Disturbance Decoupling Problem for Switched Linear Systems. A Geometric Approach

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Abstract: In this paper disturbance decoupling problem for switched linear systems is formulated under a geometrical point of view. Necessary and sufficient conditions for the problem with standardizable condition to be solvable are given.

Key–Words: Disturbance decoupling, switched linear systems, invariant subspaces.

1 Introduction

In recent years, linear switching systems are being used to study modelling control problems in diverse fields, such as electrical networks, networked control systems, power electronics aerospace, automotive technologies.

At this time, a large list of articles can be found on fundamental topics like stability, controllability and reachability of switched linear systems (see [18] and [9] for example), the authors Meng and Zhang in [14] provided necessary conditions and sufficient conditions for reachability. However, a small number of contributions can be found dealing with disturbance decoupling problem on linear switching systems.

It is well known that robustness is an important objective in control system theory because the design of plants are vulnerable to unpredicted external disturbances and noises causing always difference between the mathematical model used for design and the actual plant. Therefore, it is required to find if it is possible, a feedback to guarantee the stability and performance of the system under such uncertainties.

Different authors analyze robustness and satability problems for linear systems (see [3], [4] and [8] for example). Disturbance decoupling problems have been studied for time invariant linear systems under a geometrical point of view by using the concepts of some particular invariant subspaces associated to the systems (see [2] and [7] for example).

This concept of invariant subspaces has been generalized to various types of systems as for example singular linear systems in order to study the same kind of problems. N. Otsuka in [16] and E. Yurtseven, W.P.M.H. Heemels, M-K. Camlibel in [19], use simultaneous invariant subspaces to study families of linear systems; concretely study disturbance decoupling problem for switched systems, that is to say families of subsystems with switching rule that concerns with several environmental factors and different controllers, which many authors studied for different kind of switched systems as for example E. Feron [5], D. Liberson [13] and Z.D. Sun and S.S. Ge [18], among others.

Singular switched linear systems are an important class of switched systems that appears in many engineering problems as for example electrical networks.

Example 1

Let us consider a resistor-capacitor (RC) circuit as shown in the figure 1.

Where C represents capacitance, R load resistance and E the source voltage.

Equations when the switch S_1 is closed are:

$$\begin{pmatrix} RC & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{Q} \\ \dot{I} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & RC \end{pmatrix} \begin{pmatrix} Q \\ I \end{pmatrix} + \begin{pmatrix} C \\ -C \end{pmatrix} E \tag{1}$$

and when the switch S_2 is closed are:

$$\begin{pmatrix} RC & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{Q} \\ \dot{I} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ RC & -1 \end{pmatrix} \begin{pmatrix} Q \\ I \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} E. \tag{2}$$

We are concerned with dynamical systems described by a combination of linear dynamical systems

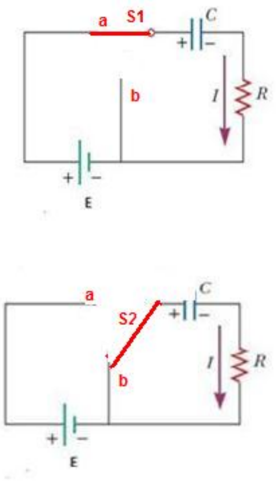

Figure 1: RC-Circuit

and discrete switching events, in the following manner.

Definition 1 *A switched singular linear system is a system which consists of several linear subsystems and a switching well-defined path σ taking values into the index set $M = \{1, \dots, \ell\}$ which indexes the different subsystems.*

$$\left.\begin{array}{rl} E_\sigma \dot{x}(t) & = A_\sigma x(t) + B_\sigma u(t), \\ y(t) & = C_\sigma x(t) \end{array}\right\} \qquad (3)$$

where $E_i, A_i \in M_n(\mathbb{R})$, $B_i \in M_{n\times m}(\mathbb{R})$, $C_i \in M_{p\times n}(\mathbb{R}$, and $\dot{x} = dx/dt$.

i) switching path $\sigma : [t_0, T) \longrightarrow M$, $t_0 < T \leq \infty$, for some initial time t_0,

ii) switching sequence of σ over $[t_0, T)$, $\{(t_0, \sigma(t_0^+)), (t_1, \sigma(t_1^+), \dots, (t_\ell, \sigma(t_\ell^+))\}$.

Remark 2 *For simplicity, the singular switched linear system 3, will be written simply as a quadruple of matrices $(E_\sigma, A_\sigma, B_\sigma, C_\sigma)$ and the standard ones as a triple of matrices $(A_\sigma, B_\sigma, C_\sigma)$. And in the case where the matrices C_σ are not involved in the problem, will be written as $(E_\sigma, A_\sigma, B_\sigma)$ for the singular case and (A_σ, B_σ) for the standard case.*

The paper is organized as follows. In section 2, the disturbance decoupling problem is presented, section 3 is devoted to define and construct simultaneously invariant subspaces. Finally, in section 4, we

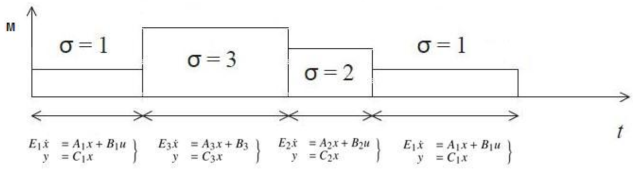

Figure 2: Switching times (discontinuities of σ for $\ell = 3$)

apply the concept of simultaneously generalized invariant subspace to obtain some conditions to solve the disturbance decoupling problem for some particular cases of singular switched linear systems.

2 Disturbance decoupling problem

Definition 3 *A switched singular linear system with "continuous" disturbance is a system which consists of several linear subsystems with disturbance and a piecewise constant map σ taking values into the index set $M = \{1, \dots, \ell\}$ which indexes the different subsystems. In the continuous case, such a system can be mathematically described by*

$$\begin{array}{rl} E_\sigma \dot{x}(t) & = A_\sigma x(t) + B_\sigma u(t) + D_\sigma d(t) \\ y(t) & = C_\sigma x(t) \end{array} \qquad (4)$$

where $E_\sigma, A_\sigma \in M_n(\mathbb{C})$, $B_\sigma \in M_{n\times m}(\mathbb{C})$, $D_\sigma \in M_{n\times q}(\mathbb{C})$, $C_\sigma \in M_{p\times n}(\mathbb{C})$ and $\dot{x} = dx/dt$.

Remark 4 *The term $d(t)$, $t \geq 0$, represents a disturbance, which may represent modeling or measuring errors, noise, or higher order terms in linearization.*

Problem 2.1 *The disturbance decoupling problem is stated as follows: find necessary and sufficient conditions under which we can choose proportional and derivative feedback such that, the matrix pencil $(E_\sigma + B_\sigma F_\sigma^E, A_\sigma + B_\sigma F_\sigma^A)$ is regular of index at most one and D_σ has no influence on the output y.*

Remark 5 *It is not sufficient that the subsystems of a switched linear system are disturbance decoupled for the switched linear system itself to be disturbance decoupled.*

Example 2

Consider a switched singular system consisting of the following two systems with disturbance

- First subsystem

$$\left.\begin{aligned}\begin{pmatrix}1&&\\&1&\\&&0\end{pmatrix}\begin{pmatrix}\dot x_1(t)\\ \dot x_2(t)\\ \dot x_3(t)\end{pmatrix} &= \begin{pmatrix}0&&\\&0&\\&&1\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{pmatrix}\\ +\begin{pmatrix}0\\0\\1\end{pmatrix}u(t) &+ \begin{pmatrix}0\\1\\0\end{pmatrix}d(t)\\ y(t) &= \begin{pmatrix}1&0&0\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{pmatrix}\end{aligned}\right\}$$

- Second subsystem

$$\left.\begin{aligned}\begin{pmatrix}1&&\\&1&\\&&0\end{pmatrix}\begin{pmatrix}\dot x_1(t)\\ \dot x_2(t)\\ \dot x_3(t)\end{pmatrix} &= \begin{pmatrix}0&&\\&0&\\&&1\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{pmatrix}\\ +\begin{pmatrix}0\\0\\1\end{pmatrix}u(t) &+ \begin{pmatrix}1\\0\\0\end{pmatrix}d(t)\\ y(t) &= \begin{pmatrix}0&1&0\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{pmatrix}\end{aligned}\right\}$$

With the following switched law:

$$\sigma(t) = \begin{cases}1 & 0 \le t < t_1\\ 2 & t_1 \le t\end{cases}$$

It is easy to compute the output at t_1 that is given by

$$y(t_1) = \begin{pmatrix}0&1&0\end{pmatrix}\begin{pmatrix}x_1(0)+\int_{t_1}^{t} d(\tau)d\tau\\ x_2(0)+\int_0^{t_1} d(\tau)d\tau\\ -u(t_1)\end{pmatrix} = x_2(0)+\int_0^{t_1} d(\tau)d\tau$$

Then the switched system is not disturbance decoupled.

Nevertheless, both subsystems are disturbance decoupled:

- First subsystem:

$$\begin{pmatrix}1&0&0\end{pmatrix}\begin{pmatrix}x_1(0)\\ x_2(0)+\int_0^t d(\tau)d\tau\\ -u(0)\end{pmatrix} = x_1(0)$$

- Second subsystem:

$$\begin{pmatrix}0&1&0\end{pmatrix}\begin{pmatrix}x_1(0)+\int_0^t d(\tau)d\tau\\ x_2(0)\\ -u(0)\end{pmatrix} = x_2(0)$$

The problem of constructing feedbacks that suppress the disturbance in the sense that $d(t)$ does not affect the input-output behavior of the system has been largely analyzed in both cases standard and singular state space systems (see [1], [15], [17] for example). In this paper we analyze the disturbance decoupling problem for standard switched systems and a particular case of singular switched systems, using geometric tools.

3 Geometric Tools

The disturbance decoupling problem of a structural control problem can be solved by geometric methods.

3.1 Invariant subspaces

Remember that a subspace $G \subset \mathbb{C}^n$ is called invariant under (A, B) (also called robust controlled invariant subspace) if if

$$AG \subset G + \operatorname{Im} B \tag{5}$$

Equivalently, we have that a subspace G is (A, B)-invariant if

$$(A + BF_A)G \subset G.$$

This definition is easily generalized to (E, A, B)-invariant subspaces in the following maner

Definition 6 *A subspace $G \subset \mathbb{C}^n$ is said invariant under (E, A, B), if*

$$AG \subset EG + \operatorname{Im} B \tag{6}$$

(For more information see [6] and [10], for example).

Remark 7 *Observe that if $E = I_n$, this definition coincides with definition of (A, B)- invariant subspace.*

We can construct (E, A, B)-invariant subspaces in the following manner. Let $H \subset \mathbb{C}^n$ be a subspace (in particular we can chose $H = \mathbb{C}^n$, we define

$$G_{k+1} = H \cap \{x \in \mathbb{C}^n \mid Ax \in EG_k + \operatorname{Im} B\},\ G_0 = H,$$

limit of recursion exists and we will denote by $G(H)$. This subspace is the supremal (E, A, B)-invariant subspace contained in H.

Example 3

Let (E, A, B) be a triple of matrices with $E = \begin{pmatrix}1&&\\&1&\\&&0\end{pmatrix}$, $A = \begin{pmatrix}0&&\\&1&\\&&1\end{pmatrix}$, $B = \begin{pmatrix}1\\0\\0\end{pmatrix}$ and $H = \{(x, y, z) \mid x = 0\}$, Computation of G_1:

$$\begin{pmatrix}0&&\\&1&\\&&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1&&\\&1&\\&&0\end{pmatrix}\begin{pmatrix}0\\\mu\\\nu\end{pmatrix}+\begin{pmatrix}\lambda\\0\\0\end{pmatrix}$$

$[(x,y,0)]\cap H=[(0,1,0)]=G_1$.

Computation of G_2:

$$\begin{pmatrix}0&&\\&1&\\&&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1&&\\&1&\\&&0\end{pmatrix}\begin{pmatrix}0\\\mu\\0\end{pmatrix}+\begin{pmatrix}\lambda\\0\\0\end{pmatrix}$$

$[(x,y,0)]\cap H=[(0,1,0)]=G_2=G_1$. Then $G=G_1$.

Obviously $AG\subset EG+\operatorname{Im}B$.

Proposition 8 *Let (E,A,B) be a triple of matrices. A subspace $G\subset\mathbb{C}^n$ is invariant under (E,A,B) if and only if is invariant under $(E+BF_E, A+BF_A, B)$ for all possible feedbacks F_E, $F_A\in M_{m\times n}(\mathbb{C})$.*

Proof:

Suppose that $AG\subset EG+\operatorname{Im}B$, then for all $x\in G$, there exists $y\in G$, $v=Bw\in\operatorname{Im}B$ such that $Ax=Ey+Bw$ so, for any $F_E,F_A\in M_{m\times n}(\mathbb{C})$, we have

$$\begin{array}{l}Ax+BF_Ax-BF_Ax=Ey+BF_Ey-BF_Ey+Bw\\(A+BF_A)x=(E+BF_E)y+B(F_Ax-F_Ey+w).\end{array}$$

Consequently, for all $x\in G$, $(A+BF_A)G\subset(E+BF_E)G+\operatorname{Im}B$.

Reciprocally. If $G\subset\mathbb{C}^n$ is invariant under $(E+BF_E, A+BF_A, B)$ for all possible feedbacks F_E, $F_A\in M_{m\times n}(\mathbb{C})$, in particular it is invariant under $(E+BF_E, A+BF_A, B)$ for $F_E=F_A=0$. □

Let $(E_1,A_1,B_1),(E_2,A_2,B_2)$ be two triples of matrices, we say that they are equivalent, if and only if, there exist invertible matrices $P,Q\in Gl(n)$ and $R\in Gl(m)$ and rectangular matrices $F_E,F_A\in M_{m\times n}$ such that

$$\begin{array}{l}(E_2,A_2,B_2)=\\(QE_1P+QB_1F_E, QA_1P+QB_1F_A, QB_1R).\end{array}$$

Proposition 9 *Let $(E_1,A_1,B_1),(E_2,A_2,B_2)$ be two equivalent triples. Then $G\subset\mathbb{C}^n$ is an invariant subspace under (E_1,A_1,B_1) if and only if $P^{-1}G$ is invariant under (E_2,A_2,B_2).*

Proof:

Suppose that $A_1G\subset E_1G+\operatorname{Im}B$.

Then,

$$\begin{array}{l}A_2P^{-1}G\\=(QA_1P+QB_1F_{A_1})P^{-1}G\\=Q(A_1G+B_1F_{A_1}P^{-1}G)\subset Q(E_1G+\operatorname{Im}B_1)\\=Q((Q^{-1}E_2P^{-1}-Q^{-1}B_2R^{-1}F_EP^{-1})G+\\\quad\operatorname{Im}Q^{-1}B_2R^{-1})\\=Q(Q^{-1}(E_2P^{-1}-B_2R^{-1}F_EP^{-1})G+\\\quad Q^{-1}\operatorname{Im}B_2R^{-1})\\=QQ^{-1}((E_2P^{-1}-B_2R^{-1}F_EP^{-1})G+\\\quad\operatorname{Im}B_2R^{-1})\\\subset(E_2-B_2R^{-1}F_E)P^{-1}G+\operatorname{Im}B_2.\end{array}$$

Now, it suffices to apply proposition 8. □

Example 4

Let (E_1,A_1,B_1) be the triple in the example 3, and G the invariant subspace obtained in it. Let $(E_2,A_2,B_2)=(QE_1P+QB_1F_E, QA_1P+QB_1F_A, QB_1R)$ be an equivalent triple with

$$P=\begin{pmatrix}1&0&1\\1&2&0\\1&0&-1\end{pmatrix},\ Q=\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix},$$

$F_E=F_A=\begin{pmatrix}1&1&1\end{pmatrix}$, $R=\begin{pmatrix}1\end{pmatrix}$.

Clearly

$$\begin{pmatrix}3&3&0\\2&2&-1\\1&0&-1\end{pmatrix}\begin{pmatrix}0\\2\lambda\\0\end{pmatrix}=\begin{pmatrix}3&3&2\\1&2&0\\0&0&0\end{pmatrix}\begin{pmatrix}0\\2\lambda\\0\end{pmatrix}.$$

Consequently, G is a (E_2,A_2,B_2)-invariant subspace.

In particular, if E is an invertible matrix, we have the following corollary.

Corollary 10 *A subspace G is (E,A,B)-invariant if and only if it is $(E^{-1}A,E^{-1}B)$-invariant.*

Proof:

It suffices to take $Q=E^{-1}$, $P=I_n$, $R=I_m$ and $F_E=F_A=0$ □

Now, we are going to present a particular case of invariant subspaces.

First of all we observe the following result.

Proposition 11 *Let (A,B) be a standard pair. Then*

$$G=[B,AB,\dots,A^{n-1}B]$$

is a (A,B)-invariant subspace.

Proof:

$$AG=A[B,AB,\dots,A^{n-1}B]=[AB,A^2B,\dots,A^nB]$$

Now, it suffices to apply the Cayley-Hamilton theorem. □

Theorem 12 *Let*

$$C_r = \begin{pmatrix} E & & & & B & & & \\ A & E & & & & B & & \\ & \ddots & \ddots & & & & \ddots & \\ & & & E & & & & B \\ & & & A & & & & B \end{pmatrix} \in M_{nr\times(n(r-1)+mr)}(\mathbb{C})$$

be the r-controllability matrix. Suppose r being the least such that $\operatorname{rank} C_r < (n(r-1)+mr)$*, and let* $\begin{pmatrix} v_1 & \dots & v_r & w_1 & \dots & w_{r+1}\end{pmatrix} \in \operatorname{Ker} C_r$ *(v_i are vectors in $\mathbb{C}^n$ and w_i vectors in $\mathbb{C}^m$). Then $G = [v_1, \dots, v_r]$ is a (E, A, B)-invariant subspace.*

Proof:

We consider $v = \lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_{r-1}v_{r-1} + \lambda_r v_r$, $Av = \lambda_1 Av_1 + \lambda_2 Av_2 + \ldots + \lambda_{r-1}Av_{r-1} + \lambda_r Av_r = \lambda_1(-Ev_2 - Bw_2) + \lambda_2(-Ev_3 - Bw_3) + \ldots + \lambda_{r-1}(-Ev_r - Bw_r) - \lambda_r Bw_{r+1} = E(\lambda_1 v_2 - \lambda_2 v_3 - \ldots - \lambda_{r-1}v_r) + B(-\lambda_1 w_2 - \lambda_2 w_3 - \ldots - \lambda_{r-1}w_r - \lambda_r w_{r+1}) \in EG + \operatorname{Im} B$. □

Definition 13 *The space sum of all spaces G in theorem before is a invariant subspace that we will call controllability subspace and we will denote it by $\mathcal{C}(E, A, B)$.*

Notice that $\mathcal{C}(E, A, B)$ is the set of states in which the system is controllable.

Corollary 14 *Let (E, A, B) be a triple with $E = I_n$. In this case the invariant subspace G obtained in the above theorem, coincides with the controllability (A, B)-invariant subspaces $[B, AB, \dots, A_{r-1}B]$.*

Proof:

Making block-row elemental transformations to the matrix C_r we obtain the equivalent matrix

$$\begin{pmatrix} I_n & & & & B & & & \\ 0 & I_n & & & -AB & B & & \\ & \ddots & \ddots & & & & \ddots & \\ & & & I_n & (-1)^{r-2}A_{r-2}B & & -AB & B \\ & & & 0 & (-1)^{r-1}A^{r-1}B & & & -AB & B \end{pmatrix}.$$

□

3.2 Simultaneously invariant subspaces

For standard case we define simultaneously invariant subspaces in the following manner.

Definition 15 *A subspace G of $\mathbb{C}^n$ is said to be simultaneously $(A_i, B_i)_{i\in M}$-invariant; if and only*

$$A_i G \subset G + \operatorname{Im} B_i, \ \forall i, 1 \le i \le \ell.$$

and we have the following result.

Proposition 16 *A subspace G of $\mathbb{C}^n$ is simultaneously $(A_i, B_i)_{i\in M}$-invariant if and only if there exist F_{A_i} such that*

$$(A_i + B_i F_{A_i})G \subset G \ \forall i, 1 \le i \le \ell..$$

In general, for singular switched linear systems, we have:

Definition 17 *A subspace G of $\mathbb{C}^n$ is said to be simultaneously $(E_i, A_i, B_i)_{i\in M}$-invariant; if and only*

$$A_i G \subset E_i G + \operatorname{Im} B_i, \ \forall i, 1 \le i \le \ell.$$

Proposition 18 *A subspace G of $\mathbb{C}^n$ is simultaneously $(E_i, A_i, B_i)_{i\in M}$-invariant if and only if, for all F_{A_i} and F_{B_i} we have*

$$(A_i + B_i F_{A_i})G \subset (E_i + B_i F_{E_i})G.$$

Proof:

It is a direct consequence of proposition 8. □

3.3 Construction of Simultaneously invariant subspaces

Analogously to the method to get invariant subspaces, we construct simultaneously invariant subspaces. A.A. Julius, A.J. van der Schaft in [11] with a similar method, constructs controlled invariant subspaces of standard switched linear systems.

i) For standard switched systems

Definition 19 *Let $H \subset \mathbb{C}^n$ be a subspace, we define:*

$$\begin{aligned} V_0 &= H, \\ V_{k+1} &= H \cap \{x \in \mathbb{C}^n \mid (A_i + B_i F_{A_i})x \\ &\quad \in V_k + \operatorname{Im} B_i, \ \forall i, 1 \le i \le \ell\}. \end{aligned}$$

Proposition 20

$$V_{k+1} \subset V_k, \ \forall k = 0, 1, 2, \dots$$

Proof:

Clearly, $V_1 \subset V_0$, and if $V_k \subset V_{k-1}$, then for all $x \in V_{k+1}$ is $x \in H$ and $\oplus(A_i + B_i F_i^A)x = \oplus(E_i + B_i F_i^E)u + \oplus B_i v_i$ with $u \in V_k \subset V_{k-1}$.

So, $\oplus(A_i + B_i F_i^A)x \in \oplus(E_i + B_i F_i^E)V_{k-1} + \oplus\operatorname{Im} B_i$, that is to say $x \in V_k$. □

Remark 21 *Limit of recursion exists and we will denote by $V(H)$. This subspace is the supremal simultaneously (A_i, B_i)-invariant subspace contained in H.*

We are interested in the case where the subspace is $\cap_\sigma \mathrm{Ker}\, C_\sigma$.

So, from now, on we consider

$$H = \cap_\sigma \mathrm{Ker}\, C_\sigma.$$

ii) For singular switched systems

Definition 22 *Let $H \subset \mathbb{C}^n$ be a subspace, we define:*

$$\begin{aligned} W_0 &= H, \\ W_{k+1} &= H \cap \{x \in \mathbb{C}^n \mid (A_i + B_i F_{A_i})x \\ &\in (E_i + B_i F_{E_i})W_k + \mathit{Im}\, B_i. \forall i, 1 \le i \le \ell\}. \end{aligned}$$

Proposition 23

$$W_{k+1} \subset W_k, \ \forall k = 0, 1, 2, \dots$$

Proof:

Analogous to proposition 20. □

Remark 24 *Limit of recursion exists and we will denote by $W(H)$. This subspace is the supremal simultaneously (E_i, A_i, B_i)-invariant subspace contained in H.*

As in the standard case, we are interested in the case where the subspace is $\cap_\sigma \mathrm{Ker}\, C_\sigma$.

So, from now on, we consider

$$H = \cap_\sigma \mathrm{Ker}\, C_\sigma.$$

Example 5

Let $(E_\sigma, A_\sigma, B_\sigma, C_\sigma)$ be the triples considered in example 2, and $H = \cap_\sigma \mathrm{Ker}\, C_\sigma = [(0, 0, 1)]$.

Computing W_1: in this particular case $E_1 = E_2$, $A_1 = A_2$ and $B_1 = B_2$, then

$$\begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \mu \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix}$$

Then, $W_1 = W_0$ and $W(H) = W_0$.

4 Solving disturbance decoupling problem

Hereinafter and in order to simplify notations, we identify D_i by $\mathrm{Im}\, D_i$.

We will use invariant subspaces constructed in the previous section to analyze the disturbance decoupling problem.

The solution for standard case can be found in [16] and [19], but we show for a better understanding of Article

Proposition 25 *Let $(A_\sigma, B_\sigma, C_\sigma)$ be a standard switched system with disturbance D_σ. Then the disturbance decoupling problem is solvable if and only if*

$$\sum D_i \subset V(H).$$

with $H = \cap_i \mathrm{Ker}\, C_i$

Proof:

Suppose that the switched system 4 is activated by the switched rule as follows.

$$(A_{i_1}, B_{i_1}, C_{i_1}, D_{i_1}) \to (A_{i_2}, B_{i_2}, C_{i_2}, D_{i_2}) \to \dots, \tag{7}$$

where $i_1, i_2, i_3, \dots \in M$.

When the first system $(A_{i_1}, B_{i_1}, C_{i_1}, D_{i_1})$ is activated the state space generated by the disturbance D_{i_1} is

$$\begin{aligned} &\langle A_{i_1} + B_{i_1} F_{i_1} \mid D_{i_1} \rangle \\ &= [D_{i_1}, (A_{i_1} + B_{i1} F_{i_1}) D_{i_1}, \dots, (A_{i_1} + B_{i_1} F_{i_1})^{n-1} D_{i_1}] \\ &= \left\{ \int_0^\tau e^{(A_{i_1} + B_{i_1} F_{i_1})(t-\tau)} D_{i_1} d(\tau) d\tau \right\}. \end{aligned}$$

If the subsystem $(A_{i_1}, B_{i_1}, C_{i_1}, D_{i_1})$ is changed to $(A_{i_2}, B_{i_2}, C_{i_2}, D_{i_2})$ by the switched rule 7, then the subspace generated by disturbances through $\langle (A_{i_1} + B_{i_1} F_{i_1}) \mid D_{i_1} \rangle$ and D_{i_2} is

$$\langle (A_{i_2} + B_{i_2} F_{i_2}) \mid \langle (A_{i_1} + B_{i_1} F_{1_1}) \mid D_{i_1} \rangle + D_{i_2} \rangle$$

Analogously, we have the following subspaces.

$$\left\langle (A_{i_j} + B_{i_j} F_{i_j}) \mid \left\langle (A_{i_{j-1}} + B_{i_{j-1}} F_{i_{j-1}}) \mid D_{i_{j-1}} \right\rangle + D_{i_j} \right\rangle, \tag{8}$$

for $j \ge 2$.

From the construction of subspaces 8 we have that

$$\begin{aligned} &\sum D_{i_j} \subseteq \langle (A_{i_1} + B_{i_1} F_{i_1}) \mid D_{i_1} \rangle \subseteq \dots \subseteq \\ &\left\langle (A_{i_j} + B_{i_j} F_{i_j}) \mid \left\langle (A_{i_{j-1}} + B_{i_{j-1}} F_{i_{j-1}}) \mid D_{i_{j-1}} \right\rangle + D_{i_j} \right\rangle \\ &\subseteq \dots \end{aligned}$$

Since they are subspaces of a finite-dimensional space, there exists a finite number ρ such that

$$\begin{aligned} &\dots \subseteq \\ &\left\langle (A_{i_\rho} + B_{i_\rho} F_{i_\rho}) \mid \left\langle (A_{i_{\rho-1}} + B_{i_{\rho-1}} F_{i_{\rho-1}}) \mid D_{i_{\rho-1}} \right\rangle + D_{i_\rho} \right\rangle \\ &= \\ &\left\langle (A_{i_\kappa} + B_{i_\kappa} F_{i_\kappa}) \mid \left\langle (A_{i_{\kappa-1}} + B_{i_{\kappa-1}} F_{i_{\kappa-1}}) \mid D_{i_{\kappa-1}} \right\rangle + D_{i_\kappa} \right\rangle \\ &= \dots \end{aligned}$$

for all $\ell \ge \rho$ and $\kappa = \ell + \rho$..

Clearly all these subspaces are simultaneously (A_i, B_i)-invariant.

Obviously, the decoupling problem has solution if and only if

$$\left\langle (A_{i_j}+B_{i_j}F_{i_j}) \mid \left\langle (A_{i_{j-1}}+B_{i_{j-1}}F_{i_{j-1}}) \mid D_{i_{j-1}}\right\rangle + D_{i_j}\right\rangle \subset \cap_i \text{Ker}\, C_i$$

So,

$$\left\langle (A_{i_j}+B_{i_j}F_{i_j}) \mid \left\langle (A_{i_{j-1}}+B_{i_{j-1}}F_{i_{j-1}}) \mid D_{i_{j-1}}\right\rangle + D_{i_j}\right\rangle \subset V(H)$$

because of $V(H)$ is the maximal simultaneously (A_i, B_i)-invariant subspace contained in $\cap_i \text{Ker}\, C_i$. □

Now, we are going to try to solve the problem for standardizable systems.

Lemma 26 *Let $(E_\sigma, A_\sigma, B_\sigma)$ be a singular switched system and suppose that* rank $(E_i, B_i) = n$ *for all $i \in \sigma$. Then the system can be reduced to a standard switched system.*

A switched system verifying this property will be called standardizable (by feedback) switched system.

The disturbance decoupling problem can be translated into the following geometric problem.

Theorem 27 *Let $(E_\sigma, A_\sigma, B_\sigma, C_\sigma)$ be a standardizable (by feedback) switched system with disturbance D_σ. Then the disturbance decoupling problem is solvable if and only if*

$$\sum (E_i + B_i F_{E_i})^{-1} D_i \subset W(H).$$

with $H = \cap_i \text{Ker}\, C_i$

Proof:

Observe that if the system is standardizable, then the index is zero.

After proposition 9, we have that the supremal simultaneously invariant subspace $W(H)$ for $(E_\sigma, A_\sigma, B_\sigma, C_\sigma)$ coincides with the supremal simultaneously invariant subspace $V(H)$ for $((E_\sigma + B_\sigma F_{E_\sigma})^{-1}A_\sigma, (E_\sigma + B_\sigma F_{E_\sigma})^{-1}B_\sigma, C_\sigma)$. And, the switched system $(E_\sigma + B_\sigma F_{E_\sigma})^{-1}A_\sigma, (E_\sigma + B_\sigma F_{E_\sigma})^{-1}B_\sigma, C_\sigma)$ with disturbance $(E_\sigma + B_\sigma F_{E_\sigma})^{-1}A_\sigma D_\sigma)$ is solvable if and only if

$$\sum (E_i + B_i F_{E_i})^{-1} D_i \subset V(H).$$

□

Following example 5, and taking

$$F_{E_i} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix},$$

we have

$$(E_i + B_i F_{E_i})^{-1} D_i = D_i,$$
$$\sum D_i \not\subset V(H).$$

But, if we consider the subsystems separately then it is easy to show that $G_i \subset V(\text{Ker}\, C_i)$ for each $i = 1, 2$.

Corollary 28 *Suppose that D_i are (E_i, B_i)-invariant. If $\sum_i D_i \subset V(H)$, then the disturbance decoupling problem of the switched system $(E_\sigma, A_\sigma, B_\sigma, C_\sigma)$ with disturbance D_σ is solvable.*

Proof:

If D_i is (E_i, B_i)-invariant then

$$D_i = (E_i + B_i F_{E_i})^{-1} D_i.$$

So,

$$\sum D_i = \sum (E_i + B_i F_{E_i})^{-1} D_i \subset V(H) = W(H).$$

□

Finally, we try to solve the problem for the case where the switched system is regularizable equisingular of index one (quite natural in applications as for example modeling a pulse-width modulator boost-converter, [12]).

Definition 29 *A switched system $(E_\sigma, A_\sigma, B_\sigma, C_\sigma)$ is called regularizable equisingular of index one, if and only if there exist matrices $Q, P \in Gl(n;\mathbb{C})$,$R \in Gl(m;\mathbb{C})$, $S \in Gl(p;\mathbb{C})$, $F_{E_i}, F_{A_i} \in M_{m\times n}(\mathbb{C})$ and $O_{E_i}, O_{A_i} \in M_{n\times p}(\mathbb{C})$, such that*

$$\begin{array}{l}(\bar{E}_i, \bar{A}_i, \bar{B}_i, \bar{C}_i) = \\ (QE_iP + QB_iF_{E_i} + O_{E_i}C_iP, QA_iP + QB_iF_{A_i} + \\ \quad O_{A_i}C_iP, SC_iP, QB_iR) = \\ (Q(E_i + B_iF_{E_i}P^{-1} + Q^{-1}O_{E_i}C_i)P, \\ \quad Q(A_i + B_iF_{A_i}P^{-1} + Q^{-1}O_{A_i}C_i)P, \\ \qquad SC_iP, QB_iR)\end{array} \tag{9}$$

with

$$\begin{array}{c}\bar{E}_i = \begin{pmatrix} I_r & \\ & 0 \end{pmatrix},\ \bar{A}_i = \begin{pmatrix} \bar{A}_{i_r} & \\ & I_{n-r} \end{pmatrix}, \\ \bar{B}_i = \begin{pmatrix} \bar{B}_{i_r} \\ 0_{n-r\times m} \end{pmatrix},\ \bar{C}_i = \begin{pmatrix} \bar{C}_{i_r} & 0_{p\times n-r} \end{pmatrix}.\end{array} \tag{10}$$

We will call *equisingular* reduced form the switched system expressed in the form 10.

(Observe that matrices Q, P, R, S are the same for all i, all subsystems are regularizable and the reduced subsystems are of index one).

In the case where we have a switched system in its equisingular reduced form with a disturbance $\bar{D}_\sigma$ we have the following corollary.

Corollary 30 *The disturbance decoupling problem for the system 10 with disturbance $\bar{D}_\sigma$ is solvable if and only if*

$$\sum D_{\sigma_r} \subset V(H).$$

where $V(H)$ is the supremal simultaneously invariant subspace corresponding to the standard switched system $(\bar{A}_{i_r}, \bar{B}_{i_r}, \bar{C}_{i_r})$, and D_{i_r} corresponds to the r first rows of D_i and $H = \cap_\sigma \mathrm{Ker}\, C_{\sigma_r}$.

5 Conclusions

In this paper disturbance decoupling problem for switched linear systems has been formulated under a geometrical point of view. Necessary and sufficient conditions in order to obtain solutions of the disturbance decoupling problem with standardizable condition are given.

It is noteworthy that all controllable systems are necessarily standardizable, therefore this condition is not very restrictive.

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References:

[1] A. Ailon, *A solution to the disturbance decoupling problem in singular systems via analogy with state-space systems*, Automatica J. IFAC, 29 (1993), pp. 1541-1545.

[2] G. Basile and G. Marro, “Controlled and conditioned invariants in linear system theory”, Prentice-Hall, 1992.

[3] S. Kr. Choudhary, *Robust Feedback Control Analysis of Magnetic Levitation System*, Wseas Transactions on Systems, **13**, pp. 285-285, 2014.

[4] Doyle, J.C., B.A. Francis and A. Tannenbaum, “Feedback Control Theory”, (1990), Macmillan publishing co.

[5] Feron, E. *Quadratic stablizibility of switched systems via state and output feedback*. MIT Technical report CICSP-468, 113, 1996.

[6] M.I. García-Planas, *Generalized Controllability Subspaces for Time-Invariant Singular Linear Systems*. Proceedings of 5th International Conference on Physics and Control (PhysCon 2011), IPACS Library. (2011).

[7] M.I. García-Planas, *Analysis of the disturbance decoupling problem for singular switched linear systems via feedback and output injection*. Recent Advances in Circuits, Systems, Signal Processing and Communications. pp. 41-45. Wseas Press, 2014.

[8] M.I. García-Planas, *Sensivity and stability of singular systems under proportional and derivative feedback*. Wseas Transactions on Mathematics. **8**, (11), pp. 635-644, 2009.

[9] M.I. García-Planas, S. Tarragona, *Analysis of functional output-controllability of time-invariant Composite linear systems*. Recent advances in systems, control and informatics. pp. 40 -47. Wseas Press, 2013.

[10] I. Gohberg, P. Lancaster, L. Rodman, “Invariant Subspaces of matrices with applications”, Wiley.(1986).

[11] A.A. Julius, A.J. van der Schaft, *The Maximal Controlled Invariant Set of Switched Linear Systems*. Proceedings of the 41st IEEE Conference on Decision and Control. pp. 3175-3179, (2002).

[12] W. L. De Koning, *Digital optimal reduced-order control of pulse-width- modulated switched linear systems*, Automatica **39**, (11),pp. 1997-2003, (2003).

[13] Liberson, D. (2003). Switching in systems and control. Systems & Control: Foundation & Applications, Birkhäuser. (2003)

[14] B. Meng, F.J. Zhang, Controllability conditions for switched linear singular systems, *Dynamics of Continuous, Discrete and Impulsive Systems Series B: Applications & Algorithms*, 14, 2007, pp. 641-657.

[15] A. S. Morse and W. M. Wonham, *Decoupling and pole assignment by dynamic compensation*, SIAM J. Control, 8 (1970), pp. 317-337.

[16] N. Otsuka, *Disturbance decoupling with quadratic stability for switched linear systems*, Systems and Control letters, **59**, pp. 349-352, (2010).

[17] M. Rakowski, *Transfer function approach to disturbance decoupling problem*, Linear Algebra for Control Theory, B. W. P. Van Dooren, ed., IMA Vol. Math. Appl. 62, Springer-Verlag, 1994, pp. 159-176.

[18] Sun, Z., Ge, S.S. “Switched Linear Systems”. London England. Springer, 2005.

[19] E. Yurtseven, W.P.M.H. Heemels, M-K. Camlibel, *Disturbance decoupling of switched linear systems*. 49th IEEE Conference on Decision and Control, pp. 6475-6480, (2010). DOI: 10.1109/CDC.2010.5717376.