Stability Analysis of Second-order RTD-based CNN

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Abstract: -This paper is concerned with the problem of global exponential stability for the second-order RTD-based CNN systems. By homeomorphism mapping, applying the fundamental solution matrix of system, some sufficient conditions that ensure the existence and exponential stability of systems. Finally, two examples are given to illustrate the effectiveness of the results.

Key–Words: -Second-order RTD-based CNN, Homeomorphism mapping, Fundamental solution matrix; Equilibrium point, Exponential stability

1 Introduction

Since the cellular neural networks (CNN) were first proposed by Chua and Yang [1,2], numerous theories and applications of CNN have been reported in recent years. The resonant tunneling diode (RTD), a class of quantum effect devices, is an excellent candidate for both analog and digital microelectronics applications because of its structural simplicity, relative ease of fabrication, inherent high speed and design flexibility. In 2000, Hănggi and Chua [3] simulated the RTD-based CNN

$$\dot{x}_{ij}(t) = -g(x_{ij}(t)) + \sum_{k,l\in N_{ij}} a_{k-i,l-j}x_{kl} + \sum_{k,l\in N_{ij}} b_{k-i,l-j}u_{kl} + z_{ij},$$

where $i, j = 1, 2, \cdots, N$,

$$g(x_{ij}(t)) = \alpha x_{ij}(t) + \gamma_1(|x_{ij}(t) - V_p| - |x_{ij}(t) - V_v|) + \gamma_1(|x_{ij}(t) + V_p| - |x_{ij}(t) + V_v|),$$

which given a circuit implementation of an RTD-based CNN. In 2001, Hănggi and Chua proposed the cellular neural networks based resonant tunneling diode[4]. In 2003, Hănggi and Chua studied the application of the RTD-based CNN, which can be used in image processing and pattern recognition[5]. Recently, the chaotic dynamics of discrete-time RTD-based cellular neural networks have been studied [6,7]. The wave propagation in RTD-based cellular neural networks has been studied [8]. Shi and Shu et al.[9] studied RTDs based cellular neural/nonlinear networks with applications in image processing. In [10] the authors investigated tunneling-based cellular nonlinear network architectures for image processing. In [11-13], Authors a novel neural network architecture is proposed and shown to be useful in approximating the unknown nonlinearities of dynamical systems, and describes an optimized artificial neural network method in order to estimate the settlements of roof, face and walls during tunneling excavation, etc., respectively. Ke and Miao [14-16] investigated the stability of BAM neural networks with inertial term and time delay, pattern memory analysis of second-order neural networks with time-delay and the existence analysis of stationary solutions for RTD-based cellular neural networks, respectively.

In 2001, Itoh, Julian and Chua [17] pointed out that the bistable RTD-based CNN exhibits good performance for a number of interesting image processing applications because of its high-speed processing and high cell density. thus, it is possible that a new generation of low power, high-speed, and large array-size CNNs appears with the introduction of the RTD-based CNN. In that paper, they gave the second-order RTD-based CNN by the following equations

$$\begin{cases} \varepsilon\frac{dx_{ij}(t)}{dt} = a_{00}x_{ij}(t) + u_{ij}(t) - g(x_{ij}(t)) \\ \qquad\qquad + \sum\limits_{k,l\in N_{ij}} b_{k-i,l-j}z_{kl} + I_{ij}, \\ \frac{du_{ij}(t)}{dt} = -x_{ij}(t) - \eta_{ij}u_{ij}(t), \end{cases} \tag{1}$$

where $i, j = 1, 2, \cdots, N$,

$$g(x_{ij}(t)) = \alpha x_{ij}(t) + r(|x_{ij}(t) - V_p| - |x_{ij}(t) - V_v|) - r(|x_{ij}(t) + V_p| - |x_{ij}(t) + V_v|), \tag{2}$$

ε, η_{ij} are positive constants, N_{ij} denotes the γ-neighborhood of cell C_{ij}, a_{00}, b_{kl} and I_{ij} denote the feedback, control and threshold template parameters, respectively. α, r are constants, V_p , V_v are the peak and valley voltages of the RTD for the positive region of x_{ij}, respectively.

Obviously, then systems (1) contains the following two systems:

Bonhoeffer-Van der Pol oscillator with an odd-symmetric three-segment piecewise -linear nonlinearity

$$\begin{cases} \frac{dx(t)}{dt} = \alpha(y(t) - f(x)), \\ \frac{dy(t)}{dt} = -x(t) - \beta y(t), \end{cases}$$

where $f(x) = bx + \frac{1}{2}(a-b)(|x+1| - |x-1|)$.

Van der Pol oscillator with an odd-symmetric three-segment piecewise -linear nonlinearity

$$\begin{cases} \frac{dx(t)}{dt} = \alpha(y(t) - f(x)), \\ \frac{dy(t)}{dt} = -x(t), \end{cases}$$

where $f(x) = bx + \frac{1}{2}(a-b)(|x+1| - |x-1|)$.

The applied background of system (1) are investigated in [17], but there are only numerical simulation. As we all know, the theoretical result on stability of system (1) has not yet seen. In this paper, we will investigate the existence and the global exponential stability of the equilibrium point for systems (1) from mathematical theory.

The initial conditions associated with system (1) are of the form

$$\begin{cases} x_{ij}(t) = \varphi_{ij}(t), \\ u_{ij}(t) = \psi_{ij}(t), \end{cases} i,j = 1,2,\cdots,N, \tag{3}$$

where $-\infty < t \leq 0, \varphi_{ij}(t)$ and $\psi_{ij}(t)(i,j = 1,2,\cdots,N)$ are bounded and continuous functions on $(-\infty, 0]$.

Let $b = \frac{\alpha - a_{00}}{\varepsilon}$,

$$f_\omega(x) = \frac{1}{2}[|x+\omega| - |x-\omega|],$$

$$q_{ij} = \frac{1}{\varepsilon}(\sum_{k,l\in N_{ij}} b_{k-i,l-j} z_{kl} + I_{ij}),$$

then, system (1) can be rewritten as

$$\begin{cases} \frac{dx_{ij}(t)}{dt} = -bx_{ij}(t) + \frac{1}{\varepsilon}u_{ij}(t) \\ \qquad -\frac{2r}{\varepsilon}[f_{V_p}(x_{ij}(t)) + f_{V_v}(x_{ij}(t))] + q_{ij}, \\ \frac{du_{ij}(t)}{dt} = -x_{ij}(t) - \eta_{ij}u_{ij}(t), \end{cases} \tag{4}$$

where $i,j = 1,2,\cdots,N$, or

$$Z'_{ij}(t) = -A_{ij}Z_{ij}(t) + F_{ij}(Z_{ij}(t)) + Q_{ij}. \tag{5}$$

where $Z_{ij}(t) = (x_{ij}(t), u_{ij}(t))^T$,

$$A_{ij} = \begin{bmatrix} b & -\frac{1}{\varepsilon} \\ 1 & \eta_{ij} \end{bmatrix}, Q_{ij} = \begin{bmatrix} q_{ij} \\ 0 \end{bmatrix}.$$

$$F_{ij}(Z_{ij}(t)) = \begin{bmatrix} -\frac{2r}{\varepsilon}[f_{V_p}(x_{ij}(t)) + f_{V_v}(x_{ij}(t))] \\ 0 \end{bmatrix}.$$

(3) can be rewritten as

$$Z_{ij}(t)) = \phi_{ij}(t), -\infty < t \leq 0, \tag{6}$$

where $\phi_{ij}(t) = (\varphi_{ij}(t), \psi_{ij}(t))^T, i,j = 1,2,\cdots,N$.

The paper is arranged as follows: In section 2, we will give some definitions. In section 3, the main results are presented. Finally, in section 4, we give two examples to illustrate our theory.

2 Definition

To obtain our results, we need introduce the following definitions.

Definition 1 *For vector $Y = (y_1, y_2, \cdots, y_n)^T$ and $n \times n$ order matrix $G = (g_{ij})_{n\times n}$, we define norm as following, respectively*

$$\|Y\| = (\sum_{i=1}^{n} |y_i|^2)^{\frac{1}{2}}, \ \ \|G\| = (\sum_{i,j=1}^{n} |g_{ij}|^2)^{\frac{1}{2}}.$$

Definition 2 *Let*
$X^* = (x_{11}^*, x_{12}^*, \cdots, x_{1N}^*, x_{21}^*, x_{22}^*, \cdots, x_{NN}^*)^T$,
$U^* = (u_{11}^*, u_{12}^*, \cdots, u_{1N}^*, u_{21}^*, u_{22}^*, \cdots, u_{NN}^*)^T$,
$Z_{ij}^* = (x_{ij}^*, u_{ij}^*)^T$. *The point $(X^{*T}, U^{*T})^T$ is called an equilibrium point of system (1), if it satisfies the following equations*

$$\begin{cases} -bx_{ij}^* + \frac{1}{\varepsilon}u_{ij}^* - \frac{2r}{\varepsilon}[f_{V_p}(x_{ij}^*) + f_{V_v}(x_{ij}^*)] + q_{ij} = 0, \\ -x_{ij}^* - \eta_{ij}u_{ij}^* = 0, i,j = 1,2,\cdots,N, \end{cases} \tag{7}$$

or

$$-A_{ij}Z_{ij}^* + F_{ij}(Z_{ij}^*) + Q_{ij} = 0. \tag{8}$$

Definition 3 *Let $(X^{*T}, U^{*T})^T$ is an equilibrium point of system (1),*
$(x_{11}, x_{12}, \cdots, x_{1N}, x_{21}, x_{22}, \cdots, x_{NN},$
$u_{11}, u_{12}, \cdots, u_{1N}, u_{21}, u_{22}, \cdots, u_{NN})^T$
is a solution of system (1) with initial value (3). The equilibrium point $(X^{*T}, U^{*T})^T$ of system (1) is said to be global exponentially stable, if there exists constants $K_{ij} > 0, \delta_{ij} \in (0,1), h_{ij} > 0, i,j = 1,2,\ldots,N$, such that

$$|x_{ij}(t) - x_{ij}^*| \leq K_{ij}\delta_{ij}^{[t/h_{ij}]},$$

$$|u_{ij}(t) - u_{ij}^*| \leq K_{ij}\delta_{ij}^{[t/h_{ij}]},$$

where $t > 0, \ i,j = 1,2,\ldots,N, \ [\cdot]$ is the integral function.

Definition 4 *([18]) A map $H : R^n \to R^n$ is a homeomorphism of R^n onto itself if H is continuous and one-to -one and its inverse map H^{-1} is also continuous.*

3 Main results

In this section, we will derive some sufficient conditions which can ensure that the equilibrium point of system (1) uniquely exist and is globally exponential stable.

Lemma 5 *[13] If $H(u) \in C^0$, and it satisfies the following conditions*
1) $H(u)$ is injective on R^n,
2) $\|H(u)\| \to +\infty$, as $\|u\| \to +\infty$,
then $H(u)$ is a homeomorphism of R^n.

Lemma 6 *For function $f_\omega(x) = \frac{1}{2}(|x+\omega| - |x-\omega|)$, we have*
$f_\omega(x)| \leq \omega, \ \ |f_\omega(x_1) - f_\omega(x_2)| \leq |x_1 - x_2|,$ *for $x_1, x_2 \in R$.*

It is clearly correct.

Lemma 7 *For matric*

$$A_{ij} = \begin{bmatrix} b & -\frac{1}{\varepsilon} \\ 1 & \eta_{ij} \end{bmatrix},$$

if $(b-\eta_{ij})^2 - 4/\varepsilon \neq 0 (i,j = 1,2,\cdots,N)$, then

$$\|exp(-A_{ij})t\| \leq M_{ij} e^{-\sigma_{ij} t}, t \geq 0,$$

where $\sigma_{ij} = \frac{(b+\eta_{ij}) - \sqrt{|(b-\eta_{ij})^2 - 4/\varepsilon|}}{2}$,
$M_{ij} = \sqrt{\frac{2(b-\eta_{ij})^2 + 2(1/\varepsilon - 1)^2}{|(b-\eta_{ij})^2 - 4/\varepsilon|}}$.

Proof. We consider the following linear differential equation

$$Z'_{ij}(t) = -A_{ij} Z_{ij}(t). \tag{9}$$

By calculation, we can obtain the eigenvalue of matric $-A_{ij}$,
$\lambda_1 = \frac{1}{2}[-(b+\eta_{ij}) + \sqrt{(b-\eta_{ij})^2 - \frac{4}{\varepsilon}}]$,
$\lambda_2 = \frac{1}{2}[-(b+\eta_{ij}) - \sqrt{(b-\eta_{ij})^2 - \frac{4}{\varepsilon}}]$,
corresponding eigenvector of the λ_1 and λ_2, respectively
$V_1 = (\lambda_1 + \eta_{ij}, -1)^T, \ \ V_2 = (\lambda_2 + \eta_{ij}, -1)^T$.
Thus, we obtain the fundamental solution matrix of system (9) is

$$\phi_{ij}(t) = \begin{bmatrix} (\lambda_1 + \eta_{ij})e^{\lambda_1 t} & (\lambda_2 + \eta_{ij})e^{\lambda_2 t} \\ -e^{\lambda_1 t} & -e^{\lambda_2 t} \end{bmatrix}.$$

By calculation, we can obtain

$$\phi_{ij}^{-1}(0) = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} -1 & -\lambda_2 + \eta_{ij} \\ 1 & \lambda_1 + \eta_{ij} \end{bmatrix}.$$

Since $exp(-A_{ij})t = \phi_{ij}(t)\phi_{ij}^{-1}(0)$, we can obtain $exp(-A_{ij})t =$

$$\frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} (\lambda_2 + \eta_{ij})e^{\lambda_2 t} - (\lambda_1 + \eta_{ij})e^{\lambda_1 t} & (\lambda_1 + \eta_{ij})(\lambda_2 + \eta_{ij})(e^{\lambda_2 t} - e^{\lambda_1 t}) \\ e^{\lambda_1 t} - e^{\lambda_2 t} & (\lambda_2 + \eta_{ij})e^{\lambda_1 t} - (\lambda_1 + \eta_{ij})e^{\lambda_2 t} \end{bmatrix}.$$

We have

$$\begin{aligned} &\|exp(-A_{ij})t\| \\ &= \tfrac{1}{|\lambda_1 - \lambda_2|}\{[(\lambda_2 + \eta_{ij})e^{\lambda_2 t} - (\lambda_1 + \eta_{ij})e^{\lambda_1 t}]^2 \\ &+ [(\lambda_1 + \eta_{ij})(\lambda_2 + \eta_{ij})(e^{\lambda_2 t} - e^{\lambda_1 t})]^2 + (e^{\lambda_1 t} - e^{\lambda_2 t})^2 \\ &+ [(\lambda_2 + \eta_{ij})e^{\lambda_1 t} - (\lambda_1 + \eta_{ij})e^{\lambda_2 t}]^2\}^{1/2} \\ &= \tfrac{1}{|\lambda_1 - \lambda_2|}\{[1 + (\lambda_1 + \eta_{ij})^2 + (\lambda_2 + \eta_{ij})^2 \\ &+ (\lambda_1 + \eta_{ij})^2(\lambda_2 + \eta_{ij})^2](e^{2\lambda_1 t} + e^{2\lambda_2 t}) \\ &- 2[1 + (\lambda_1 + \eta_{ij})(\lambda_2 + \eta_{ij})]^2 e^{(\lambda_1 + \lambda_2)t}\}^{1/2} \\ &\leq \tfrac{1}{|\sqrt{(b-\eta_{ij})^2 - \frac{4}{\varepsilon}}|}\{[(b-\eta_{ij})^2 + (1 - 1/\varepsilon)^2] \\ &\cdot (e^{2\lambda_1 t} + e^{2\lambda_2 t})\}^{1/2}. \end{aligned}$$

If $(b-\eta_{ij})^2 - \frac{4}{\varepsilon} > 0$, then

$$\|exp(-A_{ij})t\| \leq M_{ij} e^{\lambda_1 t}.$$

If $(b-\eta_{ij})^2 - \frac{4}{\varepsilon} < 0$, then

$$\|exp(-A_{ij})t\| \leq M_{ij} e^{-\frac{b+\eta_{ij}}{2} t}.$$

Thus, we have

$$\|exp(-A_{ij})t\| \leq M_{ij} e^{-\sigma_{ij} t}, t \geq 0,$$

where $\sigma_{ij} = \frac{(b+\eta_{ij}) - \sqrt{|(b-\eta_{ij})^2 - 4/\varepsilon|}}{2}$,
$M_{ij} = \sqrt{\frac{2(b-\eta_{ij})^2 + 2(1/\varepsilon - 1)^2}{|(b-\eta_{ij})^2 - 4/\varepsilon|}}$,
for $i,j = 1,2,\cdots,N$.

Lemma 8 *For the system (1), if*

$$(b-\eta_{ij})^2 - 4/\varepsilon \neq 0,$$

$$(b+\eta_{ij}) - \sqrt{|(b-\eta_{ij})^2 - 4/\varepsilon|} > 0,$$

for $i,j = 1,2,\cdots,N$, then we have

$$|x_{ij}(t)| \leq R_{ij}, \ \ |u_{ij}(t)| \leq R_{ij}, \ t \geq 0,$$

for $i,j = 1,2,\cdots,N$, where
$R_{ij} = M_{ij}[\|\phi_{ij}\|\sigma_{ij} + \frac{2|r|}{\varepsilon}(V_p + V_v) + |q_{ij}|]/\sigma_{ij}$,
$\|\phi_{ij}\| = \sup\limits_{-\infty < s \leq 0}\{\|\phi_{ij}(t)\|\}$,
σ_{ij}, M_{ij} are given of Lemma 7.

Proof. From (5) , we have

$$Z_{ij}(t) = e^{-A_{ij}t} Z_{ij}(0)$$

$$+ \int_0^t e^{-A_{ij}(t-s)} [F_{ij}(Z_{ij}(s)) + Q_{ij}] ds, t > 0. \quad (10)$$

By Lemma 7, from (10) we have

$$\|Z_{ij}(t)\| \le M_{ij} \|Z_{ij}(0)\| e^{-\sigma_{ij} t}$$
$$+ M_{ij} \int_0^t e^{-\sigma_{ij}(t-s)} [\|F_{ij}(Z_{ij}(s))\| + \|Q_{ij}\|] ds$$
$$\le M_{ij} \|Z_{ij}(0)\| e^{-\sigma_{ij} t}$$
$$+ M_{ij} [\tfrac{2|r|}{\varepsilon}(V_p + V_v) + |q_{ij}|] \int_0^t e^{-\sigma_{ij}(t-s)} ds$$
$$\le M_{ij} \|Z_{ij}(0)\| e^{-\sigma_{ij} t}$$
$$+ M_{ij} [\tfrac{2|r|}{\varepsilon}(V_p + V_v) + |q_{ij}|] / \sigma_{ij} (1 - e^{-\sigma_{ij} t}).$$

If $(b + \eta_{ij}) - \sqrt{|(b - \eta_{ij})^2 - 4/\varepsilon|} > 0$, i.e., $\sigma_{ij} > 0 (i, j = 1, 2, \cdots, N)$, we have

$$\|Z_{ij}(t)\| \le M_{ij} \|\phi_{ij}\|$$
$$+ M_{ij} [\tfrac{2|r|}{\varepsilon}(V_p + V_v) + |q_{ij}|] / \sigma_{ij} = R_{ij},$$

for $t \ge 0, \ i, j = 1, 2, \cdots, N$.
Thus, we can obtain

$$|x_{ij}(t)| \le R_{ij}, \ \ |u_{ij}(t)| \le R_{ij},$$

for $t \ge 0, \ i, j = 1, 2, \ldots, N$,
where

$$R_{ij} = M_{ij} [\|\phi_{ij}\| \sigma_{ij} + \tfrac{2|r|}{\varepsilon}(V_p + V_v) + |q_{ij}|] / \sigma_{ij},$$
$$\|\phi_{ij}\| = \sup_{-\infty < s \le 0} \{\|\phi_{ij}(t)\|\},$$

σ_{ij}, M_{ij} are given of Lemma 7.

Theorem 9 *For the system (1), if*

$$1 + \frac{8|r|}{\varepsilon} + \frac{1}{\varepsilon} - 2b < 0,$$

$$1 + \frac{1}{\varepsilon} - 2\eta_{ij} < 0,$$

for $i, j = 1, 2, \cdots, N$, *then has a unique equilibrium point for system (1).*

Proof. Let

$$X = (x_{11}, x_{12}, \cdots, x_{1N}, x_{21}, x_{22}, \cdots, x_{NN})^T,$$
$$U = (u_{11}, u_{12}, \cdots, u_{1N}, u_{21}, u_{22}, \cdots, u_{NN})^T,$$
$$H(X, U) = (H_{11}, H_{12}, \cdots H_{1N}, H_{21}, H_{22}, \cdots H_{NN},$$
$$\bar{H}_{11}, \bar{H}_{12}, \cdots, \bar{H}_{1N}, \bar{H}_{21}, \bar{H}_{22}, \cdots, \bar{H}_{NN})^T,$$

where

$$H_{ij}(X, U) = -b x_{ij}(t) + \tfrac{1}{\varepsilon} u_{ij}(t) - \tfrac{2r}{\varepsilon} [f_{V_p}(x_{ij}(t)) + f_{V_v}(x_{ij}(t))] + q_{ij},$$
$$\bar{H}_{ij}(X, U) = -x_{ij}(t) - \eta_{ij} u_{ij}(t),$$

for $i, j = 1, 2 \cdots, N$.

It is known that the solutions of $H(X, U) = 0$ are equilibriums of system (1). If the mapping $H(X, U)$ is a homeomorphism on $R^{2(N \times N)}$, then there exists a unique point (X^*, U^*), such that $H(X^*, U^*) = 0$, i.e., system (1) has a unique equilibrium point (X^*, U^*). In the following, we shall prove that $H(X, U)$ is a homeomorphism.

Firstly, we prove that $H(X, U)$ is an injective mapping on $R^{2(N \times N)}$.

In fact, if there exist

$$(X, U) = (x_{11}, x_{12}, \cdots, x_{1N}, x_{21}, x_{22}, \cdots, x_{NN}, u_{11}, u_{12}, \cdots, u_{1N}, u_{21}, u_{22}, \cdots, u_{NN})^T,$$
$$(\bar{X}, \bar{U}) = (\bar{x}_{11}, \bar{x}_{12}, \cdots, \bar{x}_{1N}, \bar{x}_{21}, \bar{x}_{22}, \cdots, \bar{x}_{NN}, \bar{u}_{11}, \bar{u}_{12}, \cdots, \bar{u}_{1N}, \bar{u}_{21}, \bar{u}_{22}, \cdots, \bar{u}_{NN})^T,$$

such that $H(X, U) = H(\bar{X}, \bar{U})$ for $(X, U) \ne (\bar{X}, \bar{U})$, then

$$-b(x_{ij}(t) - \bar{x}_{ij}(t)) + \tfrac{1}{\varepsilon}(u_{ij}(t) - \bar{u}_{ij}(t)) - \tfrac{2r}{\varepsilon}[f_{V_p}(x_{ij}(t)) - f_{V_p}(\bar{x}_{ij}(t))$$
$$+ f_{V_v}(x_{ij}(t)) - f_{V_v}(\bar{x}_{ij}(t))] = 0. \quad (11)$$

$$-(x_{ij}(t) - \bar{x}_{ij}(t)) - \eta_{ij}(u_{ij}(t) - \bar{u}_{ij}(t)) = 0, \quad (12)$$

for $i, j = 1, 2 \cdots, N$.
From (11) and (12), we obtain

$$-b(x_{ij}(t) - \bar{x}_{ij}(t))^2 + \tfrac{1}{\varepsilon}(u_{ij}(t) - \bar{u}_{ij}(t))(x_{ij}(t) - \bar{x}_{ij}(t))$$
$$- \tfrac{2r}{\varepsilon}(x_{ij}(t) - \bar{x}_{ij}(t))[f_{V_p}(x_{ij}(t)) - f_{V_p}(\bar{x}_{ij}(t))$$
$$+ f_{V_v}(x_{ij}(t)) - f_{V_v}(\bar{x}_{ij}(t))] = 0. \quad (13)$$

$$-(x_{ij}(t) - \bar{x}_{ij}(t))(u_{ij}(t) - \bar{u}_{ij}(t))$$
$$- \eta_{ij}(u_{ij}(t) - \bar{u}_{ij}(t))^2 = 0, \quad (14)$$

for $i, j = 1, 2 \cdots, N$.
From (13) and (14), we obtain

$$-b(x_{ij}(t) - \bar{x}_{ij}(t))^2 + \tfrac{1}{2\varepsilon}[(u_{ij}(t) - \bar{u}_{ij}(t)))^2$$
$$+ (x_{ij}(t) - \bar{x}_{ij}(t))^2 + \frac{4r}{\varepsilon}(x_{ij}(t) - \bar{x}_{ij}(t))^2 \ge 0. \quad (15)$$

$$\tfrac{1}{2}[(x_{ij}(t) - \bar{x}_{ij}(t))^2 + (u_{ij}(t) - \bar{u}_{ij}(t))^2]$$
$$- \eta_{ij}(u_{ij}(t) - \bar{u}_{ij}(t))^2 \ge 0, \quad (16)$$

for $i, j = 1, 2 \cdots, N$.
From (15) and (16), we have

$$(1 + \tfrac{8|r|}{\varepsilon} + \tfrac{1}{\varepsilon} - 2b)(x_{ij}(t) - \bar{x}_{ij}(t))^2$$
$$+ (1 + \frac{1}{\varepsilon} - 2\eta_{ij})(u_{ij}(t) - \bar{u}_{ij}(t))^2 \ge 0, \quad (17)$$

for $i, j = 1, 2, \cdots, N$.
Since

$$1 + \frac{8|r|}{\varepsilon} + \frac{1}{\varepsilon} - 2b < 0,$$

$$1 + \frac{1}{\varepsilon} - 2\eta_{ij} < 0,$$

for $i, j = 1, 2, \cdots, N$.
From (17), it is easy to see that $x_{ij} = \bar{x}_{ij}, u_{ij} = \bar{u}_{ij}$, for $i, j = 1, 2 \cdots, N$, which contradict $(X, U) \ne$

$(\bar{X},\bar{U})$. So $H(X,U)$ is an injective mapping on $R^{2(N\times N)}$.

Secondly, we prove that $\|H(X,U)\| \to +\infty$ as $\|(X,U)\| \to +\infty$. Let

$$\begin{aligned}H^*(X,U) &= H(X,U) - H(0,0)\\ &= (H^*_{11}, H^*_{12}, \cdots, H^*_{1N}, H^*_{21}, H^*_{22}, \cdots, H^*_{NN},\\ &\quad \bar{H}^*_{11}, \bar{H}^*_{12}, \cdots, \bar{H}^*_{1N}, \bar{H}^*_{21}, \bar{H}^*_{22}, \cdots, \bar{H}^*_{NN})^T,\end{aligned}$$

where

$$\begin{aligned}H^*_{ij}(X,U) &= -bx_{ij}(t) + \tfrac{1}{\varepsilon}u_{ij}(t)\\ &\quad -\tfrac{2r}{\varepsilon}[f_{V_p}(x_{ij}(t)) - f_{V_p}(x_{ij}(0))\\ &\quad +f_{V_v}(x_{ij}(t)) - f_{V_v}(x_{ij}(0))]. \end{aligned} \tag{18}$$

$$\bar{H}^*_{ij}(X,U) = -x_{ij}(t) - \eta_{ij}u_{ij}(t), \tag{19}$$

for $\quad i,j = 1,2\cdots,N$.

By (18) and (19), we can find

$$\begin{aligned}&(X,U)^T H^*(X,U)\\ &= \sum_{i,j=1}^{N} x_{ij}H^*_{ij}(X,U) + \sum_{i,j=1}^{N} u_{ij}\bar{H}^*_{ij}(X,U)\\ &= \sum_{i,j=1}^{N} x_{ij}\{-bx_{ij}(t) + \tfrac{1}{\varepsilon}u_{ij}(t)\\ &\quad - \tfrac{2r}{\varepsilon}[f_{V_p}(x_{ij}(t)) - f_{V_p}(x_{ij}(0))\\ &\quad + f_{V_v}(x_{ij}(t)) - f_{V_v}(x_{ij}(0))]\}\\ &\quad + \sum_{i,j=1}^{N} u_{ij}\{-x_{ij}(t) - \eta_{ij}u_{ij}(t)\}\\ &\le \sum_{i,j=1}^{N}\{-bx^2_{ij}(t) + \tfrac{1}{\varepsilon}|u_{ij}(t)||x_{ij}| + \tfrac{4|r|}{\varepsilon}x^2_{ij}(t)]\}\\ &\quad + \sum_{i,j=1}^{N}\{|u_{ij}(t)||x_{ij}| - \eta_{ij}u^2_{ij}(t)\}\end{aligned}$$

$$\le \sum_{i,j=1}^{N}\{(1+\frac{8|r|}{\varepsilon}+\frac{1}{\varepsilon}-2b)x^2_{ij}(t)+(1+\frac{1}{\varepsilon}-2\eta_{ij})u^2_{ij}(t)\}. \tag{20}$$

Using the Schwartz inequality

$$-X^TY \le |X^TY| \le \|X\|\cdot\|Y\|, \tag{21}$$

where $\|X\|, \|Y\|$ are the norms of vectors X and Y, respectively. From (20) and (21), we get

$$\begin{aligned}&\|(X,U)\|\cdot\|H^*(X,U)\|\\ &\ge \sum_{i,j=1}^{N}(2b - 1 - \tfrac{8|r|}{\varepsilon} - \tfrac{1}{\varepsilon})\|x_{ij}(t)\|\\ &\quad + \min_{1\le i,j\le N}\{2\eta_{ij} - 1 - \tfrac{1}{\varepsilon}\}\sum_{i,j=1}^{N}\|u_{ij}(t)\|\\ &\ge M(\|X\|^2 + \|U\|^2) = M\|(X,U)\|^2,\end{aligned}$$

where

$M = \min\{(2b-1-\frac{8|r|}{\varepsilon}-\frac{1}{\varepsilon}), \min_{1\le i,j\le N}\{2\eta_{ij}-1-\frac{1}{\varepsilon}\}\}$.

When $\|(X,U)\| \ne 0$, we have

$$\|H^*(X,U)\| \ge M\|(X,U)\|.$$

Therefore $\|H^*(X,U)\| \to +\infty$ as $\|(X,U)\| \to +\infty$, which implies that $\|H(X,U)\| \to +\infty$ as $\|(X,U)\| \to +\infty$. By Lemma 5, we know that $H(X,U)$ is a homeomorphism on R^{n+m}. Thus , system (1) has a unique equilibrium point.

Theorem 10 *For the system (1), if* $\sigma_{ij} > 4|r|/\varepsilon$,

$$(b-\eta_{ij})^2 - 4/\varepsilon \ne 0,$$

$$1 + \frac{8|r|}{\varepsilon} + \frac{1}{\varepsilon} - 2b < 0,$$

$$41 + \frac{1}{\varepsilon} - 2\eta_{ij} < 0,$$

for $\ i,j = 1,2,\cdots,N$, *then the equilibrium point of system (1) is globally exponentially stably, where* σ_{ij} *is given of Lemma 7.*

Proof. Obviously, the condition of Theorem 9 holds, from Theorem 9 system (1) has a unique equilibrium point. In the following, we will prove the equilibrium point $(X^{*T}, U^{*T})^T$ is globally exponential stable. Let

$$W_{ij}(t) = \begin{bmatrix} x_{ij}(t) - x^*_{ij}\\ u_{ij}(t) - u^*_{ij}\end{bmatrix},$$

$$\bar{F}_{ij}(W_{ij}(t)) = \begin{bmatrix} \overline{f}_{ij}(W_{ij}(t))\\ 0\end{bmatrix}.$$

where

$$\begin{aligned}\overline{f}_{ij}(W_{ij}(t)) &= -\tfrac{2r}{\varepsilon}[(f_{V_p}(x_{ij}(t)) - f_{V_p}(x^*_{ij}))\\ &\quad + (f_{V_v}(x_{ij}(t)) - f_{V_v}(x^*_{ij})].\end{aligned}$$

From (5) and (8) , we have

$$W'_{ij}(t) = -A_{ij}W_{ij}(t) + \bar{F}_{ij}(W_{ij}(t)). \tag{22}$$

From (22), we get

$$\frac{d}{dt}(e^{A_{ij}t}W_{ij}(t)) = e^{A_{ij}t}\bar{F}_{ij}(W_{ij}(t)). \tag{23}$$

Integrating both sides of (23) in $[0,t](t>0)$, we have

$$\begin{aligned}W_{ij}(t)) &= e^{-A_{ij}t}W_{ij}(0))\\ &\quad + \int_0^t e^{-A_{ij}(t-s)}\bar{F}_{ij}(W_{ij}(s))ds. \end{aligned}\tag{24}$$

If $\sigma_{ij} > 4|r|/\varepsilon > 0$, then

$(b+\eta_{ij}) - \sqrt{|(b-\eta_{ij})^2 - 4/\varepsilon|} > 0$,

for $\ i,j = 1,2,\cdots,N$.

By Lemma 7 and Lemma 8, we have

$$|x_{ij}(t) \le R_{ij}, \ \ |u_{ij}(t) \le R_{ij}, t \ge 0,$$

and x^*_{ij}, u^*_{ij} are bounded, then there exist constants $\bar{R}_{ij} > 0$, such that

$\|W_{ij}(t)\|$

$= \sqrt{(x_{ij}(t)-x_{ij}^*)^2+(u_{ij}(t)-u_{ij}^*)^2} \le \bar{R}_{ij},$
for $t \ge 0, \ i,j=1,2,\cdots,N.$
From (24) we have

$$\begin{aligned}&\|W_{ij}(t)\| \le M_{ij}\|W_{ij}(0)\|e^{-\sigma_{ij}t}\\&+M_{ij}\int_0^t e^{-\sigma_{ij}(t-s)}\|\bar{F}_{ij}(W_{ij}(s))\|ds\\&\le M_{ij}\bar{R}_{ij}e^{-\sigma_{ij}t}+\tfrac{4|r|}{\varepsilon}M_{ij}\bar{R}_{ij}\int_0^t e^{-\sigma_{ij}(t-s)}ds\\&\le M_{ij}\bar{R}_{ij}[e^{-\sigma_{ij}t}+\tfrac{4|r|}{\varepsilon\sigma_{ij}}(1-e^{-\sigma_{ij}t})]\end{aligned}$$

$$\le RM_{ij}[e^{-\sigma_{ij}t}+\frac{4|r|}{\varepsilon\sigma_{ij}}(1-e^{-\sigma_{ij}t})], \qquad (25)$$

where $i,j=1,2,\cdots,N, R=\max_{1\le i,j\le N}\{\bar{R}_{ij}\}.$
Let

$$\rho_{ij}(t)=M_{ij}[e^{-\sigma_{ij}t}+\frac{4|r|}{\varepsilon\sigma_{ij}}(1-e^{-\sigma_{ij}t})]. \qquad (26)$$

If $\sigma_{ij}>4|r|/\varepsilon$, then

$$\rho_{ij}'(t)=M_{ij}[-\sigma_{ij}+\frac{4|r|}{\varepsilon}]e^{-\sigma_{ij}t}<0,$$

i.e., $\rho_{ij}(t)$ is monotone decreasing function.
If we select $T_{ij}=\max\{0,\frac{1}{\sigma_{ij}}ln\frac{(\varepsilon\sigma_{ij}-4|r|)M_{ij}}{\varepsilon\sigma_{ij}-4|r|M_{ij}}\}$, then from (26) we have

$$\rho_{ij}(t)=M_{ij}[e^{-\sigma_{ij}t}+\frac{4|r|}{\varepsilon\sigma_{ij}}(1-e^{-\sigma_{ij}t})]<1, \quad (27)$$

for $t>T_{ij}, i,j=1,2,\cdots,N.$
From (27), there exist $h_{ij}>T_{ij}$, such that
$\Gamma_{ij}=M_{ij}[e^{-\sigma_{ij}h_{ij}}+\frac{4|r|}{\varepsilon\sigma_{ij}}(1-e^{-\sigma_{ij}h_{ij}})]<1,$
for $t>h_{ij}, \ i,j=1,2,\cdots,N.$
From (25), we obtain

$$|W_{ij}(t)\| \le R\Gamma_{ij}, \ t\ge h_{ij}, i,j=1,2,\cdots,N. \quad (28)$$

Integrating both sides of (23) in $[h_{ij},t](t>h_{ij})$, by (28), we have

$$\begin{aligned}&\|W_{ij}(t)\| \le M_{ij}R\Gamma_{ij}[e^{-\sigma_{ij}(t-h_{ij})}\\&+\tfrac{4|r|}{\varepsilon\sigma_{ij}}(1-e^{-\sigma_{ij}(t-h_{ij})})], \ \ t\ge h_{ij},\end{aligned}$$

for $i,j=1,2,\cdots,N.$
Thus we can obtain

$$\|W_{ij}(t)\| \le R\Gamma_{ij}^2, \ t\ge 2h_{ij}, i,j=1,2,\cdots,N. \quad (29)$$

Repeat the above process, we obtain

$$\begin{aligned}&\|W_{ij}(t)\| \le M_{ij}R\Gamma_{ij}^{k-1}[e^{-\sigma_{ij}(t-kh_{ij})}\\&+\tfrac{4|r|}{\varepsilon\sigma_{ij}}(1-e^{-\sigma_{ij}(t-kh_{ij})})]\\&\le RM_{ij}\Gamma_{ij}^{k-1}[1+\tfrac{4|r|}{\varepsilon\sigma_{ij}}], \ \ t\ge kh_{ij},\end{aligned}$$

for $i,j=1,2,\cdots,N.$
We have

$$\|W_{ij}(t)\| \le RM_{ij}[1+\frac{4|r|}{\varepsilon\sigma_{ij}}]\Gamma_{ij}^{[t/h_{ij}]}, \ \ t>0, \quad (30)$$

for $i,j=1,2,\cdots,N.$
From (30), we can obtain

$$|x_{ij}(t)-x_{ij}^*| \le K_{ij}\Gamma_{ij}^{[t/h_{ij}]}, \ \ t>0, \ \ i,j=1,2,\cdots,N,$$

$$|u_{ij}(t)-u_{ij}^*| \le K_{ij}\Gamma_{ij}^{[t/h_{ij}]}, \ \ t>0, \ \ i,j=1,2,\cdots,N,$$

where $K_{ij}=RM_{ij}[1+\frac{4|r|}{\varepsilon\sigma_{ij}}], \Gamma_{ij}<1, h_{ij}>0$, by Definition 3, the equilibrium point $(X^{*T},U^{*T})^T$ of system (1) is globally exponentially stable.

Theorem 11 *For the system (1), if* $1+4|r|-b\varepsilon<0, \eta_{ij}>1$*, and*

$$1+\frac{8|r|}{\varepsilon}+\frac{1}{\varepsilon}-2b<0,$$

$$1+\frac{1}{\varepsilon}-2\eta_{ij}<0,$$

for $i,j=1,2,\cdots,N$*, then the unique equilibrium point of system (1) is globally exponentially stable.*

Proof. Obviously, the condition of Theorem 9 holds, by Theorem 9, system (1) exists unique equilibrium point $(X^{*T},U^{*T})^T$. Let
$y_{ij}(t)=x_{ij}(t)-x_{ij}^*, \ \ z_{ij}(t)=u_{ij}(t)-u_{ij}^*,$
$f_{v_p}(y_{ij}(t))=f_{v_p}(x_{ij}(t))-f_{v_p}(x_{ij}^*),$
$f_{v_v}(y_{ij}(t))=f_{v_v}(x_{ij}(t))-f_{v_v}(x_{ij}^*).$
From (4) and (7), we derive

$$\begin{cases}\frac{dy_{ij}(t)}{dt}=-by_{ij}(t)+\frac{1}{\varepsilon}z_{ij}(t)\\ \qquad -\frac{2r}{\varepsilon}[f_{v_p}(y_{ij}(t))+f_{v_v}(y_{ij}(t))],\\ \frac{dz_{ij}(t)}{dt}=-y_{ij}(t)-\eta_{ij}z_{ij}(t),\end{cases} \quad (31)$$

for $i,j=1,2,\cdots,N.$
From (31), by Lemma 6 we can obtain

$$\begin{aligned}\frac{d|y_{ij}(t)|}{dt}&=sgn(y_{ij}(t))\{-by_{ij}(t)+\tfrac{1}{\varepsilon}z_{ij}(t)\\&-\tfrac{2r}{\varepsilon}[f_{v_p}(y_{ij}(t))+f_{v_v}(y_{ij}(t))]\}\\&\le -b|y_{ij}(t)|+\tfrac{1}{\varepsilon}|z_{ij}(t)|+\tfrac{4|r|}{\varepsilon}|y_{ij}(t)|\end{aligned}$$

$$=(\frac{4|r|}{\varepsilon}-b)|y_{ij}(t)|+\frac{1}{\varepsilon}|z_{ij}(t)|, \quad (32)$$

$$\frac{d|z_{ij}(t)|}{dt}=sgn(z_{ij}(t))\{-y_{ij}(t)-\eta_{ij}z_{ij}(t)\}$$

$$\le |y_{ij}(t)|-\eta_{ij}|z_{ij}(t)|, \quad (33)$$

for $i, j = 1, 2, \cdots, N$.

From (32) and (33), we can obtain

$$|y_{ij}(t)| \le e^{(4|r|/\varepsilon - b)t}|y_{ij}(0)| + \frac{1}{\varepsilon}\int_0^t e^{(4|r|/\varepsilon - b)(t-s)}|z_{ij}(s)|ds, \quad (34)$$

$$|z_{ij}(t)| \le e^{-\eta_{ij}t}|z_{ij}(0)| + \int_0^t e^{-\eta_{ij}(t-s)}|y_{ij}(s)|ds, \quad (35)$$

for $i, j = 1, 2, \cdots, N$.

We considering the functions $g(\xi)$ and $G_{ij}(\xi)$, which are given by

$$g(\xi) = \xi\varepsilon + 1 + 4|r| - b\varepsilon, \ \ G_{ij}(\xi) = \xi + 1 - \eta_{ij},$$

for $i, j = 1, 2, \cdots, N$.
By the condition of Theorem, obviously

$$\frac{dg(\xi)}{d\xi} > 0, \ \lim_{\xi \to +\infty} g(\xi) = +\infty, \ \ g(0) < 0,$$

$$\frac{dG_{ij}(\xi)}{d\xi} > 0, \ \lim_{\xi \to +\infty} G_{ij}(\xi) = +\infty, \ \ G_{ij}(0) < 0,$$

for $i, j = 1, 2, \cdots, N$.
Therefore, there exist constants $\sigma_1, \sigma_{ij} \in (0, +\infty)$, such that

$$g(\sigma_1) = 0, \quad G_{ij}(\sigma_{ij}) = 0, \ \ i, j = 1, 2, \ldots, N.$$

We choose $\bar{\xi} = min\{\sigma_1, \sigma_{11}, \sigma_{12}, \ldots, \sigma_{NN}\}$, then $\bar{\xi} > 0$, when $0 < \sigma < \bar{\xi}$, we have

$$\sigma\varepsilon + 1 + 4|r| - b\varepsilon < 0, \quad \sigma + 1 - \eta_{ij} < 0, \quad (36)$$

for $i, j = 1, 2, \cdots, N$.

Since the initial values $\varphi_{ij}(s), \psi_{ij}(s)$ are bounded and continuous functions, then exist $N_1, N_2 > 0$, such that

$$|\varphi_{ij}(t)| \le N_1, \ \ |\psi_{ij}(t)| \le N_2,$$

for $i, j = 1, 2, \cdots, N, t \in (-\infty, 0]$.

Let $L = \max\{N_1, N_2\}$, we will show that for any sufficiently small constant $\delta > 0$,

$$|y_{ij}(t)| < (L+\delta)e^{-\sigma t}, \ \ |z_{ij}(t)| < (L+\delta)e^{-\sigma t}, \quad (37)$$

where $t \ge 0, \ i, j = 1, 2, \cdots, N, 0 < \sigma < \bar{\xi}$.

Considering the method of contrary. If (37) does not hold, there exists some $k, l \in \{1, 2, \cdots, N\}$ and $t_1 \ge 0$, such that

$$\begin{cases} |y_{kl}(t_1)| = (L+\delta)e^{-\sigma t_1}, \\ |y_{ij}(t)| < (L+\delta)e^{-\sigma t}, \ t \in [0, t_1), \\ |z_{ij}(t)| < (L+\delta)e^{-\sigma t}, \ t \in [0, t_1], \end{cases} \quad (38)$$

or

$$\begin{cases} |y_{kl}(t_1)| = |z_{kl}(t_1)| = (L+\delta)e^{-\sigma t_1}, \\ |y_{ij}(t)| < (L+\delta)e^{-\sigma t}, \ t \in [0, t_1), \\ |z_{ij}(t)| < (L+\delta)e^{-\sigma t}, \ t \in [0, t_1), \end{cases} \quad (39)$$

or

$$\begin{cases} |z_{kl}(t_1)| = (L+\delta)e^{-\sigma t_1}, \\ |y_{ij}(t)| < (L+\delta)e^{-\sigma t}, \ t \in [0, t_1], \\ |z_{ij}(t)| < (L+\delta)e^{-\sigma t}, \ t \in [0, t_1), \end{cases} \quad (40)$$

for $i, j = 1, 2, \cdots, N$.

From (34) and (38)(or 39), we obtain

$$\begin{aligned} |y_{ij}(t_1)| &= (L+\delta)e^{-\sigma t_1} \le e^{(4|r|/\varepsilon - b)t_1}|y_{ij}(0)| \\ &\quad + \tfrac{1}{\varepsilon}\textstyle\int_0^{t_1} e^{(4|r|/\varepsilon - b)(t_1 - s)}|z_{ij}(s)|ds \\ &\le (L+\delta)[e^{(4|r|/\varepsilon - b)t_1} \\ &\quad + \tfrac{1}{\varepsilon}\textstyle\int_0^{t_1} e^{(4|r|/\varepsilon - b)(t_1 - s) - \sigma s}ds] \\ &\le (L+\delta)[e^{(4|r|/\varepsilon - b)t_1} \\ &\quad + \tfrac{1}{\varepsilon}e^{(4|r|/\varepsilon - b)t_1}\tfrac{1}{b - 4|r|/\varepsilon - \sigma}(e^{(b - 4|r|/\varepsilon - \sigma)t_1} - 1)] \\ &= (L+\delta)[e^{(4|r|/\varepsilon - b)t_1} \\ &\quad + \frac{1}{\varepsilon(b - 4|r|/\varepsilon - \sigma)}(e^{-\sigma t_1} - e^{(4|r|/\varepsilon - b)t_1})], \end{aligned} \quad (41)$$

for $\imath, j = 1, 2, \cdots, N$.

By (36), we obtain $\frac{1}{\varepsilon(b - 4|r|/\varepsilon - \sigma)} < 1$, from (41), we have

$$L + \delta < L + \delta,$$

which is a contradiction.

By (35) and (40), we obtain

$$\begin{aligned} |z_{ij}(t_1)| &= (L+\delta)e^{-\sigma t_1} \\ &\le e^{-\eta_{ij}t_1}|z_{ij}(0)| + \textstyle\int_0^{t_1} e^{-\eta_{ij}(t_1 - s)}|y_{ij}(s)|ds \\ &\le (L+\delta)[e^{-\eta_{ij}t_1} + \textstyle\int_0^{t_1} e^{-\eta_{ij}(t_1 - s) - \sigma s}ds] \\ &\le (L+\delta)[e^{-\eta_{ij}t_1} + \tfrac{e^{-\eta_{ij}t_1}}{\eta_{ij} - \sigma}(e^{(\eta_{ij} - \sigma)t_1} - 1)] \\ &= (L+\delta)[e^{-\eta_{ij}t_1} + \frac{1}{\eta_{ij} - \sigma}(e^{-\sigma t_1} - e^{-\eta_{ij}t_1})], \end{aligned} \quad (42)$$

for $i, j = 1, 2, \cdots, N$.

By (36), we obtain $\frac{1}{\eta_{ij} - \sigma} < 1$, from (42), we have

$$L + \delta < L + \delta,$$

which is a contradiction.

Thus (37) holds, let $\delta \to 0$, we have

$$|y_{ij}(t) \le Le^{-\sigma t}, \ \ |z_{ij}(t)| \le Le^{-\sigma t}, \quad (43)$$

where $t \ge 0, \sigma > 0, i, j = 1, 2, \cdots, N$.
From (43), there exist constants $M > 0$, $\delta = e^{-1} \in (0, 1)$, and $\sigma > 0$, such that

$$|x_{ij}(t) - x^*_{ij}| \le M\delta^{\sigma t}, \ \ t > 0,$$

$$|u_{ij}(t) - u^*_{ij}| \le M\delta^{\sigma t}, \ \ t > 0,$$

for $i, j = 1, 2, \cdots, N$.
This implies that the equilibrium of system (1) is globally exponentially stable.

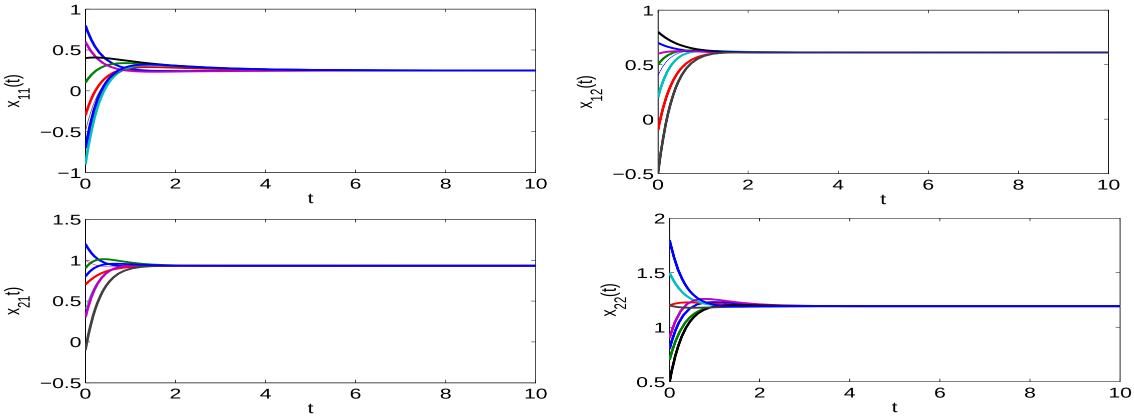

Fig.1. Transient response of state variables $x_{11}(t), x_{12}(t), x_{21}(t), x_{22}(t)$ of Example 4.1

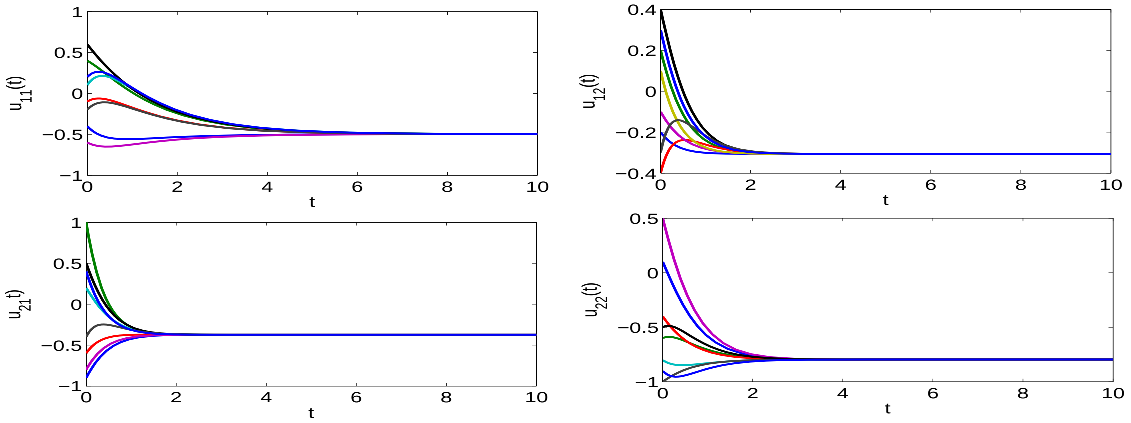

Fig.2. Transient response of state variables $u_{11}(t), u_{12}(t), u_{21}(t), u_{22}(t)$ of Example 4.1

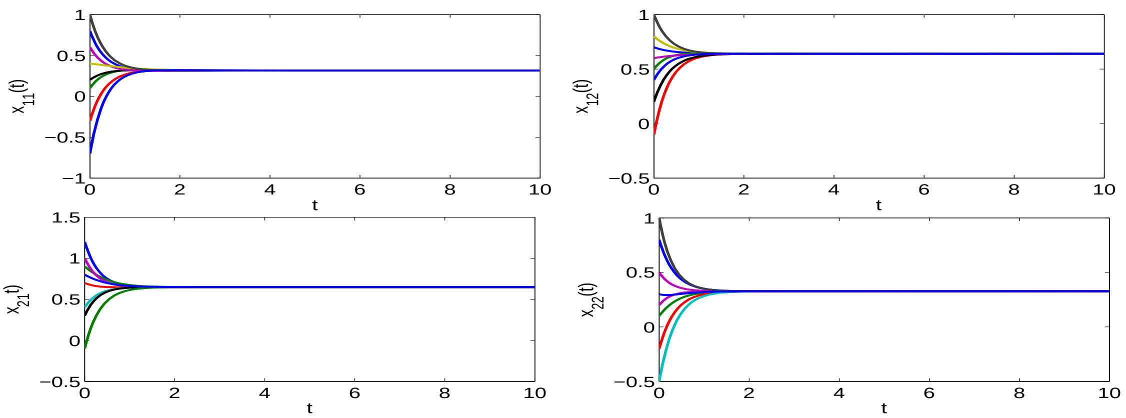

Fig.3. Transient response of state variables $x_{11}(t), x_{12}(t), x_{21}(t), x_{22}(t)$ of Example 4.2

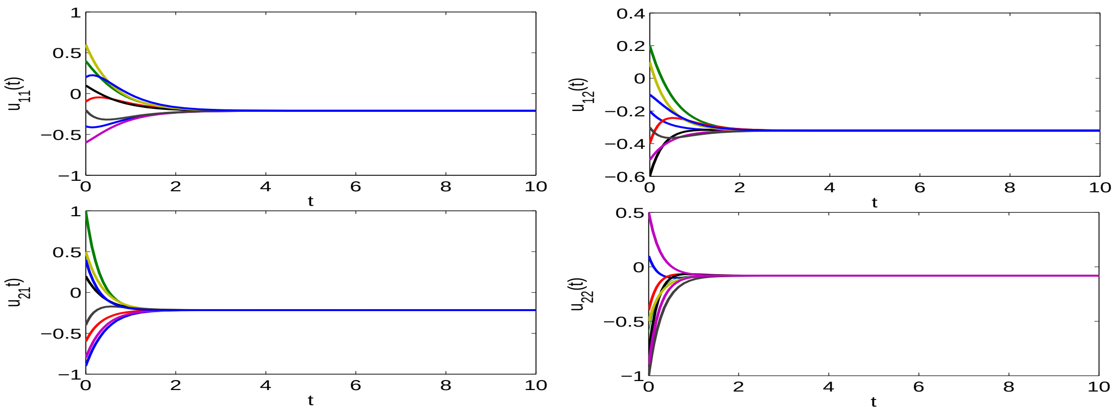

Fig.4. Transient response of state variables $u_{11}(t), u_{12}(t), u_{21}(t), u_{22}(t)$ of Example 4.2

4 Numerical simulation

In this Section, we give two examples to demonstrate our results.

Examples. Consider the following second-order RTD-based CNN systems($N = 2$)

$$
\begin{cases}
\varepsilon \frac{dx_{11}(t)}{dt} = a_{00}x_{11}(t) + u_{11}(t) - g(x_{11}(t)) + q_{11}, \\
\varepsilon \frac{dx_{12}(t)}{dt} = a_{00}x_{12}(t) + u_{12}(t) - g(x_{12}(t)) + q_{12}, \\
\varepsilon \frac{dx_{21}(t)}{dt} = a_{00}x_{21}(t) + u_{21}(t) - g(x_{21}(t)) + q_{21}, \\
\varepsilon \frac{dx_{22}(t)}{dt} = a_{00}x_{22}(t) + u_{22}(t) - g(x_{22}(t)) + q_{22}, \\
\frac{du_{11}(t)}{dt} = -x_{11}(t) - \eta_{11}u_{11}(t), \\
\frac{du_{12}(t)}{dt} = -x_{12}(t) - \eta_{12}u_{12}(t), \\
\frac{du_{21}(t)}{dt} = -x_{21}(t) - \eta_{21}u_{21}(t), \\
\frac{du_{22}(t)}{dt} = -x_{22}(t) - \eta_{22}u_{22}(t),
\end{cases} \tag{44}
$$

where

$$
\begin{aligned}
g(x_{ij}(t)) = {} & \alpha x_{ij}(t) + \gamma(|x_{ij}(t) - V_p| - |x_{ij}(t) - V_v|) \\
& - \gamma(|x_{ij}(t) + V_p| - |x_{ij}(t) + V_v|),
\end{aligned}
$$

$$q_{ij} = \sum_{k,l \in N_{ij}} b_{k-i,l-j} z_{kl} + I_{ij}, i, j = 1, 2.$$

Example 4.1 For system (44), let $\varepsilon = 2$, $a_{00} = 3$, $\alpha = 9$, $\gamma = 0.01$, $q_{11} = 1$, $q_{12} = 2$, $q_{21} = 3$, $q_{22} = 4$, $\eta_{11} = 0.5$, $\eta_{12} = 2$, $\eta_{21} = \frac{5}{2}$, $\eta_{22} = \frac{3}{2}$, $V_p = 1, V_v = 2$.

For numerical simulation, the following eight cases are given with the initial state

$$
\begin{aligned}
& [\varphi_{11}(0), \varphi_{12}(0), \varphi_{21}(0), \varphi_{22}(0), \\
& \quad \psi_{11}(0), \psi_{12}(0), \psi_{21}(0), \psi_{22}(0)] = \\
& [0.8; 0.7; 1.2; 0.8; -0.4; -0.2; -0.9; 0.1]; \\
& [0.1; 0.5; 0.9; 0.7; 0.4; 0.2; 1; -0.6]; \\
& [-0.3; -0.1; 0.7; 1.2; -0.1; -0.4; -0.6; -0.4]; \\
& [-0.9; 0.2; 0.4; 1.5; 0.1; 0.4; 0.2; -0.8]; \\
& [0.6; 0.6; 1; 0.9; -0.6; -0.1; -0.8; 0.5]; \\
& [0.4; 0.8; 0.3; 0.5; 0.6; 0.1; 0.5; -0.5]; \\
& [-0.5; -0.5; -0.1; 1.2; -0.2; -0.3; -0.4; -1]; \\
& [-0.7; 0.4; 0.8; 1.8; 0.2; 0.3; 0.4; -0.9].
\end{aligned}
$$

Figs.1-Figs.2 depict the time responses of state variables of $x_{11}(t), x_{12}(t), x_{21}(t), x_{22}(t), u_{11}(t), u_{12}(t), u_{21}(t), u_{22}(t)$ of system in example 4.1, respectively.

On the other hand, we have the following results by simple calculation, we can obtain

$$b = \frac{\alpha - a_{00}}{\varepsilon} = 3, \quad t > 0 (b - \eta_{ij})^2 - 4/\varepsilon \neq 0,$$

$$\sigma_{ij} = \frac{(b + \eta_{ij}) - \sqrt{|(b - \eta_{ij})^2 - 4/\varepsilon|}}{2} > 4|r|/\varepsilon,$$

$$1 + \frac{8|r|}{\varepsilon} + \frac{1}{\varepsilon} - 2b < 0, \quad 1 + \frac{1}{\varepsilon} - 2\eta_{ij} < 0,$$

for $i, j = 1, 2$, thus the conditions of Theorem 10 hold.

From (44), we can get the equation of the equilibriums

$$
\begin{cases}
-3x_{11} + \frac{1}{2}u_{11} - \frac{1}{100}[f_{V_p}(x_{11}) + f_{V_v}(x_{11})] + 1 = 0, \\
-3x_{12} + \frac{1}{2}u_{12} - \frac{1}{100}[f_{V_p}(x_{12}) + f_{V_v}(x_{12})] + 2 = 0, \\
-3x_{21} + \frac{1}{2}u_{21} - \frac{1}{100}[f_{V_p}(x_{21}) + f_{V_v}(x_{21})] + 3 = 0, \\
-3x_{22} + \frac{1}{2}u_{22} - \frac{1}{100}[f_{V_p}(x_{22}) + f_{V_v}(x_{22})] + 4 = 0, \\
x_{11} + 0.5u_{11} = 0, \\
x_{12} + 2u_{12} = 0, \\
x_{21} + \frac{5}{2}u_{21} = 0, \\
x_{22} + \frac{3}{2}u_{22} = 0,
\end{cases} \tag{45}
$$

where

$$f_{V_p}(x_{ij}) = \frac{1}{2}[|x_{ij} + 1| - |x_{ij} - 1|], i, j = 1, 2,$$

$$f_{V_v}(x_{ij}) = \frac{1}{2}[|x_{ij} + 2| - |x_{ij} - 2|], i, j = 1, 2.$$

Obviously, from (45), we can solve the unique equilibrium point

$$
\begin{aligned}
Z^* &= (x_{11}^*, x_{12}^*, x_{21}^*, x_{22}^*, u_{11}^*, u_{12}^*, u_{21}^*, u_{22}^*)^T \\
&= (\tfrac{25}{88}, \tfrac{200}{327}, \tfrac{150}{161}, \tfrac{1191}{1000}, -\tfrac{25}{44}, -\tfrac{100}{327}, -\tfrac{60}{161}, -\tfrac{1191}{1500})^T.
\end{aligned}
$$

Evidently, this consequence is coincident with the results of numerical simulation.

Example 4.2 For system (44), let $\varepsilon = 4$, $a_{00} =$

$1,\ \alpha = 9,\ r = 1,\ q_{11} = 1,\ q_{12} = 2,\ q_{21} = 2,\ q_{22} = 1,\ \eta_{11} = \frac{3}{2},\ \eta_{12} = 2,\ \eta_{21} = 3,\ \eta_{22} = 4,\ V_p = 1, V_v = 2.$

For numerical simulation, the following eight cases are given with the initial state

$[\varphi_{11}(0), \varphi_{12}(0), \varphi_{21}(0), \varphi_{22}(0),$
$\psi_{11}(0), \psi_{12}(0), \psi_{21}(0), \psi_{22}(0)] =$
$[0.8; 0.7; 1.2; 0.8; -0.4; -0.2; -0.9; 0.1];$
$[0.1; 0.5; 0.9; 0.1; 0.4; 0.2; 1; -0.6];$
$[-0.3; -0.1; 0.7; -0.2; -0.1; -0.4; -0.6; -0.4];$
$[0.2; 0.2; 0.4; -0.5; 0.1; -0.6; 0.2; -0.8];$
$[0.6; 0.6; 1; 0.2; -0.6; -0.5; -0.8; 0.5];$
$[0.4; 0.8; 0.3; 0.5; 0.6; 0.1; 0.5; -0.5];$
$[1; 1; -0.1; 1; -0.2; -0.3; -0.4; -1];$
$[-0.7; 0.4; 0.8; 0.3; 0.2; -0.1; 0.4; -0.9].$

Figs.3-Figs.4 depict the time responses of state variables of $x_{11}(t), x_{12}(t), x_{21}(t), x_{22}(t), u_{11}(t), u_{12}(t), u_{21}(t), u_{22}(t)$ of system in example 4.2, respectively.

On the other hand, we have the following results by simple calculation, we can obtain

$$b = \frac{\alpha - a_{00}}{\varepsilon} = 2,\ \ 1 + 4|r| - b\varepsilon < 0,\ \ \eta_{ij} > 1,$$

$$1 + \frac{8|r|}{\varepsilon} + \frac{1}{\varepsilon} - 2b < 0,\ \ 1 + \frac{1}{\varepsilon} - 2\eta_{ij} < 0,\ \ i, j = 1, 2.$$

Thus the conditions of Theorem 11 hold.

From (44), we can get the equation of the equilibriums

$$\begin{cases} -2x_{11} + \frac{1}{4}u_{11} - \frac{1}{2}[f_{V_p}(x_{11}) + f_{V_v}(x_{11})] + 1 = 0, \\ -2x_{12} + \frac{1}{4}u_{12} - \frac{1}{2}[f_{V_p}(x_{12}) + f_{V_v}(x_{12})] + 2 = 0, \\ -2x_{21} + \frac{1}{4}u_{21} - \frac{1}{2}[f_{V_p}(x_{21}) + f_{V_v}(x_{21})] + 2 = 0, \\ -2x_{22} + \frac{1}{4}u_{22} - \frac{1}{2}[f_{V_p}(x_{22}) + f_{V_v}(x_{22})] + 1 = 0, \\ x_{11} + \frac{3}{2}u_{11} = 0, \\ x_{12} + 2u_{12} = 0, \\ x_{21} + 3u_{21} = 0, \\ x_{22} + 4u_{22} = 0, \end{cases} \tag{46}$$

where

$$f_{V_p}(x_{ij}) = \frac{1}{2}[|x_{ij} + 1| - |x_{ij} - 1|], i, j = 1, 2,$$

$$f_{V_v}(x_{ij}) = \frac{1}{2}[|x_{ij} + 2| - |x_{ij} - 2|], i, j = 1, 2.$$

Obviously, from (46), we can solve the unique equilibrium point

$Z^* = (x_{11}^*, x_{12}^*, x_{21}^*, x_{22}^*, u_{11}^*, u_{12}^*, u_{21}^*, u_{22}^*)^T$
$= (\frac{2}{7}, \frac{16}{25}, \frac{24}{37}, \frac{16}{49}, -\frac{4}{21}, -\frac{8}{25}, -\frac{8}{37}, -\frac{4}{49})^T.$

Evidently, this consequence is coincident with the results of numerical simulation.

Remark 12 *Example 4.1 and Example 4.2 showed their equilibrium point are exponential stable. In Example 4.1, there is* $\eta_{11} = 0.5 < 1$. *But this condition isn't satisfied Theorem 11. While in Example 4.2, there is* $(b - \eta_{21})^2 - 4/\varepsilon = 0$. *This condition isn't satisfied Theorem 10. It showed that theorem 10 and theorem 11 have different applications.*

In fact, the parameter η_{ij} *in Theorem 10 must be satisfy* $(b - \eta_{ij})^2 - 4/\varepsilon \neq 0$, *allow* $\eta_{ij} \leq 1$. *But for Theorem 11 it is required to satisfy* $\eta_{ij} > 1$, *allow* $(b - \eta_{ij})^2 - 4/\varepsilon = 0$. *Therefore, Theorems 10 and Theorem 11 can solve different problems.*

5 Conclusions

In this paper, we give three theorems to ensure the existence and the exponential stability of the equilibrium point for second-order RTD-based CNN system. Novel existence and stability conditions are stated with simple algebraic forms and their verification and applications are straightforward and convenient. Especially, we give different conditions in Theorems 10 and Theorems 11 to ensure the exponential stability of the equilibrium point, which have different advantages in different problems and applications. Finally two examples illustrate the effectiveness in different conditions.

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