \rho \xi(t)$$ \hspace{1cm} (106)

$$V_2(t) = \int_0^d \int_0^t \xi(s)^T \bar{F}_d^T \bar{P} \bar{F} \xi(s) ds d\theta$$ \hspace{1cm} (107)

$$V_3(t) = \int_0^d \int_0^t \xi(s)^T \bar{F}_d^T \bar{P} \bar{F} \xi(s) ds d\theta$$ \hspace{1cm} (108)

In order to establish sufficient conditions for existence of (3) according to (4), we should verify the inequality (93).

The derivative of the functional $V(t)$ is:

$$\dot{V}(t) = \dot{V}_1(t) + d\dot{V}_2(t) + d\dot{V}_3(t)$$ \hspace{1cm} (109)

According to equations (61) and (99), we have:

$$\dot{V}_1(t) = \xi(t)^T [\bar{F}_d^T P + P \bar{F}] \xi(t)$$

$$+ \xi(t-d)^T \bar{F}_d^T P \xi(t)$$

$$+ \xi(t)^T P \bar{F} \xi(t-d)$$

$$+ w^T(t) \bar{B}^T P \xi(t)$$

$$+ \xi^T(t) P \bar{B} w(t)$$ \hspace{1cm} (110)

Then,

$$\dot{V}_2(t) = \int_0^d [\xi(t)^T \bar{F}_d^T P \bar{F} \xi(t)$$

$$- \xi(t) \bar{F}_d^T P \bar{F} \xi(t)]d\theta$$ \hspace{1cm} (111)

so,

$$\dot{V}_2(t) = d\xi(t)^T \bar{F}_d^T P \bar{F} \xi(t)$$

$$- \int_0^d \xi(t) \bar{F}_d^T P \bar{F} \xi(t)]d\theta$$ \hspace{1cm} (112)

Let’s

$$\Upsilon(t - \theta) = -\bar{F} \xi(t - \theta) \in \mathbb{R}^n$$ \hspace{1cm} (113)

so we write:

$$\dot{V}_2(t) = d\xi(t)^T \bar{F}_d^T P \bar{F} \xi(t)$$

$$- \int_0^d \Upsilon(t - \theta)^T P \Upsilon(t - \theta)]d\theta$$ \hspace{1cm} (114)

and

$$\dot{V}_3(t) = \int_0^d [\xi(t)^T \bar{F}_d^T P \bar{F} \xi(t)$$

$$- \xi(t) \bar{F}_d^T P \bar{F} \xi(t)]d\theta$$ \hspace{1cm} (115)

Let’s:

$$\Upsilon_d(t - \theta) = -\bar{F} \xi(t - \theta) \in \mathbb{R}^n$$ \hspace{1cm} (116)

so,

$$\dot{V}_3(t) = d\xi(t)^T \bar{F}_d^T P \bar{F} \xi(t)$$

$$- \int_0^d \Upsilon_d(t - \theta)^T P \Upsilon_d(t - \theta)]d\theta$$ \hspace{1cm} (117)

Uniform asymptotic stability implies that:

$$\lim_{t \to +\infty} \dot{V}(t) \leq 0$$ \hspace{1cm} (118)

As $\theta$ is bounded, the quantities $\Upsilon(t - \theta)$ and $\Upsilon_d(t - \theta)$, respectively, given by (113) and (116) satisfy:

$$\lim_{t \to +\infty} \Upsilon(t - \theta) = \lim_{t \to +\infty} \Upsilon_d(t - \theta)$$ \hspace{1cm} (119)

and,

$$\lim_{t \to +\infty} \Upsilon_d(t - \theta) = \lim_{t \to +\infty} \Upsilon_d(t)$$ \hspace{1cm} (120)

Consequently,

$$\lim_{t \to +\infty} \left(\int_0^d \Upsilon(t - \theta)^T P \Upsilon(t - \theta)d\theta\right)$$

$$= d \lim_{t \to +\infty} \Upsilon(t)^T P \Upsilon(t) \hspace{1cm} (121)$$

and,

$$\lim_{t \to +\infty} \left(\int_0^d \Upsilon_d(t - \theta)^T P \Upsilon_d(t - \theta)d\theta\right)$$

$$= d \lim_{t \to +\infty} \Upsilon_d(t)^T P \Upsilon_d(t) \hspace{1cm} (122)$$

We set the variable’s changes:

$$\gamma_v = \lim_{t \to +\infty} \Upsilon(t)$$ \hspace{1cm} (123)

$$\nu = \lim_{t \to +\infty} \Upsilon_d(t)$$ \hspace{1cm} (124)
The equations (121) and (122) can be written as:

\[
\lim_{t \to +\infty} \left(\int_{0}^{d} \gamma(t-\theta)^{T} \mathcal{Y}(t-\theta) d\theta\right) = d\gamma_{c}^{T} P_{\gamma} \tag{125}
\]

and,

\[
\lim_{t \to +\infty} \left(\int_{0}^{d} \gamma(t-\theta)^{T} \mathcal{Y}_{d}(t-\theta) d\theta\right) = dv^{T} P_{v} \tag{126}
\]

We suppose that \( \xi = \lim_{t \to +\infty} \xi(t) \), we have:

\[
\lim_{t \to +\infty} \dot{V}(t) = \left[\xi^{T} \left[\left(\tilde{F} + \tilde{F}_{d}\right)^{T} P + P(\tilde{F} + \tilde{F}_{d})\right]\right] \xi + \tau_{v}^{2} \xi^{T} \tilde{F}_{d}^{T} P \tilde{F}_{d} \xi + \tau_{v}^{2} \xi^{T} \tilde{F}_{d}^{T} P \tilde{F}_{d} \xi + \left[\xi^{T}(t) P \tilde{B} \bar{w}(t) + w^{T}(t) \tilde{B} \bar{P} \xi(t)\right] - \tau_{v}^{2} \gamma_{v} \tau_{v} - \tau_{v}^{2} \nu^{T} P \nu \tag{127}
\]

Therefore, according to equation (98) we have:

\[
\lim_{t \to +\infty} \dot{V}(t) \leq \left[\xi^{T} \left[\left(\tilde{F} + \tilde{F}_{d}\right)^{T} P + P(\tilde{F} + \tilde{F}_{d})\right]\right] \xi + \tau_{v}^{2} \xi^{T} \tilde{F}_{d}^{T} P \tilde{F}_{d} \xi + \tau_{v}^{2} \xi^{T} \tilde{F}_{d}^{T} P \tilde{F}_{d} \xi + \left[\xi^{T}(t) P \tilde{B} \bar{w}(t) + w^{T}(t) \tilde{B} \bar{P} \xi(t)\right] - \tau_{v}^{2} \gamma_{v} \tau_{v} - \tau_{v}^{2} \nu^{T} P \nu \tag{128}
\]

Then, according to (87) we have:

\[
H(\xi, t) < \begin{bmatrix} \xi^{T} & \gamma_{v}^{T} & \nu^{T} \end{bmatrix} \Psi \begin{bmatrix} \xi \\ \gamma_{v} \\ \nu \end{bmatrix} < 0 \tag{129}
\]

where

\[
\Psi = \begin{pmatrix} \beta_{c} & 0 & 0 & P \tilde{B} \\ 0 & -\tau_{v}^{2} P & 0 & 0 \\ 0 & 0 & \tau_{v}^{2} P & 0 \\ \tilde{B}^{T} P & 0 & 0 & -\gamma_{v}^{2} I_{m} \end{pmatrix} \tag{130}
\]

and

\[
\beta_{c} = \tilde{F}^{T} P + P \tilde{F} + \tilde{F}_{d}^{T} P + P \tilde{F}_{d} + \tau_{v}^{2} \tilde{F}^{T} P \tilde{F} + \tau_{v}^{2} \tilde{F}_{d}^{T} P \tilde{F}_{d} + I \tag{131}
\]

Since \( \Psi \) is a symmetric matrix then relation (129) is equivalent to:

\[
\Psi < 0 \tag{132}
\]

In order to avoid the quadratic form present in \( \beta_{c} \), we propose to transform the inequality given by (132) in another form according to the Schur Lemma (See [3]).

In fact, matrix \( \Psi \) can be written as:

\[
\Psi = U - Q_{v}^{T} \Gamma^{-1} Q_{v} \tag{133}
\]

where

\[
\Gamma = \Xi^{-1} \tag{134}
\]

According to the Schur lemma, \( \Psi < 0 \) and \( \Gamma < 0 \) if and only if:

\[
\Omega_{v} = \begin{pmatrix} \Gamma & Q_{v} \\ Q_{v}^{T} & U \end{pmatrix} < 0 \tag{135}
\]

where

\[
Q_{v} = \begin{pmatrix} \tau_{v}^{2} \tilde{F} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \tau_{v}^{2} \tilde{F}_{d} & 0 & I & 0 \\ 0 & 0 & 0 & \gamma_{v} \end{pmatrix} \tag{136}
\]

Now, we apply a congruence transformation to \( \Omega_{v} \) such that:

\[
\Pi = T^{T} \Omega_{v} T < 0 \tag{137}
\]

Where \( T \) is a non singular matrix given by:

\[
T = \begin{pmatrix} P & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & P & \cdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix} \tag{138}
\]

Then, by replacing \( P, F, F_{d}, \alpha \) and \( \beta \) by their expressions given, respectively, by (100), (42), (43), (23) and (24) in (137) and considering equation (103), theorem 4 holds.

\[
\square
\]

### 4 Filter-based controller design in the frequency domain

In this section and based on time domain results, we propose the filter-based controller design procedure that operates in the frequency domain, dependently and independently form the delay, using left co-prime factorization of a transfer matrix [9, 24]. So, the filter transfer function is given by the following theorem:
Theorem 6. The frequency domain description of the $H_{\infty}$ functional filter-based controller (3) for the linear singular delayed system (1) is given by:

$$u(s) = T_1(s) \times y(s) + T_2(s)e^{-ds} \times y(s)$$  \hspace{1cm} (139)

where

$$T_1(s) = [I - N_1^{-1}(s)M_1(s) - N_2^{-1}(s)M_2(s)] \times N_3^{-1}(s)M_3(s)$$

$$= [I - T_c(s)(H + H_d e^{-ds})]^{-1} \times [M + T_c(s)L_1]$$  \hspace{1cm} (140a)

$$T_2(s) = [I - N_1^{-1}(s)M_1(s) - N_2^{-1}(s)M_2(s)] \times N_4^{-1}(s)M_4(s)$$

$$= [I - T_c(s)(H + H_d e^{-ds})]^{-1} \times T_c(s)L_2$$  \hspace{1cm} (140b)

with

$$T_c(s) = sI_{m_z} - F_x(s)$$  \hspace{1cm} (141)

and

$$F_x(s) = F + F_d e^{-ds}$$  \hspace{1cm} (142)

and where, using left coprime factorization [9], all matrices implemented in this design are given by:

$$N_1(s) = -(sI_{m_z} - F_x(s) + X_1)^{-1} + I_{m_z}$$  \hspace{1cm} (143)

$$M_1(s) = (sI_{m_z} - F_x(s) + X_1)^{-1}H$$  \hspace{1cm} (144)

$$N_2(s) = (sI_{m_z} - F_x(s) - X_2)^{-1}X_2 + I_{m_z}$$  \hspace{1cm} (145)

$$M_2(s) = (sI_{m_z} - F_x(s) - X_2)^{-1}H_d e^{-ds}$$  \hspace{1cm} (146)

$$N_3(s) = -(sI_{m_z} - F_x(s) + X_3)^{-1} + I_{m_z}$$  \hspace{1cm} (147)

$$M_3(s) = (sI_{m_z} - F_x(s) + X_3)^{-1}(L_1 - X_3M) + M$$  \hspace{1cm} (148)

$$N_4(s) = (sI_{m_z} - F_x(s) - X_4)^{-1}X_4 + I_{m_z}$$  \hspace{1cm} (149)

$$M_4(s) = (sI_{m_z} - F_x(s) - X_4)^{-1}L_2 e^{-ds}$$  \hspace{1cm} (150)

Note that $X_1, X_2, X_3$ and $X_4$ are matrices of appropriate dimensions such that, respectively, det$(sI_{m_z} - F_x(s) + X_1)$, det$(sI_{m_z} - F_x(s) - X_2)$, det$(sI_{m_z} - F_x(s) + X_3)$, and det$(sI_{m_z} - F_x(s) - X_4)$ are Hurwitz.

Proof 5. By applying the Laplace transform to (3a) and taking into accounts (141), we write:

$$\chi(s) = (sI_{m_z} - F_x(s))^{-1}Hu(s)$$

$$= (sI_{m_z} - F_x(s))^{-1}H_d e^{-ds}u(s)$$

$$+ (sI_{m_z} - F_x(s))^{-1}L_2 e^{-ds}y(s)$$

$$+ (sI_{m_z} - F_x(s))^{-1}L_1y(s)$$  \hspace{1cm} (151)

By replacing in (3b) $\chi(s)$ by its expression in (151), we have:

$$u(s) = [I_{m_z} - T_c(s)(H + H_d e^{-ds})]^{-1}$$

$$\times [M + T_c(s)L_1]y(s)$$

$$+ [I_{m_z} - T_c(s)(H + H_d e^{-ds})]^{-1}$$

$$\times T_c(s)L_2 e^{-ds}y(s)$$  \hspace{1cm} (152)

So, the proposed frequency domain description holds.

5 filter-based controller design steps summary

5.1 State Feedback Synthesis

Step 1) Compute matrices $X_c, Y_c$ and $Z_c$ using (11) and (12).

Step 2) Compute matrix $K_c$ using (16).

5.2 Time Domain Functional Filter-Based Controller Design

Step 1) Compute matrix $S$ using (2).

Step 2) Compute matrices $\Theta$ and $\Sigma$ using (34) and (35).

Step 3) Compute matrices $F_{11}, F_{22}, F_{d11}, F_{d22}, J_{11}, J_{22}, J_{d11}$ and $J_{d22}$ using (40), (41), (44), (45), (47), (48), (50) and (51).

5.2.1 Time Domain Design Independent from the state delay

Step 4) Compute matrix $L$ using (82).

Step 5) Compute matrices $F_s, Q_s$ and $Y_s$ by solving the LMIs given by (76), (77) and (80).

Step 6) Compute matrix gain $Z_l$ using (81).

Step 7) Compute matrix gain $Z_c$ using (67).

Step 8) Compute $F$ and $F_d$ using equations (42) and (43).

Step 9) Compute $J, J_d$ and $K$ using respectively equations (46), (49) and (52).

Step 10) Get matrices $L_1$ and $L_2$ from (29) and (31).

Step 11) Get matrices $H$ and $H_d$ using, respectively, conditions iii) and iv) from theorem 2.

5.2.2 Time Domain Design Dependent on the state delay

Step 4) Compute matrix $L$ using (104).

Step 5) Compute matrices $P, Q$ and $Y$ by solving the LMIs given by (99), (102) and (169).

Step 6) Compute matrix gain $Z_l$ using (103).

Step 7) Compute matrix gain $Z_c$ using (67).

Step 8) Compute $F$ and $F_d$ using equations (42) and (43).
We have

\[ e = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad a_0 = \begin{bmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix}, \quad b_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad c_0 = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad d_0 = 0. \]

Figure 1 illustrates the used bounded disturbance such that \( \|w\|_2 = 4109.6 \text{ units} \).
So, the filter-based controller matrices values are given as follows:

\[ F = -541.0033, \]
\[ F_d = -378.8023, \]
\[ L_1 = 70.7845 \]
\[ L_2 = 35.3923, \]
\[ H = 23.5221, \]
\[ H_d = 0.2182, \]
\[ K = 23.6312 \]

Figures 2 and 3 show a comparison between the desired law control defined by \( K_c x(t) \) and the output of the filter-based controller given by equation (3b).

Figures 4 and 5 represent the evolution of the estimation error. We can remark as it shown in figure (4a) that the response time of the filter based controller is relatively small, so it has a rapid estimation dynamic. Then the disturbance effect on the estimation error (Figure (4b)) is not noticed.

6.3 Filter-based controller design dependent on the delay

Similarly to paragraph (6.2), we suppose that:

\[ \gamma = 7.07, \quad d = 1s. \]

Then, we can consider, as an application of theorem 4 on a constant known delay, according to equation (98) that:

\[ \tau^* = 1s. \]

Using equation (102), we get:

\[ P = \begin{pmatrix} 0.5349 & -2.9705 \times 10^{-6} \\ -2.9705 \times 10^{-6} & 41.9921 \end{pmatrix}, \]
\[ Y = \begin{pmatrix} 0.58 & 0.41 & 0.02 & -0.08 & -0.04 \end{pmatrix}, \]

and
\[ Z = \begin{pmatrix} 1.10 & 0.77 & 0.05 & -0.15 & -0.07 \end{pmatrix}, \]

So, the filter-based controller matrices values are given as follows:

\[ F = -1.0599, \quad F_d = -0.842, \]
\[ L_1 = 0.0993, \quad L_2 = 0.0496, \]
\[ H = -0.0428, \quad H_d = 0.2277, \]
\[ K = 0.071 \]

Using the same disturbance function \( w(t) \) as used in paragraph (6.2), we draw the estimation error and the evolution of the law control.

Figure 6 shows the disturbance effect on the estimation error where figure 7 represents a comparison between the real law control and the estimated one during the Transient phase, permanent phase and the Transient duration when disturbance is applied.

\[ \begin{align*}
\text{SNR} &= 20 \log_{10} \left( \frac{\|y\|_2}{\|w\|_2} \right) = 7.7223 \text{dB} \\
\|H_{ew}\|_\infty &= 3.8624 < \gamma.
\end{align*} \]

Comparison between the design dependent on the delay and the independent from delay technique:

Figures 8 and 9 represent a comparison between the estimation errors using the two mentioned methods. Then, we note a quicker dynamic when using the independent from state delay technique but with a greater magnitude (figure 9a).

It’s obvious in (figure 9b) that the dependent on delay method leads to a better error magnitude during the permanent phase.

\[ \begin{align*}
\tau^*_{1} &= 0.4s. \\
\tau^*_{2} &= 1s.
\end{align*} \]

Then, we have:

\[ P = \begin{pmatrix} 0.5312 & -3.1338 \times 10^{-6} \\ -3.1338 \times 10^{-6} & 42.0079 \end{pmatrix}, \]

6.4 Filter-based controller design Dependent on the delay: Application on a linear singular system with variable state delay

In this paragraph, we switch the constant state delay to a variable one \( d(t) \) such that:

\[ d(t) = 0.3\sin(t) + 0.7 \]

Then, we have:

\[ \tau^*_{1} = 0.4s. \quad \tau^*_{2} = 1s. \]

Using equation (102), we get:
\[
Y = \begin{pmatrix} 0.60 & 0.42 & 0.02 & -0.07 & -0.04 \end{pmatrix},
\]
and
\[
Z = \begin{pmatrix} 1.13 & 0.79 & 0.04 & -0.14 & -0.07 \end{pmatrix},
\]
So, the filter-based controller matrices values are given as follows:
\[
F = -1.0924, \quad F_d = -0.8647,
\]
\[
L_1 = 0.0994, \quad L_2 = 0.0497,
\]
\[
H = -0.0396, \quad H_d = 0.2182,
\]
\[
K = 0.0695
\]
When using the same disturbance function \( w(t) \) as used in paragraph (6.2), we draw the estimation error as shown in Figure 11 and the evolution of the function \( u(t) \) and \( K_c x(t) \):

![Figure 10: The Control Laws Evolution](image)

![Figure 11: The Estimation Error](image)

6.5 Filter-based controller synthesis in the frequency domain

6.5.1 Filter-based controller design independent from the delay

The considered controller is the same as in paragraph (6.2). By using the left co-prime factorization, matrices of the frequency domain description of the filter-based controller for singular system (1) are given by:
\[
X_1 = X_3 = 0.9575,
\]
\[
X_2 = X_4 = 0.9649
\]
\[
N_1(s) = \frac{s + 378.8e^{-s} + 540.9}{s + 378.8e^{-s} + 541.9};
\]
\[
M_1(s) = 23.52 \frac{s + 378.8e^{-s} + 541.9}{s + 378.8e^{-s} + 540}
\]
Then:
\[
N_1^{-1}M_1(s) = \frac{23.52}{s + 378.8e^{-s} + 540.9};
\]
and we have:
\[
N_2(s) = \frac{s + 365.5e^{-s} + 522}{s + 378.8e^{-s} + 540};
\]
\[
M_2(s) = 0.2e^{-s} \frac{s + 378.8e^{-s} + 540}{s + 378.8e^{-s} + 540}
\]
Then:
\[
N_2^{-1}M_2(s) = \frac{0.2e^{-s}}{s + 365.5e^{-s} + 522};
\]
Similarly to \( N_1(s) \) and \( M_1(s) \), we get:
\[
N_3(s) = N_1(s)
\]
\[
M_3(s) = \frac{0.06s + 22.3e^{-s} + 38.8}{s + 378.8e^{-s} + 541.9}
\]
Then:
\[
N_3^{-1}M_3(s) = \frac{0.06s + 22.3e^{-s} + 38.8}{s + 378.8e^{-s} + 540.9};
\]
And finally:
\[
N_4(s) = N_2(s)
\]
\[
M_4(s) = \frac{35.4e^{-s}}{s + 378.8e^{-s} + 540.03}
\]

The signal \((y(t))\) to noise \((w(t))\) ratio is evaluated by:
\[
SNR = 20\log_{10}\left(\frac{\|y\|_2}{\|w\|_2}\right) = 7.5782 dB
\]
and the norm \(H_\infty\) of the transfer function of the error to the disturbance is evaluated by:
\[
\|H_{cw}\|_\infty = 3.6831 < \gamma.
\]
Then
\[ N_4^{-1}M_4(s) = \frac{35.4e^{-s}}{s + 365.5e^{-s} + 522}; \]
The singular values plot is given by figure 12.

![Figure 12: The singular values plot](image)

6.5.2 Filter-based controller design dependent on the delay

The considered controller is the same as in paragraph (6.3).

By using the left co-prime factorization, matrices of the frequency domain description of the filter-based controller for singular system (1) are given by:

\[ X_1 = X_3 = 0.1576, \]
\[ X_2 = X_4 = 0.9706 \]
\[ N_1(s) = \frac{s + 0.74e^{-s} + 0.079}{s + 0.74e^{-s} + 1.079}; \]
\[ M_1(s) = \frac{-0.047}{s + 378.8e^{-s} + 541.9}; \]

Then:
\[ N_1^{-1}M_1(s) = \frac{-0.047}{s + 0.74e^{-s} + 0.079}; \]
and we have:
\[ N_2(s) = \frac{0.97s + 0.72e^{-s} + 0.95}{s + 0.74e^{-s} - 0.0492}, \]
\[ M_2(s) = \frac{0.22e^{-s}}{s + 0.74e^{-s} - 0.0492}; \]

Then:
\[ N_2^{-1}M_2(s) = \frac{0.22e^{-s}}{0.97s + 0.72e^{-s} + 0.95}; \]

Similarly to \( N_1(s) \) and \( M_1(s) \), we get:
\[ N_3(s) = N_1(s) \]
\[ M_3(s) = \frac{0.086s + 0.064e^{-s} + 0.034}{s + 0.74e^{-s} + 1.079}; \]

Then:
\[ N_3^{-1}M_3(s) = \frac{0.086s + 0.064e^{-s} + 0.034}{s + 0.74e^{-s} + 0.079}; \]

And finally:
\[ N_4(s) = N_2(s) \]
\[ M_4(s) = \frac{0.038e^{-s}}{s + 0.74e^{-s} - 0.0492}; \]

Then:
\[ N_4^{-1}M_4(s) = \frac{0.038e^{-s}}{0.97s + 0.72e^{-s} + 0.95}; \]

The singular values plot is given by figure 13.

![Figure 13: The Singular values plot](image)

In this section, we show the filter-based controller designs effectiveness in numerical examples. So, we highlight the effectiveness of the design techniques independently from the state delay and dependently on the delay with an application on a variable state delay.

7 Conclusion

In this paper, we have studied the problem of controller design based on a functional \( H_\infty \) filter for singular systems with delay in both state and input vector. The controller is set in time and frequency domains. The time domain method begin with computing the feedback gain for the control law design with the respect to the admissibility problem and a \( H_\infty \) criteria by means of LMIs. Then, a functional filter techniques are used to reconstruct this control
law. Note that the filter synthesis verifies a LMI condition dependently and independently from the delay and based on Lyapunov-Krasovskii theory. The frequency domain approach is based on the time domain result. So using some useful MFDs, functional $H_\infty$ filter description is given. The proposed approaches have been applied on a numerical example and they show their effectiveness.

References:


Appendix A : Theorem 4 equations:

$$\alpha_{13} = P_{1s}F_{d1} - Y_y \Delta F_{d22} + (P_{1s}F_{11} - Y_y \Delta F_{d22})^T + Q_1 + I_n \tag{155}$$

$$\alpha_{14} = -P_{1s}F_{d1}K_e b_0 D_2 - Y_y \Delta F_{d22}K_e b_0 D_2 - Y_y \Delta J_{d22} D_2 + P_{1s}J_{d11}D_2 \tag{156}$$

$$\alpha_{15} = P_{1s}L^T \tag{157}$$

$$\alpha_{22} = -LP_{1s}K_e a_0 D_1 - LP_{1s}K_{n1}c_0 D_1 + LP_{1s}J_{d11}D_2 - LP_{1s}F_{11}K_e b_0 D_2 + LY_y \Delta F_{d22}K_e b_0 D_2 - LY_y \Delta J_{d22} D_2 + (LP_{1s}a_0 D_1 - LP_{1s}K_{n1}c_0 D_1 - LP_{1s}F_{11}K_e b_0 D_2 + LY_y \Delta F_{d22}K_e b_0 D_2 + LP_{1s}J_{d11}D_2 - LY_y \Delta J_{d22} D_2 - LP_{1s}L^T)^T - P_{3s} - P_{3s}^T + Q_4 + I_m \tag{158}$$

$$\alpha_{23} = LP_{1s}F_{d1} - LY_y F_{d22} \tag{159}$$

$$\alpha_{24} = LP_{1s}F_{d1} K_e b_0 D_2 - LY_y \Delta F_{d22}K_e b_0 D_2 + LY_y \Delta J_{d22} D_2 - LP_{1s}J_{d11}D_2 \tag{160}$$

$$\alpha_{25} = P_{3s} \tag{161}$$

$$\alpha_{33} = -Q_1 \tag{162}$$

$$\alpha_{34} = -Q_2 \tag{163}$$

$$\alpha_{35} = 0_{n \times m} \tag{164}$$

$$\alpha_{34} = -Q_4 \tag{165}$$

$$\alpha_{45} = 0_m \tag{166}$$

$$\alpha_{55} = -(\gamma^2 - 1)I_m \tag{167}$$

and

$$\Delta = (I - \zeta_{22} \zeta_{22}^T) \tag{168}$$

Appendix B: Theorem 5 equations:

$$\Xi = \begin{pmatrix} -P & 0 & 0 & 0 \\ -P & 0 & 0 & 0 \\ 0 & -P & 0 & 0 \\ 0 & 0 & -P & 0 \end{pmatrix} \tag{169}$$

$$U_{11} = h_1 + h_1^T + I_{m_z} \tag{170}$$

$$U_{22} = h_2 + h_2^T + I_m \tag{171}$$

$$U_{33} = U_{55} = U_{34} = -(1 + \tau \tau^T) P_1 \tag{172}$$

$$U_{44} = U_{66} = -(1 + \tau \tau^T) P_3 \tag{173}$$

$$U_{77} = -2h_2^2 I_m \tag{174}$$

$$U_{56} = (1 + \tau \tau^T) P_1 N_{11} D_2 \tag{175}$$

$$U_{12} = h_3 + h_4 + h_5 L^T \tag{176}$$

$$U_{13} = U_{35} = 0_{m_z \times m_z} \tag{177}$$

$$U_{14} = U_{36} = U_{57} = 0_{m_z \times m} \tag{178}$$

$$U_{15} = -h_5 \tag{179}$$
\[
U_{16} = -h_5 L^T 
\]  
(180)  
\[
U_{17} = -P_1 N_{11} D_2 
\]  
(181)  
\[
U_{23} = U_{45} = 0_{m \times m} 
\]  
(182)  
\[
U_{24} = U_{46} = U_{67} = 0_m 
\]  
(183)  
\[
U_{25} = -h_4^T T 
\]  
(184)  
\[
U_{26} = -h_4^T L^T 
\]  
(185)  
\[
U_{27} = P_3 
\]  
(186)  

with

\[
h_1 = P_1 F_{11} - Y \Delta F_{22} + P_1 F_{d11} - Y \Delta F_{d22} 
\]  
(187)  
\[
h_2 = -LP_1 K_c a_0 D_1 - LP_1 K_{11} c_0 D_1 + L \Delta F_{22} K_c b_0 D_2 - LP_1 F_{11} K_c b_0 D_2 - L \Delta J_{22} D_2 + (LP_1 a_0 D_1 - LP_1 K_{11} c_0 D_1 - LP_1 F_{11} K_c b_0 D_2 - LP_1 L^T + L \Delta F_{22} K_c b_0 D_2 + LP_1 J_{11} D_2 - L \Delta J_{22} D_2)^T 
\]  
(188)  
\[
h_3 = -P_1 K_c a_0 D_1 - P_1 K_{11} c_0 D_1 - P_1 F_{11} K_c b_0 D_2 + Y \Delta F_{22} K_c b_0 D_2 + Q_2 + (LP_1 F_{11} - L \Delta F_{22})^T - P_1 L^T - Y \Delta J_{22} D_2 + P_1 J_{11} D_2 
\]  
(189)  
\[
h_4 = P_1 F_{d11} K_c b_0 D_2 - Y \Delta F_{d22} K_c b_0 D_2 - Y \Delta J_{d22} D_2 + P_1 J_{d11} D_2 
\]  
(190)  
\[
h_5 = (P_1 F_{d11} - Y F_{d22})^T 
\]  
(191)  

and

\[
Q_{11} = \tau^* 2 P_1 F_{11} - \tau^* 2 Y \Delta F_{22} 
\]  
(192)  
\[
Q_{12} = \tau^* 2 (-P_1 K_c a_0 D_1 - P_1 K_{11} c_0 D_1 - P_1 F_{11} K_c b_0 D_2 + Y \Delta F_{22} K_c b_0 D_2 - Y \Delta J_{22} D_2 + P_1 L^T + P_1 J_{11} D_2) 
\]  
(193)  
\[
Q_{21} = \tau^* 2 (LP_1 F_{11} - L \Delta F_{22}) 
\]  
(194)  
\[
Q_{22} = \tau^* 2 (-LP_1 K_c a_0 D_1 - LP_1 K_{11} c_0 D_1 - LP_1 F_{11} K_c b_0 D_2 + L \Delta F_{22} K_c b_0 D_2 + LP_1 J_{11} D_2 - L \Delta J_{22} D_2 - P_3) 
\]  
(195)  
\[
Q_{51} = \tau^* 2 (P_1 F_{d11} - Y \Delta F_{d22}) 
\]  
(196)  
\[
Q_{52} = \tau^* 2 (-P_1 F_{d11} K_c b_0 D_2 - Y \Delta F_{d22} K_c b_0 D_2 - Y \Delta J_{d22} D_2 + P_1 J_{d11} D_2) 
\]  
(197)  
\[
Q_{61} = \tau^* 2 (LP_1 F_{d11} - L \Delta F_{d22}) 
\]  
(198)  
\[
Q_{62} = \tau^* 2 (LP_1 F_{d11} K_c b_0 D_2 - L \Delta F_{d22} K_c b_0 D_2 + L \Delta J_{d22} D_2 - LP_1 J_{d11} D_2) 
\]  
(199)  

\[
\Delta = (I - \zeta_{22} \zeta_{22}^+) 
\]  
(200)