Methods to study the stability of dynamical systems. Applications

DUMITRU BĂLĂ
Department of Applied Mathematics
University of Craiova- University Centre of Drobeta Turnu Severin
1, Călugăreni Street, Drobeta-Turnu Severin
ROMANIA
dumitru_bala@yahoo.com

Abstract: - In this paper we propose to study the stability of dynamical systems described by a differential equation or by a system of differential equations. These dynamic systems have applications in technical and economy. Dynamic systems are taken from the specialty literature, but the study of stability has, by the following methods, original elements. Stability study has been my field of study in the last 15 years.

Key-Words: - dynamic system, stability, Leapunov function, instability, Lagrangian, Hamiltonian

1 Introduction

The study of economic or technical processes described by a differential equation or a system of differential equations is difficult. Most times we can not find the exact solution. We propose then to see if the dynamic system is stable or not.

Further we mention the notions of: stable solution, solution asymptotically stable, positive defined function, negative defined function and Leapunov function. These concepts are known in the specialty literature and can be deepened in [3], [5], [6] and [8]. We consider the differential system

$\dot{x} = f(t, x)$

(1)

where $f : \Omega \rightarrow R^n$ is a function verifying the following conditions:

i) f is continuous in the (t, x)

ii) f is locally Lipschitz the x on the set $\Omega = \{(t, x) \in R^+ \text{ x } R^n ; \|x\| \prec a\}$

From the theorems of existence, uniqueness and extension results that, for any point $(t_o, x_o) \in \Omega$, the system (1) with initial condition $x(t_o) = x_o$ admits a unique solution $x = x(t, t_o, x_o)$, defined on a maximal interval $[t_o, T)$.

Let $x = \phi(t)$ a solution of system (1) semi-axis defined by $[t_o, +\infty]$.

Definition 1. Solution ϕ is called stable if for any $\varepsilon > 0$, there exists $\delta(\varepsilon, t_o) > 0$, so that $x(t,t_o,x_o)$ is defined on $[t_o, +\infty]$ and the inequality

$\|x(t, t_o, x_o) - \varphi(t)\| \leq \varepsilon \quad \forall t \in [t_o, +\infty]$

(2)

For every $\|x_o\| < a$ checking the condition $\|x_o - \varphi(t_o)\| \leq \delta(\varepsilon, t_o)$

Definition 2. Solution ϕ is called asymptotic stable if it is stable and there is $\mu(t_o) > 0$, so that

$\lim_{t \to \infty} \|x(t, t_{o,} x_o) - \varphi(t)\| = 0$

(3)

as soon as $\|x_o - \varphi(t_o)\| \leq \mu(t_o)$

Definition 3. Real function V defined on $\Omega = \{(t, x) \in R^+ \times R^n ; \|x\| < a \leq \infty\}$ is called definite positive if there is a monotonic not decreasing and continuous function $w : \overline{R}^+ \rightarrow \overline{R}^+$ such that w(0) = 0; w(r) >0 for $r \neq 0$ and $V(t,x) \geq w(\|x\|) \forall (t, x) \in \Omega$.

Definition 4. We say that V is negative definite if -V is positive definite.

Definition 5. Function V is called Leapunov function associated system $\dot{x} = f(t, x)$ (non-autonomous system) if it satisfies the following conditions:

j) V is continuous together with its derivatives of the first order (in t and x) on the field Ω.

jj) V is positive definite on Ω and V(t,o) = o $\forall t \geq 0$

jjj) $\frac{\partial V}{\partial t}(t,x)+(grad_x V(t,x), f(t,x)) \leq 0 \ \forall (t,x) \in \Omega$

We denoted by (,) the scalar product in the space R^n and by $grad_x V$ the gradient function V as a function of x, ie the vector $(\frac{\partial V}{\partial x_1},...,\frac{\partial V}{\partial x_n})$.

We also present three theorems of which the most used is Leapunov stability theorem for autonomous systems. These three theorems we encounter in [3], [5], [6] and [8].

Leapunov's stability theorem for autonomous systems. We consider the differential system $\dot{x} = f(t,x)$ where $f:\Omega \to R^n$ R is a function verifying the following conditions:

i) f is continuous in the (t, x)

ii) f f is locally Lipschitz in x on the set $\Omega = \{(t,x) \in R^+ \text{ x } R^n; \|x\| < a\}$

If there is a Leapunov function V(t,x) for this system, the trivial solution is stable.

Assume further that the function

$w(t,x) = \frac{\partial V}{\partial t}(t,x) + (grad_x V(t,x), f(t,x))$ is negative definited on Ω and the inequality occurs: $V(t,x) \leq \mu(\|x\|), \ \forall (t,x) \in \Omega$ where μ is a continuous and positive function, which cancels itself in origin. Then the trivial solution of the system $\dot{x} = f(t,x)$ is asymptotically stable.

Theorem 1. If $\Omega = R^+ \text{ x } R^n$ and the function w from the condition $V(t,x) \geq w(\|x\|) \quad \forall (t,x) \in \Omega$ has the property $\lim_{r\to\infty}$ w(r)=+ ∞ $(w:\overline{R}^+ \to \overline{R}^+; w(o)=0; w(r)>0$ for r ≠ 0; w monotonically decreasing), then the assumptions of Theorem Leapunov stability of autonomous systems, the trivial solution of the system $\dot{x} = f(t,x)$ is globally asymptotically stable.

Definition 6. Function V: D → R is called Lyapunov function on D associated autonomous differential system $\dot{x} = f(x)$ if it fulfills the following conditions:

e)V ∈ C' (D) and V(o) = 0

ee) V(x) >0; $\forall x \neq 0$

eee) (grade V(x), f(x)) ≤ 0∀ x ∈ D

We add that $D = \{x \in R^n, \|x\| \prec a \leq +\infty\}$; $f: D \to R^n$; f locally Lipschitz; (,)Is the scalar product in R^n.

$grad_x$ V is the vector $\left(\frac{\partial V}{\partial x_1},...,\frac{\partial V}{\partial x_n}\right)$

Leapunov's stability theorem for autonomous systems. If there is an associated Leapunov autonomous differential system $\dot{x} = f(x)$ then the trivial solution of the system $\dot{x} = f(x)$ is stable. In addition, if the condition is checked

(grad V(x), f(x)) <0 $\quad \forall x \neq 0$

then the null solution is asymptotically stable.

If $\lim_{\|x\|\to\infty} V(x) = +\infty$ then the trivial solution is globally asymptotically stable.

Example 1. The study of the stability of the the neoclassical model of the economical increase

The economic model is described by the differential equation:

$\dot{r} = sr^{\alpha} - (n+\delta)r$

(4)

Sizes that appear are:

$r = \frac{K}{L}$, K is the capital and L is the work force

de $s = \frac{S}{Y}$, S are the savings and Y are incomes

α is a coefficient between 0 and 1 that appears considering that the production function is Cobb-Douglas function $Y = K^{\alpha}L^{1-\alpha}, 0 \langle \alpha \langle 1$

$\dot{L} = nL$, n is the rate of increase in the work force

Technical progress occurs by introducing coefficient δ. It will be calculated as the ratio of capital and stock per unit time.

$\dot{r} + (n+\delta)r = sr^{\alpha}$

(5)

By substitution $y = r^{1-\alpha}$ we obtain the differential equation:

$$\dot{z} + (n+\delta)(1-\alpha)z = s(1-\alpha) \tag{6}$$

The solution of this equation is:

$$z = Ce^{-(n+\delta)(1-\alpha)t} + \frac{s}{n+\delta} \tag{7}$$

For $t \to \infty$, $z(t) \to C + \dfrac{s}{n+\delta}$

Knowing that n, s can be positive constants and C is determined from the initial conditions is observed that the studied system is stable. This is an example of a dynamic system for the stability study is done using the definition of stability in the qualitative theory of differential equations as specified in [5].In conclusion, we note that the capital work force ratio is stable over time and depends on the increase rate of work force, technical progress, savings and income. Stability to the neoclassical model of economic increase has been studied in [10], but in this paper there are drawn some conclusions and made statements on the method of study. The practical application of the definition in the study of stability is very difficult because we can not find the exact solution.

Example 2

The model of urban dynamic of type Lorenz is described by the system of differential equations (8).

$$\begin{cases} \dfrac{dx_1}{dt}_1 = a_1(a_2x_2 - a_3x_1) \\ \dfrac{dx_2}{dt} = c_1(c_2x_1 - c_3x_2) - c_4x_1x_3 \\ \dfrac{dx_3}{dt} = d_1x_1x_2 - d_2x_3 \end{cases} \tag{8}$$

The economical measures that appear are: x_1 - production of the urban system, x_2 - the number of residents, x_3 - the rent land and a_i, c_i, d_i are positive parameters.

We solve the equation det[A-λi]=0 ie det[A-λi]=

$$\lambda^3 + \lambda^2(-c_1c_3 - a_1a_3 + d_2) +$$
$$\lambda(-a_1a_3d_2 - c_1c_3d_2 - a_1a_3c_1c_3 +$$
$$c_1c_2a_1a_2)$$
$$- a_1a_3c_1c_3d_2 = 0$$

For the system to be stable, the real part of the roots of the characteristic equation have to be negative. There are few cases where the characteristic equation can be solved exactly and the resolution can lead to a situation where we can not say anything about stability.

The stability study using the characteristic equation can be seen in [13].

Example 3

The evolution model of the capital of a firm is described by the system of differential equations (9).

$$\begin{cases} \dfrac{dx_1}{dt} = cx_1{}^2x_2 + bx_1 \\ \dfrac{dx_2}{dt} = x_1 + ax_2 - 1 \end{cases} \tag{9}$$
$$x_1 = x_1(t), x_2 = x_2(t)$$

The economical measures that appear are:

x_1 - the capital of the firm and x_2 - the volume of the work force.

The mathematical model implies that the firm is characterized by a function of production in the form Cobb-Douglas. Solving the system:

$$\begin{cases} cx_1{}^2x_2 + bx_1 = 0 \\ x_1 + ax_2 - 1 = 0 \end{cases} \tag{10}$$

We obtain the set of the equilibrium points which we note by Ω . So Ω is the set of the solutions of the system (10). We study the stability of the system (9) on $R^2 - \Omega$. We study the stability of the system (9) using Leapunov function for autonomous systems.

We take as a Leapunov function

$$V(x_1,x_2)=$$
$$\frac{1}{2}(cx_1{}^2x_2+bx_1)^2+\frac{1}{2}(x_1+ax_2-1)^2 \quad (11)$$

We verify the conditions which must be fulfilled by the V function for the system to be stable.

V≥0

(gradV,F)

$$=(cx_1{}^2x_2+bx_1)^2(2cx_1x_2+b)+$$
$$(x_1+ax_2-1)^2a$$

If $(2cx_1x_2+b)\le 0, a\le 0$ the system is stable. In the system (9) a, b, c are constants. The system is an autonomous system and for the study of the stability we apply the method of Leapunov function for autonomous systems. The conditions of the theorems have been verified and we obtained the condition for the system (9) to be stable. In conclusion, if the product of the capital of a firm and the volume of the work force less than or equal to

$-\frac{b}{2c}$ şi $a\le 0$ the system (9) is stable.

This method of construction of the Leapunov function can be seen in [11], [12] and [13]. For the stability condition here is notable the economic sense.

Example 4

The model request-offer was studied by Beckmann and Ryder in 1969 and by Collel in 1986 and it presents the reaction that the price p produces on the quantity q and the opposite. This model is given by the system of differential equations:

$$\begin{cases}\dot{p}=kF(p)-kq\\ \dot{q}=\mu p-\mu C(q)\end{cases} \quad (12)$$

where F(p) is the excess of request and C(q) is the cost of q.

The system of differential equations (12) is stable on the set $R^2-\Omega$ if the following conditions are fulfilled:

$$\mu k(pF(p)-qC(q))\le 0 \quad (13)$$

We study the stability of the system of differential equations (12)

$$V=\mu\frac{p^2}{2}+k\frac{q^2}{2} \quad (14)$$

Is a prime integral and we take it as a Leapunov function. We verify the conditions from the theorem of stability Leapunov for autonomous systems. V is continuous together with its first degree derivatives and so the first condition is fulfilled. V is positively defined if we consider μ and k positives and so the second condition is fulfilled.

If (gradV(x),f(x)) $\le$ 0 (consider the system in the form $\dot{x}=f(x)$) then the third condition is also fulfilled and the system of differential equations given by (13) is stable.

$$grad(v,f)=\mu k(pF(p)-qC(q))\le 0 \quad (15)$$

So the system is stable if the quantity q and the price p are direct proportional with the excess of request F(p) and the cost of q C(q).

We applied the method of Leapunov function for autonomous systems. The six stability theorems for autonomous and non-autonomous systems can be found in [5]. It is noted that the function has the form of an energy.

The study of stability of this example has been presented in [12]. Here it is used in order to highlight the author's original method to build the Leapunov function.

Example 5

One of the measures that must be fulfilled in order to increase the threading capacity for the milling machine for wheel gear FD-320 is the diminishing as much as possible of the vibrations that appear in the threading process. The increase of the threading capacity but also of the quality of the worked surfaces leads to the increase of the production capacity. The vibrations of the supple system are described by the solutions of system of differential equations (16) where J_{me}, C_S, C_P are constants representing the moment of inertia and the last two constants of elasticity. The choice of the indix was done taking into consideration the electrical engine, the tool and the part.

We start with the cinematic system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \dfrac{C_{me}}{J_{me}} \dfrac{C_S(x_3 - x_1) + C_P(x_5 - x_1)}{C_{me} + C_S + C_P} \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = \dfrac{C_S}{J_S} \dfrac{C_{me}(x_1 - x_3) + C_P(x_5 - x_3)}{C_{me} + C_S + C_P} \\ \dot{x}_5 = x_6 \\ \dot{x}_6 = \dfrac{C_P}{J_P} \dfrac{C_{me}(x_1 - x_5) + C_S(x_3 - x_5)}{C_{me} + C_S + C_P}. \end{cases}$$

(16)

We build the Lagrange extension following the method of Prof.dr. Constantin Udrişte. For this we introduce the vectorial field "milling machine" which has six components

$$\begin{cases} X_1(x_1,x_2,x_3,x_4,x_5,x_6) = x_2 \\ X_2(x_1,x_2,x_3,x_4,x_5,x_6) = \dfrac{C_{me}}{J_{me}} \dfrac{C_S(x_3 - x_1) + C_P(x_5 - x_1)}{C_{me} + C_S + C_P} \\ X_3(x_1,x_2,x_3,x_4,x_5,x_6) = x_4 \\ X_4(x_1,x_2,x_3,x_4,x_5,x_6) = \dfrac{C_S}{J_S} \dfrac{C_{me}(x_1 - x_3) + C_P(x_5 - x_3)}{C_{me} + C_S + C_P} \\ X_5(x_1,x_2,x_3 x_4,x_5,x_6) = x_6 \\ X_6(x_1,x_2,x_3,x_4,x_5,x_6) = \dfrac{C_P}{J_P} \dfrac{C_{me}(x_1 - x_5) + C_S(x_3 - x_5)}{C_{me} + C_S + C_P}, \end{cases}$$

(17)

where $X = (X_1, X_2, X_3, X_4, X_5, X_6)$ and $x = (x_1, x_2, x_3, x_4, x_5, x_6)$

The system (16) is an autonomous system in the form

$$\frac{dx_i}{dt} = X_i(x_1, x_2, x_3, x_4, x_5, x_6) \qquad i = \overline{1,6}.$$

(18)

The equilibrium points of the system (16) are in form x = (a,0,a,0,a,0) where a is a constant.

The f function is given by

$$f = \frac{1}{2}\sum_{i=1}^{6} X_i^{\ 2}$$

(19)

Represents the energy density associated to the vectorial field ”milling machine”and the euclidian structure δ_{ij}. The geometrical dynamic associated to the supple system is described by the differential system of second degree

$$\frac{d^2 x_i}{dt^2} = \frac{\partial f}{\partial x_i} + \sum_j \left(\frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i}\right)\frac{dx_j}{dt}, \quad i,\ j = \overline{1,6},$$

(20)

That proves to be an Euler-Lagrange extension. In other words, the lagrangian

$$L = \frac{1}{2}\sum_{i=1}^{6}\left(\frac{dx_i}{dt} - X_i\right)^2$$

(21)

or

$$L = \frac{1}{2}\sum_{i=1}^{6}\left(\frac{dx_i}{dt}\right)^2 - \sum_{i=1}^{6} X_i \frac{dx_i}{dt} + f$$

(22)

Determines this second degree system with six degrees of freedom, whose trajectories contain also the solutions of the system (16). Depending on the constant values that appear in the system given by (16),we apply the stability theorems Leapunov for autonomous systems. As a Leapunov function we can take the energy density, a langragian or a prime integrale. If we take as a Leapunov function V = f given by (19) we observe that:

e)V $\in$ C' (D) because f is an elementary function and it is easy to check that V(o) = 0;

ee) V(x) >0; $\forall x \neq 0$ because it has the form (19), which is a sum of squares;

eee) The condition (grad V(x), f(x)) $\leq 0 \forall$ x $\in$ D gives the stability condition. The stability condition is a long but not complicated expression and depends on the constants. J_{me}, C_{me}, C_S, C_P.

The elements of differential geometry for this example are presented in [10] and [14] but details of the stability and the construction of the Leapunov function can be found only in this paper.

Example 6

The study of the stability of the system of the model given:

$$\begin{cases} m\ddot{x}_1 + c\dot{x}_1 + k(x_1 - x_2) = 0 \\ m\ddot{x}_2 + c\dot{x}_2 + k(x_2 - x_1) = 0 \end{cases}$$

(23)

Gathering part with part the system equations (23) we obtain:

$m(\ddot{x}_1+\ddot{x}_2)+c(\dot{x}_1+\dot{x}_2)=0 \quad \Rightarrow$

$\frac{d}{dt}[m(\dot{x}_1+\dot{x}_2)+c(x_1+x_2)]=0 \Rightarrow$

$m(\dot{x}_1+\dot{x}_2)+c(x_1+x_2)=C$ is a first integral.

We make the substitutions: $y_1 = x_1$, $y_2 = x_2$, $y_3 = \dot{x}_1$, $y_4 = \dot{x}_2$.

The system (23) becomes:

$$\begin{cases} \dot{y}_1 = y_3 \\ \dot{y}_2 = y_4 \\ \dot{y}_3 = -\frac{c}{m}y_3 - \frac{k}{m}(y_1-y_2) \\ \dot{y}_4 = -\frac{c}{m}y_4 - \frac{k}{m}(y_2-y_1) \end{cases} \quad (24)$$

$V(y_1,y_2,y_3,y_4) = [m(y_3+y_4)+c(y_1+y_2)]^2$ is Leapunov function for the system (24) so it fulfills the conditions e – eee from the Leapunov function definition for autonomous systems.

Applying the Leapunov theorem of stability for autonomous systems ,it ensues that the system (23) is stable. The system given by the relation (23) is a mechanical system, where m is the mass, c is the damping coefficient, and k is the elasticity constant. Expressions having the following form mean:

$m\ddot{x}$ force in general;

$c\dot{x}$ damping force;

kx elastic force.

Elements on the stability of this example can be found in [5], in this paper there are made some remarks on the physical quantities that appear and some calculations.

Exemplul 7

One considers the vibropercutant system that has a degree of freedom and a percutant coupling which can be studied with the *q* coordinate. Between two successive collisions the differential equation of movement will be

$$\ddot{q}+\varphi(\dot{q},q,t)=0$$

(25)

the function $\varphi(\dot{q},q,t)$ being recurring in comparison with t, with T. Collisions occur in positions $q=-q_0$ at moments t=t (k=1,2,…), and can be described with the help of the one-sided function

$$f(q)=q+q_0 \geq 0$$

(26)

of interest, however, are the recurrent periods of movement of the vibropercutant system. These movements are made with $t_{k+1}-t_k=rT$ (r=1,2,…). The law of recurrent movement $\bar{q}=\bar{q}(t)$ is the solution to the differential equation (25) which is obtained for the initial conditions

$$\bar{q}(t_k+0)=-q_0, \dot{\bar{q}}(t_k+0)=\dot{\bar{q}}_c'=-R\dot{q}_c$$

(27)

and the final conditions,

$$\bar{q}(t_{k+1}-0)=-q_0, \dot{\bar{q}}(t_{k+1}-0)=\dot{q}_c$$

(28)

$\dot{q}_c$ being the speed at the beginning of the collision, and R, the restore coefficient.

We will study the stability of the linear system, with a degree of freedom, with differential equation of movement

$$m\ddot{q}+c\dot{q}+kq=Q(t)$$

(29)

in which m, c, k are constants, and Q(t) is a recurrent function with period T. Besides, the system is subjected to a one-sided link of form (26). The study of stability is based on the equation in variations

$$m\ddot{x}+c\dot{x}+kx=0$$

(30)

The general integral of this equation depends on the ratio between constants *c* and $c_{cr}=2\sqrt{mk}$. The study of stability for case $c\langle c_{cr}$ was tackled in [28]. We will further analyze the other two cases.

Case I $c=c_{cr}$

The solution to the equation (6) will be

$$x=e^{-\frac{c}{2m}(t-t_k)}[x_1(t-t_k)+x_2]$$

(31)

x_1, x_2 being integration constants which can be determined from the initial conditions.

We obtain:

$$x_1 = \Delta\dot{q}_k{}' - (\frac{c}{2m}\dot{q}_c{}' + \ddot{q}_c{}')\Delta t_k$$

$$x_2 = -\dot{q}_c{}'\Delta t_k$$

Taking into considering the final conditions:

$$x(t_{k+1} - 0) = -\dot{q}_c \Delta t_{k+1}$$

$$\dot{x}(t_{k+1} - 0) = \Delta\dot{q}_{k+1} - \ddot{q}_c \Delta t_{k+1}$$

(32)

in solution (31) we obtain linear equations in perturbations.

$$e^{-\frac{c}{2m}rT} rT(-R\Delta\dot{q}_k) - e^{-\frac{c}{2m}rT}\left[\left(\frac{c}{2m}rT + 1\right)\cdot\left(-R\dot{q}_c\right) + rT\ddot{q}_c'\right]\Delta t_k = \dot{q}_c \Delta t_{k+1} \quad (33)$$

$$-R\Delta\dot{q}_k e^{-\frac{c}{2m}rT}\left(1 - \frac{c}{2m}rT\right) + \Delta t_k \left\{\begin{matrix} \frac{c}{2m}rT\left[\frac{c}{2m}\left(-R\dot{q}_c\right) + \ddot{q}_c'\right] \\ -e^{-\frac{c}{2m}rT}.\ddot{q}_c' \end{matrix}\right\} = \Delta\dot{q}_{k+1} - \ddot{q}_c \Delta t_{k+1} \quad (34)$$

For the system of equations with finite differences (33+34) one considers solutions of the form

$$\Delta\dot{q}_k = v.\beta^k; \quad \Delta t_k = \theta\beta^k$$

On condition that the linear and homogenous system in v and θ admits non-nil solutions, one obtains an equation characteristic in β of the form

$$-\beta^2\dot{q}_c + \beta\left[2R\dot{q}_c + RrT\ddot{q}_c - rT\ddot{q}_c'\right].e^{-\frac{c}{2m}rT} + e^{-\frac{c}{m}rT}\left[2RrT\ddot{q}_c' + R^2.\dot{q}_c.\frac{c}{2m}rT - R^2\dot{q}_c\right] + e^{-\frac{c}{2m}rT}\left[\frac{c^2r^2T^2R^2}{4m^2}\dot{q}_c - \frac{cr^2T^2R}{2m}\ddot{q}_c'\right] = 0 \quad (35)$$

where $\ddot{q}_c$ and $\ddot{q}_c'$ are calculated from the differential equation of movement.

$$m\ddot{q} + c\dot{q} + kq = Q(t)$$

(29)

The movement is stable if the roots of the equation (35) is sub-unit in module. So, by applying Schur's criterion, the conditions of stability are obtained.

$$\left|2R\dot{q}_c + RrT\ddot{q}_c - rT\ddot{q}_c'\right|e^{-\frac{c}{2m}rT} \prec 2\left|-\dot{q}_c\right| \quad (36)$$

Case II. $c \rangle c_{cr}$.

The solution to the equation in this case will be:

$$x = X_1 e^{-\beta_1(t-t_k)} + X_2 e^{-\beta_2(t-t_k)}$$

$$-\beta_1 = -\left(\frac{c}{2m}\right) - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \prec 0$$

$$-\beta_2 = -\left(\frac{c}{2m}\right) - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \prec 0 \quad (37)$$

X_1 and X_2 are constants of integration which are determined from the initial conditions.

$$x(t_k + 0) = -\dot{q}_c'\Delta t_k$$

$$\dot{x}(t_k + 0) = \Delta\dot{q}_k' - \ddot{q}_c'\Delta t_k$$

(38)

What is obtained is

$$X_1 = \frac{\Delta t_k(\beta_2\dot{q}_c' + \ddot{q}_c') - \Delta\dot{q}_k'}{\beta_1 - \beta_2}$$

$$X_2 = \frac{\Delta\dot{q}_k' - (\ddot{q}_c' + \beta_1\dot{q}_c')\Delta t_k}{\beta_1 - \beta_2}$$

Giving the final conditions:

$$x(t_{k+1} - 0) = -\dot{q}_c\Delta t_{k+1}$$

$$\dot{x}(t_{k+1} - 0) = \Delta\dot{q}_{k+1} - \ddot{q}_c\Delta t_{k+1}$$

(32)

From solution (37), linear equations in perturbations result.

$$\left[\frac{\Delta t_k(\beta_2\dot{q}_c' + \ddot{q}_c') - \Delta\dot{q}_k'}{\beta_1 - \beta_2}\right]e^{-\beta_1 rT} + \left[\frac{\Delta\dot{q}_k' - (\ddot{q}_c' + \beta_1\dot{q}_c')\Delta t_k}{\beta_1 - \beta_2}\right]e^{-\beta_2 rT} = \frac{-\dot{q}_c\Delta t_{k+1}.(\beta_1 - \beta_2)}{\beta_1 - \beta_2} \quad (39)$$

$$-\beta_1 \cdot \left[\frac{\Delta t_k(\beta_2 \dot{q}'_c + \ddot{q}'_c) - \Delta \dot{q}'_k}{\beta_1 - \beta_2}\right] e^{-\beta_1 rT} -$$

$$-\beta_2 \left[\frac{\Delta \dot{q}'_k - (\ddot{q}'_c + \beta_1 \dot{q}'_c)\Delta t_k}{\beta_1 - \beta_2}\right] e^{-\beta_2 rT} =$$

$$= \frac{(\Delta \dot{q}_{k+1} - \ddot{q}_c \Delta t_{k+1})(\beta_1 - \beta_2)}{\beta_1 - \beta_2} \quad (40)$$

For the system of equations with finite differences [(39)+(40)] one considers solutions of the form

$\Delta \dot{q}_k = v.\beta^k; \quad \Delta t_k = \theta.\beta^k$

On condition that the linear and homogenous system in v and θ allows non-nil solutions, the characteristic equation in β is obtained. The movement is stable if the roots of this equation are sub-unit in module. So, by applying Schur's criterion, the condition of stability is obtained.

$$\Big| -(\beta_1 - \beta_2)^2 \Big[R\dot{q}_c (\beta_2 - \beta_1)^2 \cdot (e^{-\beta_1 rT} + e^{-\beta_2 rT}) + (\ddot{q}'_c + R\ddot{q}_c)\left(e^{-\beta_2 rT} - e^{-\beta_1 rT}\right) \Big] \Big| \prec 2(\beta_1 - \beta_2)^2 . |\dot{q}_c|$$

(41)

The study of the stability of vibropercutant systems is extremely difficult. The difficulty originates in the existence of both collisions and different speeds before and after collision. When considering the initial and the final conditions, differentials to the right and to the left can be different.

The study of stability of the dynamic system of Example 7 was made for the case $c \langle c_{cr}$ in [28] and for the cases $c = c_{cr}$ and $c \rangle c_{cr}$ in [1]. The paper [1] was recently presented WSEAS Conference held in Brasov on 26-28 June 2014.

3. Presenting the methods of the stability system

In this paper are analyzed seven dynamic models of economical and technical processes. Economical and technical processes are described using differential equations and systems we considered that this description is correct as it appears in the specialty literature and the models are have familiar names. The study of the stability of systems of differential equations is a concern of the author in the last fifteen years. In fact, the original elements consist of two methods for the construction of the Leapunov function. We calculate the Lagrangian L, the Hamiltonian H and the energy density f as these functions can be used to build Leapunov function.

There is no general method for determining the function Leapunov and therefore we can say that the work has elements of originality. If you find a function that satisfies certain conditions, which in the specialty literature is called Leapunov function and is usually denoted by V, we apply the theorems of stability for autonomous differential systems. As a Leapunov function can sometimes take energy density and Hamiltonian. Also as a Leapunov function can take a full premium or its square. There are various ways of finding prime integrals, one of them being that we use the Lagrangian system.

To find the two methods we have relied on imagination but any step can be argued mathematically rigorous. Applying the two methods require mastery of several knowledge on stability, computational, imagination, and is the result of studying the stability of the many examples.

4. Conclusions

Example 1 presents a dynamical system given by a differential equation in which the study is done by applying the definition of stability. This example is rare because the dynamic systems which describe an economical or technical process are complicated and the exact solution to apply the definitions of stability can not be found.

Example 2 presents a dynamic system given by three differential equations. The system is autonomous. To study the stability, the equation has to be found and solved and if its roots have the real part negative then the system is stable. Even if we can solve the equation we can not always tell if the system is stable.

In example 3 we have a dymanical system described by two differential equations. The study of stability is done using the Leapunov function. We take as a Leapunov function an energy density. Finding the Leapunov function is difficult and is an original idea.

In example 4 we have a dymanical system described by two differential equations. The study of stability is done using the Leapunov function. We take as a Leapunov function an energy density. Finding the Leapunov function is difficult and is an original idea. To find the Leapunov function we can start from the liniarized original system. We find the Leapunov function linearized for the system and see if it fulfills the conditions of the stability theorems. So the function we found can be a Leapunov function for the not liniarized system, so for the initial system which is more complicated.

In example 5 we have a dynamical system described by the six differential equations. The stability study is done using the Leapunov function. We take an energy density as a Leapunov function. Finding the Leapunov function is made using the method of teacher Constantine Udrişte to find the energy density, the Lagrangian and Hamiltonian. Not always the energy density, Hamiltonian or Lagrangian helps us to study the stability.

In example 6 we have a dynamic system described by two differential equations. It gathers two equations and is an integral first. The stability study is made using the method of the Leapunov function. We take the first full the Leapunov function to the power of two. Finding first integral the Leapunov function is difficult and is an original idea.

In example 7 we study the movement of an vibropercutant system. Here the system is subject to links and we consider movement periodical. To study the stability we use a special technique described in detail from Example 7. We arrive at the characteristic equation and then apply Schur's criterion and find the conditions of stability.

In conclusion, the study of the stability of dynamical systems is a complicated issue. There are several types of stability. The most known is the stability in disturbances or stability the Leapunov.

Theory tells us that if there is a Leapunov function satisfying certain conditions we can say something about the stability of the system. There is no general method to find the Leapunov function. Finding the Leapunov function is an element of originality. At vibropercutante systems appear collisions and the stability study technique is special and very difficult and involves many calculations.

In conclusion, the paper aims to present some stability study techniques study involving elements of originality.

It can be seen that quantitative and qualitative methods in the study of dynamical systems is a concern of the author and important results were published in WSEAS Conferences resulting work [1], [2], [4], [22], [23], [24] and [25].

Acknowledgment: This work was partially supported by the grant number 3C/27.01.2014, awarded in the internal grant competition of the University of Craiova.

References:

[1] Bălă Dumitru, A Study of the Stability of the System with Linear Characteristic Subjected to One-Sided Connection, *Proceedings of the 16-th International Conference on Automatic Control, Modelling & Simulation (ACMOS '14),* Braşov, România, June 26-28, 2014, pp.46-50.

[2] Bălă Dumitru, Discontinuous Dynamical Systems with Applications in Economics, *Proceedings of the Ist International Conference on Tourism and Economic Development (TED'11),* Drobeta Turnu Severin, România, October, 2011, pp. 119-123.

[3] Bălă Dumitru, *Quantitative methods in market studies*, STEF Publishing House, Drobeta Turnu Severin, România, 2010.

[4] Bălă Dumitru, Mazilu Mirela, Marinescu Cristina Roxana, Tourism and Statistics, Inseparable Binomial, *Proceedings of the 5-th WSEAS International Conference on Economy and Management Transformation (EMT'10),* Volume I, Timişoara, România, October 24-26, 2010, pp. 39-43.

[5] Bălă Dumitru, *Elements of mathematics and statistics. Theory and Applications*, Universitaria Publishing House, Craiova, România, 2009.

[6] Bălă Dumitru, *Applied Mathematics in Economics*, Universitaria Publishing House, Craiova, România, 2007.

[7] Bălă Dumitru, *Geometrical Methods in the Study of Vibrant and Vibropercutante Systems Movement*, Universitaria Publishing House, Craiova, 2006.

[8] Bălă Dumitru, Silaş Gheorghe, *Problems on the stability of dynamics and of mechanical systems*, Şcoala Mehedinţiului Publishing House, Drobeta Turnu Severin, România, 2001.

[9] Bălă Dumitru, Geometrical Dynamics, *Proceedings of the XI-th International Conference on Vibration Engineering, Timişoara, România, September 27-30, 2005, Scientific Buletin of the " Politehnica " University of Timişoara, România, Transaction on Mechanics*, pp. 37-40.

[10] Bălă Dumitru, Geometrical dynamics in the study of some dynamical systems, *Proceedings of the X-th Symposium, Acoustics and Vibrations of Mechanical Structures, AVMS-2009,* Timişoara, 2009, pp. 137-141.

[11] Bălă Dumitru, The study of dynamic systems with applications in technology and in economy, *Proceedings of the XI-th Symposium, Acoustics and Vibrations of Mechanical Structures, AVMS-2011*, Timişoara, 2011, pp. 147-152.

[12] Bălă Dumitru, The study of the stability of some dynamical systems with application in economy using geometrical methods, *Proceedings of the twelfth Symposium of mathematics and its applications, Timişoara, November, 5-7, 2009*, pp. 288-293.

[13] Bălă Dumitru, Geometrical methods in the study of some dynamical systems with applications in economy, *Theoretical and Applied Economics/Economie Teoretică şi Aplicată,* Anul XV, Nr.7(524), 2008, pp. 55-60.

[14] Bălă Dumitru, Geometric dynamics of cutting machin, *ROMAI Journal*, Volume I, Number 2, Year 2005, pp. 7-10.

[15] Bălă Dumitru, Geometric methods in study of the stability of some dynamical systems, *An. Şt. Univ. Ovidius Constanţa,* Vol.17(3), 2009, pp. 27-35.

[16] Bălă Dumitru, The study of the stability of some dynamical systems with applications in economy, *An. Univ. from Craiova, Series: Economic Sciences*, Year XXXV, Nr.35, 2007, Volumul 7, 2007. pp. 1541-1552.

[17] Bălă Dumitru, The role of dynamical systems theory in economics, *Economics, Management, and Financial Markets,* Volume 5,Number 3, 2010, pp. 249-254.

[18] Bălă Dumitru, Optimum productivity, performance, and milling machine, *Official journal of the contemporary science association-New York, Economics, Management, and Financial Markets,* Volume 5, Number 2-June, 2010, pp. 316-321.

[19] Bălă Dumitru, The application of mathematical models in describing real economic processes, *Official journal of the contemporary science association-New York, Economics, Management, and Financial Markets*, Volume 5, Number 1-March, 2010, pp. 164-169.

[20] Danca Marius-Florin, *Dynamic Discountinuous Systems,* Risoprint Publishing House, Cluj Napoca, 2004.

[21] Georgescu Adelina, Moroianu Mihnea, Oprea Iuliana, *The Theory of Bifurcation. Principles and Applications* , Pitesti University Publishing House, Piteşti, 1999.

[22] Marinca Vasile, Herişanu Nicolae, Bălă Dumitru, On an parametric iteration method for thin film flow, *Proceedings of the 4th European Computing Conference (ECC'10),* Bucharest, România, April 20-22, 2010, pp. 34-38.

[23] Marinca Vasile, Herişanu Nicolae, Bălă Dumitru, Some optimal aproximate methods with application to thin film flow, *WSEAS Transactions on Systems*, Issue 7, Volume 9, July 2010, pp. 744-753.

[24] Mazilu Mirela, Bălă Dumitru, Severineanu Roxana, The statistical Analysis of Romanian Tourism in the Context of The World Economic Crisis, *Recent Researches in Mechanics,* Corfu Island, July 14-16, 2011, pp.340-346.

[25] Pop Camelia, Petrişor Camelia, Bălă Dumitru, A Geometric Approach of some Dynamics Arising from Engineering and Biology, *Selected Topics in Mathematical Methods and Computational Techniques in Electrical Engineering*, ISSN:1792-5967, ISBN:978-960--474-238-7, pp. 78-82.

[26] Pop Camelia, Petrişor Camelia, Bălă Dumitru, Hamilton-Poisson Realizations for the Lü System, *Mathematical Problems in Engineering* , vol. 2011, http://www.hindawi.com/journals/mpe/2011/842325/.

[27] Pop Camelia, Anania Aron, Petrişor Camelia, Geometrical aspects of the ball-plate problem, *Balkan Journal of Geometry and Its Applications,* Vol.16, No 2, 2011, pp. 114-121.

[28] Silaş Gheorghe, Brîndeu Liviu, *Vibropercutante Systems,* Technological Publishing House Editura Tehnică, Bucureşti, 1986.
[29] Udrişte Constantin, *Atlas of Magnetic Geometric Dynamics*, Geometry Balkan Press, Bucarest, Romania, 2001.