Almost Periodic Solution of Neutral-Type Neural Networks with Time Delay in the Leakage Term on Time Scales

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Abstract: In this paper, based on the theory of calculus on time scales, by using the exponential dichotomy of linear dynamic equations and Banach's fixed point theorem as well as some mathematical methods, some sufficient conditions are obtained for the existence and exponential stability of almost periodic solution of neutral-type neural networks with time-varying delay in the leakage term on time scales. These results have important leading significance in designs and applications of such neural networks. Finally, an example is given to illustrate the feasibility and effectiveness of the results.

Key–Words: Almost periodic solution; Exponential stability; Neutral delay neural networks; Leakage term; Time scale.

1 Introduction

In the past few years, different types of neural networks have been extensively studied since they can be applied in many different fields such as pattern recognition, image processing, optimization problems and so on; see, for example, [1-8] and the references therein. As we know, in applications, there are many neural networks whose developing processes are both continuous and discrete. Hence, using the only differential equation or difference equation can't accurately describe the law of their developments. Therefore, there is a need to establish correspondent dynamic models on new time scales. Recently, neural networks with periodic or almost periodic coefficients on time scales received more researchers' special attention; see, for example, [9-13].

Since neurons from attenuation process is not instantaneous, when neurons and neural network and the external input disconnected, reset to the isolation static state takes time, so, time-varying delay in the leakage term need to be considered. In fact, time delays in the leakage terms are difficult to handle, and the leakage term has great impact on the dynamical behavior of neural networks [14-17]. Therefore, it is important and, in effect, necessary to study neural networks with time-varying delay in the leakage term, which plays an important role in designs and applications of such neural networks.

To the best of our knowledge, there are few papers published on the existence and stability of almost periodic solution of neutral-type neural networks with time-varying delays in the leakage term on time scales.

Motivated by the above, in the present paper, we shall study an almost periodic neutral-type neural networks with time-varying delay in the leakage term on time scales as follows:

$$\begin{aligned} x_i^{\Delta}(t) &= -\delta_i(t)x_i(t-\tau_i(t)) \\ &\quad +\sum_{j=1}^{n} a_{ij}(t)f_j(x_j(t-\alpha_{ij}(t))) \\ &\quad +\sum_{j=1}^{n} b_{ij}(t)g_j(x_j^{\Delta}(t-\beta_{ij}(t)))+I_i(t), \\ &\quad i=1,2,\dots,n, \end{aligned} \tag{1}$$

where $t \in \mathbb{T}$, $\mathbb{T}$ is an almost periodic time scale, $0 \in \mathbb{T}$; $x_i(t)$ denotes the potential (or voltage) of cell i at time t; $\delta_i(t) > 0$ represents the rate with the ith unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs at time t; $a_{ij}(t)$ and $b_{ij}(t)$ represent the delayed strengths of connectivity and neutral delayed strengths of connectivity between cell i and j at time t, respectively; f_j and g_j are the activation functions in system (1); $I_i(t)$ is an external input on the ith unit at time t; $\tau_i(t) \geq 0$ denote the leakage time delay, $\alpha_{ij}(t) \geq 0$ and $\beta_{ij}(t) \geq 0$ correspond to the transmission delay of the ith unit along the axon of the jth unit at time t.

The initial condition associated with system (1) is of the form

$$x_i(t) = \varphi_i(t),\ x_i^{\Delta}(t) = \varphi_i^{\Delta}(t),$$
$$t \in [-\tau, 0]_{\mathbb{T}}, i = 1, 2, \dots, n.$$

where $\varphi_i(\cdot)$ denotes a bounded Δ-differentiable function defined on $[-\tau, 0]_{\mathbb{T}}$, and $\tau = \max_{1\le i,j\le n} \{\sup_{t\in\mathbb{T}} \tau_i(t), \sup_{t\in\mathbb{T}} \alpha_{ij}(t), \sup_{t\in\mathbb{T}} \beta_{ij}(t)\}$.

The main purpose of this paper is to use the exponential dichotomy of linear dynamic equations on time scales and Banach's fixed point theorem as well as some mathematical methods to study the existence and global exponential stability of almost periodic solution of system (1).

In this paper, for each $x = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n$, when it comes to that x is continuous, delta derivative, delta integrable, and so forth; we mean that each element x_i is continuous, delta derivative, delta integrable, and so forth.

The organization of this paper is as follows. In Section 2, we introduce some notations and definitions and state some preliminary results needed in later sections. In Section 3, we establish some existence and exponential stability results for system (1). In Section 4, an example is given to illustrate that our results are feasible and more general.

2 Preliminaries

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \mu(t) = \sigma(t) - t$.

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided that it is continuous at each right-dense point and has a left-sided limit at each point. A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$. Denote $\mathcal{R}^+ = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0, \forall\, t \in \mathbb{T}\}$.

If r is a regressive function, then the generalized exponential function e_r is defined by

$$e_r(t, s) = \exp\left\{ \int_s^t \xi_{\mu(\tau)}(r(\tau))\Delta\tau \right\}$$

for all $s, t \in \mathbb{T}$, with the cylinder transformation

$$\xi_h(z) = \begin{cases} \frac{\mathrm{Log}(1+hz)}{h}, & \text{if } h \neq 0, \\ z, & \text{if } h = 0. \end{cases}$$

Let $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions, define

$$p \oplus q = p + q + \mu pq,\ \ominus p = -\frac{p}{1 + \mu p},\ p \ominus q = p \oplus (\ominus q).$$

Lemma 1. ([17]) *Assume that $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions, then*
(i) $e_0(t, s) \equiv 1$ and $e_p(t, t) \equiv 1$;
(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
(iii) $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$;
(iv) $e_p(t, s)e_p(s, r) = e_p(t, r)$;
(v) $e_p(t, s)e_q(t, s) = e_{p\oplus q}(t, s)$;
(vi) $(e_{\ominus p}(t, s))^{\Delta} = (\ominus p)(t)e_{\ominus p}(t, s)$.

The basic theories of almost periodic differential equation on time scales, see [18,19].

Definition 2. ([18]) *Let $A(t)$ be an $n \times n$ rd-continuous matrix on $\mathbb{T}$, the linear system*

$$x^{\Delta}(t) = A(t)x(t),\ t \in \mathbb{T} \tag{2}$$

is said to admit an exponential dichotomy on $\mathbb{T}$ if there exist positive constant k, α, projection P and the fundamental solution matrix $X(t)$ of (2), satisfying

$$\|X(t)PX^{-1}(\sigma(s))\|_0 \le ke_{\ominus\alpha}(t, \sigma(s)),$$
$$s, t \in \mathbb{T}, t \ge \sigma(s),$$
$$\|X(t)(I - P)X^{-1}(\sigma(s))\|_0 \le ke_{\ominus\alpha}(\sigma(s), t),$$
$$s, t \in \mathbb{T}, t \le \sigma(s),$$

where $|\cdot|_0$ is a matrix norm on $\mathbb{T}$.

Consider the following almost periodic system

$$x^{\Delta}(t) = A(t)x(t) + f(t),\ t \in \mathbb{T}, \tag{3}$$

where $A(t)$ is an almost periodic matrix function, $f(t)$ is an almost periodic vector function.

Let $A(t) = \big(a_{ij}(t)\big)_{n\times n}, \bar{A} = \big(\sup(a_{ij}(t))\big)_{n\times n}$, $1 \le i, j \le n,\ t \in \mathbb{T}$.

Lemma 3. ([19]) *If the linear system (2) admits an exponential dichotomy, $-\bar{A}$ is an M-matrix, then system (3) has a unique almost periodic solution $x(t)$, and*

$$\begin{aligned} x(t) &= \int_{-\infty}^{t} X(t)PX^{-1}(\sigma(s))f(s)\Delta s \\ &\quad - \int_{t}^{+\infty} X(t)(I - P)X^{-1}(\sigma(s))f(s)\Delta s, \end{aligned}$$

where $X(t)$ is the fundamental solution matrix of (2).

Lemma 4. *Suppose that $f(t)$ is an rd-continuous function and $p(t)$ is a positive rd-continuous function satisfying $-p \in \mathcal{R}^+$. Let*

$$x(t) = \int_{t_0}^{t} e_{-p}(t, \sigma(s)) f(s)\Delta s,$$

then

$$x^{\Delta}(t) = f(t) + \int_{t_0}^{t} [-p(t)e_{-p}(t, \sigma(s)) f(s)]\Delta s.$$

Proof. By a direct calculation,

$$\begin{aligned} x^{\Delta}(t) &= \left[e_{-p}(t,0) \int_{t_0}^{t} e_{-p}(0, \sigma(s)) f(s)\Delta s \right]^{\Delta} \\ &= e_{-p}(\sigma(t), 0)e_{-p}(0, \sigma(t)) f(t) \\ &\quad -p(t)e_{-p}(t, 0) \int_{t_0}^{t} e_{-p}(0, \sigma(s)) f(s)\Delta s \\ &= f(t) + \int_{t_0}^{t} [-p(t)e_{-p}(t, \sigma(s)) f(s)]\Delta s. \end{aligned}$$

This completes the proof. □

Let $AP(\mathbb{T})$ be the set of all $\mathbb{R}^n$-valued almost periodic functions on almost time scales $\mathbb{T}$, and $Y = \{x \in C^1(\mathbb{T}, \mathbb{R}^n) : x, x^{\Delta} \in AP(\mathbb{T})\}$. Set $X = \{\varphi = (\varphi_1, \varphi_2 \dots, \varphi_n)^T : \varphi_i \in Y\}$ with the norm defined by $\|\varphi\|_X = \max\{\|\varphi\|_0, \|\varphi^{\Delta}\|_0\}$, where

$$\|\varphi\|_0 = \max_{1\le i\le n} \sup_{t\in\mathbb{T}} |\varphi_i(t)|, \|\varphi^{\Delta}\|_0 = \max_{1\le i\le n} \sup_{t\in\mathbb{T}} |\varphi_i^{\Delta}(t)|,$$

then X is a Banach space.

Definition 5. *The almost periodic solution $x = (x_1, x_2, \dots, x_n)^T$ of system* (1) *with initial value $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n)^T$ is said to be globally exponentially stable, if there exist positive constants λ with $\ominus\lambda \in \mathcal{R}^+$ and $M > 1$ such that*

$$\|x - \bar{x}\|_X \le M e_{\ominus\lambda}(t, 0)\|\psi\|_X, \ \forall t \in [0, +\infty)_{\mathbb{T}},$$

where $\bar{x} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)^T$ is an arbitrary almost periodic solution of system (1) *with initial value $\bar{\varphi} = (\bar{\varphi}_1, \bar{\varphi}_2, \dots, \bar{\varphi}_n)^T$, and*

$$\begin{aligned} \|x - \bar{x}\|_X &= \max\{\|x - \bar{x}\|_0, \|(x - \bar{x})^{\Delta}\|_0\}, \\ \|\psi\|_X &= \max\{\|\varphi - \bar{\varphi}\|_0, \|(\varphi - \bar{\varphi})^{\Delta}\|_0\}. \end{aligned}$$

3 Existence and exponential stability

In this section, we shall study the existence and global exponential stability of almost periodic solution of system (1). For convenience, denote $f^- = \inf_{i\in\mathbb{T}} f(t)$, $f^+ = \sup_{\in\mathbb{T}} f(t)$, where $f(t)$ be any bounded function defined on $\mathbb{T}$.

Firstly, we make the following assumptions:

(H_1) $\delta_i(t) > 0, \tau_i(t) \ge 0, \alpha_{ij}(t) \ge 0, \beta_{ij}(t) \ge 0$, $a_{ij}(t), b_{ij}(t)$ and $I_i(t)$ are almost periodic functions on $\mathbb{T}, -\delta_i \in \mathcal{R}^+, t - \tau_i(t), t - \alpha_{ij}(t), t - \beta_{ij}(t) \in \mathbb{T}$ for $t \in \mathbb{T}, i, j = 1, 2, \dots, n$.

(H_2) There exist positive constants $L_i > 0$, $l_i > 0$ such that for $i = 1, 2, \dots, n$,

$$|f_i(x) - f_i(y)| \le L_i|x - y|,$$
$$|g_i(x) - g_i(y)| \le l_i|x - y|$$

for all $x, y \in \mathbb{R}$, and $f_i(0) = g_i(0) = 0$.

Theorem 6. *Assume that* (H_1), (H_2) *and*

(H_3) $r = \max_{1\le i\le n} \left\{ \frac{1}{\delta_i^-}, 1 + \frac{\delta_i^+}{\delta_i^-} \right\} \left(\delta_i^+ \tau_i^+ + \sum_{j=1}^{n} a_{ij}^+ L_j + \sum_{j=i}^{n} b_{ij}^+ l_j \right) < 1$

hold, then there exists exactly one almost periodic solution of system (1) *in the region $X_0 = \left\{ \phi(t) \middle| \|\phi - \phi_0\|_X \le \frac{rR}{1-r} \right\}$, where*

$$\begin{aligned} R &= \max_{1\le i\le n} \left\{ \frac{I_i^+}{\delta_i^-}, I_i^+ \left(1 + \frac{\delta_i^+}{\delta_i^-} \right) \right\}, \\ \phi_0 &= \left(\int_{-\infty}^{t} I_1(s)e_{-\delta_1}(t, \sigma(s))\Delta s, \dots, \right. \\ &\quad \left. \int_{-\infty}^{t} I_n(s)e_{-\delta_n}(t, \sigma(s))\Delta s \right)^T. \end{aligned}$$

Proof. System (1) can be written as

$$\begin{aligned} x_i^{\Delta}(t) &= -\delta_i(t)x_i(t) + \delta_i(t) \int_{t-\tau_i(t)}^{t} x_i^{\Delta}(s)\Delta s \\ &\quad + \sum_{j=1}^{n} a_{ij}(t) f_j(x_j(t - \alpha_{ij}(t))) \\ &\quad + \sum_{j=1}^{n} b_{ij}(t) g_j(x_j^{\Delta}(t - \beta_{ij}(t))) + I_i(t), \\ &\quad i = 1, 2, \dots, n. \end{aligned}$$

For any $\phi \in X$, we consider the following system

$$\begin{aligned} x_i^{\Delta}(t) &= -\delta_i(t)x_i(t) + \delta_i(t) \int_{t-\tau_i(t)}^{t} \phi_i^{\Delta}(s)\Delta s \\ &\quad + \sum_{j=1}^{n} a_{ij}(t) f_j(\phi_j(t - \alpha_{ij}(t))) \\ &\quad + \sum_{j=1}^{n} b_{ij}(t) g_j(\phi_j^{\Delta}(t - \beta_{ij}(t))) + I_i(t), \\ &\quad i = 1, 2, \dots, n. \end{aligned} \tag{4}$$

Since $\min\limits_{1\le i\le n}\left\{\inf\limits_{t\in\mathbb{T}}\delta_i(t)\right\} > 0$, it follows from Lemma 3 that system (4) has a unique almost periodic solution which can be expressed as follows:

$$X^{\phi}(t)=\left(x_1^{\phi}(t),x_2^{\phi}(t),\ldots,x_n^{\phi}(t)\right)^T \tag{5}$$

where

$$\begin{aligned}\phi_i(t) = &\int_{-\infty}^{t} e_{-\delta_i}(t,\sigma(s))\Big[\delta_i(s)\int_{s-\tau_i(s)}^{s}\phi_i^{\Delta}(u)\Delta u\\ &+\sum_{j=1}^{n}a_{ij}(s)f_j(\phi_j(s-\alpha_{ij}(s)))\\ &+\sum_{j=1}^{n}b_{ij}(s)g_j(\phi_j^{\Delta}(s-\beta_{ij}(s)))+I_i(s)\Big]\Delta s,\\ &i=1,2,\ldots,n.\end{aligned}$$

Define a mapping $\Phi : X \to X$ by $(\Phi\phi)(t) = X^{\phi}(t), \forall\phi\in X$. By the definition of $\|\cdot\|_X$, we have

$$\begin{aligned}&\|\phi_0\|_X\\ =\ &\max\left\{\|\phi_0\|_0,\|\phi_0^{\Delta}\|_0\right\}\\ =\ &\max\Big\{\max_{1\le i\le n}\sup_{t\in\mathbb{T}}\Big|\int_{-\infty}^{t}I_i(s)e_{-\delta_i}(t,\sigma(s))\Delta s\Big|,\\ &\max_{1\le i\le n}\sup_{t\in\mathbb{T}}\Big|I_i(t)\\ &-\int_{-\infty}^{t}\delta_i(t)I_i(s)e_{-\delta_i}(t,\sigma(s))\Delta s\Big|\Big\}\\ \le\ &\max\left\{\max_{1\le i\le n}\left\{\frac{I_i^+}{\delta_i^-}\right\},\max_{1\le i\le n}\left\{I_i^+(1+\frac{\delta_i^+}{\delta_i^-})\right\}\right\}\\ =\ &R.\end{aligned} \tag{6}$$

Hence, for any $\phi\in X_0=\left\{\phi|\phi\in X,\|\phi-\phi_0\|_X\le\frac{rR}{1-r}\right\}$, one has

$$\|\phi\|_X\le\|\phi_0\|_X+\|\phi-\phi_0\|_X\le R+\frac{rR}{1-r}=\frac{R}{1-r}.$$

Next, we will show that $\Phi(X_0)\subset X_0$. In fact, for any $\phi\in X_0$, we have

$$\begin{aligned}&\|\Phi\phi-\phi_0\|_0\\ =\ &\max_{1\le i\le n}\sup_{t\in\mathbb{T}}\Big\{\Big|\int_{-\infty}^{t}e_{-\delta_i}(t,\sigma(s))\\ &\times\Big[\delta_i(s)\int_{s-\tau_i(s)}^{s}\phi_i^{\Delta}(u)\Delta u\\ &+\sum_{j=1}^{n}a_{ij}(s)f_j(\phi_j(s-\alpha_{ij}(s)))\\ &+\sum_{j=1}^{n}b_{ij}(s)g_j(\phi_j^{\Delta}(s-\beta_{ij}(s)))\Big]\Delta s\Big|\Big\}\\ \le\ &\max_{1\le i\le n}\sup_{t\in\mathbb{T}}\Big\{\int_{-\infty}^{t}e_{-\delta_i}(t,\sigma(s))\Big[\delta_i^+\|\phi^{\Delta}\|_0\tau_i^+\\ &+\sum_{j=1}^{n}a_{ij}^+L_j\|\phi\|_0+\sum_{j=1}^{n}b_{ij}^+l_j\|\phi^{\Delta}\|_0\Big]\Delta s\Big\}\\ \le\ &\max_{1\le i\le n}\sup_{t\in\mathbb{T}}\Big\{\int_{-\infty}^{t}e_{-\delta_i}(t,\sigma(s))\Big(\delta_i^+\tau_i^+\\ &+\sum_{j=1}^{n}a_{ij}^+L_j+\sum_{j=1}^{n}b_{ij}^+l_j\Big)\|\phi\|_X\Delta s\Big\}\\ \le\ &\max_{1\le i\le n}\left\{\frac{1}{\delta_i^-}\Big(\delta_i^+\tau_i^++\sum_{j=1}^{n}a_{ij}^+L_j+\sum_{j=1}^{n}b_{ij}^+l_j\Big)\right\}\|\phi\|_X\end{aligned}$$

and

$$\begin{aligned}&\|(\Phi\phi-\phi_0)^{\Delta}\|_0\\ =\ &\max_{1\le i\le n}\sup_{t\in\mathbb{T}}\Big\{\Big|\delta_i(t)\int_{t-\tau_i(t)}^{t}\phi^{\Delta}(u)\Delta u\\ &+\sum_{j=1}^{n}a_{ij}(t)f_j(\phi_j(t-\alpha_{ij}(t)))\\ &+\sum_{j=1}^{n}b_{ij}(t)g_j(\phi_j^{\Delta}(t-\beta_{ij}(t)))\\ &-\int_{-\infty}^{t}\delta_i(t)e_{-\delta_i}(t,\sigma(s))\Big[\delta_i(s)\int_{s-\tau_i(s)}^{s}\phi_i^{\Delta}(u)\Delta u\\ &+\sum_{j=1}^{n}a_{ij}(s)f_j(\phi_j(s-\alpha_{ij}(s)))\\ &+\sum_{j=1}^{n}b_{ij}(s)g_j(\phi_j^{\Delta}(s-\beta_{ij}(s)))\Big]\Delta s\Big|\Big\}\\ \le\ &\max_{1\le i\le n}\sup_{t\in\mathbb{T}}\Big\{\Big(\delta_i^+\tau_i^++\sum_{j=1}^{n}a_{ij}^+L_j+\sum_{j=1}^{n}b_{ij}^+l_j\Big)\|\phi\|_X\\ &+\delta_i^+\int_{-\infty}^{t}e_{-\delta_i}(t,\sigma(s))\Big(\delta_i^+\tau_i^++\sum_{j=1}^{n}a_{ij}^+L_j\\ &+\sum_{j=1}^{n}b_{ij}^+l_j\Big)\|\phi\|_X\Delta s\Big\}\\ \le\ &\max_{1\le i\le n}\Big\{\Big(1+\frac{\delta_i^+}{\delta_i^-}\Big)\Big(\delta_i^+\tau_i^++\sum_{j=1}^{n}a_{ij}^+L_j\\ &+\sum_{j=1}^{n}b_{ij}^+l_j\Big)\Big\}\|\phi\|_X.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}&\|\Phi\phi-\phi_0\|_X\\ =\ &\max\left\{\|\Phi\phi-\phi_0\|_0,\|(\Phi\phi-\phi_0)^{\Delta}\|_0\right\}\\ \le\ &\max_{1\le i\le n}\left\{\frac{1}{\delta_i^-},1+\frac{\delta_i^+}{\delta_i^-}\right\}\Big(\delta_i^+\tau_i^++\sum_{j=1}^{n}a_{ij}^+L_j\end{aligned}$$

$$+\sum_{j=1}^{n} b_{ij}^{+} l_j\Big)\|\phi\|_X$$
$$= r\|\phi\|_X \le \frac{rR}{1-r},$$

which implies $(\Phi\phi) \in X_0$, so the mapping Φ is a self-mapping from X_0 to X_0.

Finally, we prove that Φ is a contraction mapping. Taking $\phi, \psi \in X_0$, we have that

$$\begin{aligned}
&\|\Phi\phi - \Phi\psi\|_0 \\
&\le \max_{1\le i\le n}\Big\{\frac{1}{\delta_i^-}\Big(\delta_i^+\tau_i^+ + \sum_{j=1}^{n} a_{ij}^+ L_j \\
&\quad + \sum_{j=1}^{n} b_{ij}^+ l_j\Big)\Big\}\|\phi-\psi\|_X \\
&\le r\|\phi-\psi\|_X
\end{aligned}$$

and

$$\begin{aligned}
&\|(\Phi\phi - \Phi\psi)^\Delta\|_0 \\
&\le \max_{1\le i\le n}\Big\{\Big(1+\frac{\delta_i^+}{\delta_i^-}\Big)\Big(\delta_i^+\tau_i^+ + \sum_{j=1}^{n} a_{ij}^+ L_j \\
&\quad + \sum_{j=1}^{n} b_{ij}^+ l_j\Big)\Big\}\|\phi-\psi\|_X \\
&\le r\|\phi-\psi\|_X.
\end{aligned}$$

Noticing that $r < 1$, it means that Φ is a contraction mapping. Therefore, there exists a unique fixed point $\phi \in X_0$ such that $\Phi\phi = \phi$. Then system (1) has a unique almost periodic solution in the region $X_0 = \big\{\phi(t) \in X \big| \|\phi - \phi_0\| \le \frac{rR}{1-r}\big\}$. This completes the proof. □

Theorem 7. *Assume that* $(H_1) - (H_3)$ *hold, then system* (1) *has a unique almost periodic solution which is globally exponentially stable.*

Proof. From Theorem 6, we see that system (1) has at least one almost periodic solution $\bar{x}(t) = (\bar{x}_1(t), \bar{x}_2(t), \dots, \bar{x}_n(t))^T$. Suppose that $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ is an arbitrary solution. Set $y_i(t) = x_i(t) - \bar{x}_i(t), i = 1, 2, \dots, n$, then it follows from system (1) that

$$\begin{aligned}
&y_i^\Delta(t) \\
&= x_i^\Delta(t) - \bar{x}_i^\Delta(t) \\
&= -\delta_i(t)x_i(t-\tau_i(t)) + \sum_{j=1}^{n} a_{ij}(t) f_j(x_j(t-\alpha_{ij}(t))) \\
&\quad + \sum_{j=1}^{n} b_{ij}(t) g_j(x_j^\Delta(t-\beta_{ij}(t))) + I_i(t) \\
&\quad + \delta_i(t)\bar{x}_i(t-\tau_i(t)) - \sum_{j=1}^{n} a_{ij}(t) f_j(\bar{x}_j(t-\alpha_{ij}(t))) \\
&\quad - \sum_{j=1}^{n} b_{ij}(t) g_j(\bar{x}_j^\Delta(t-\beta_{ij}(t))) - I_i(t) \\
&= -\delta_i(t)y_i(t-\tau_i(t)) + \sum_{j=1}^{n} a_{ij}(t)\big[f_j(x_j(t-\alpha_{ij}(t))) \\
&\quad - f_j(\bar{x}_j(t-\alpha_{ij}(t)))\big] \\
&\quad + \sum_{j=1}^{n} b_{ij}(t)\big[g_j(x_j^\Delta(t-\beta_{ij}(t))) \\
&\quad - g_j(\bar{x}_j^\Delta(t-\beta_{ij}(t)))\big] \\
&= -\delta_i(t)y_i(t-\tau_i(t)) + \sum_{j=1}^{n} a_{ij}(t)\big[f_j(y_j(t-\alpha_{ij}(t)) \\
&\quad + \bar{x}_j(t-\alpha_{ij}(t))) - f_j(\bar{x}_j(t-\alpha_{ij}(t)))\big] \\
&\quad + \sum_{j=1}^{n} b_{ij}(t)\big[g_j(y_j^\Delta(t-\beta_{ij}(t)) + \bar{x}_j^\Delta(t-\beta_{ij}(t))) \\
&\quad - g_j(\bar{x}_j^\Delta(t-\beta_{ij}(t)))\big] \\
&= -\delta_i(t)y_i(t-\tau_i(t)) + \sum_{j=1}^{n} a_{ij}(t) F_j(y_j(t-\alpha_{ij}(t))) \\
&\quad + \sum_{j=1}^{n} b_{ij}(t) G_j(y_j^\Delta(t-\beta_{ij}(t))),
\end{aligned} \tag{7}$$

where $i = 1, 2, \dots, n$ and for $i, j = 1, 2, \dots, n$,

$$\begin{aligned}
&F_j(y_j(t-\alpha_{ij}(t))) \\
&= f_j\big(y_j(t-\alpha_{ij}(t)) + \bar{x}_j(t-\alpha_{ij}(t))\big) \\
&\quad - f_j(\bar{x}_j(t-\alpha_{ij}(t))), \\
&g_j(y_j^\Delta(t-\beta_{ij}(t))) \\
&= g_j\big(y_j^\Delta(t-\beta_{ij}(t)) + \bar{x}_j^\Delta(t-\beta_{ij}(t))\big) \\
&\quad - g_j(\bar{x}_j^\Delta(t-\beta_{ij}(t))).
\end{aligned}$$

It follows from (H_2) that for $i, j = 1, 2, \dots, n$,

$$\begin{aligned}
\big|F_j(y_j(t-\alpha_{ij}(t)))\big| &\le L_j\big|y_j(t-\alpha_{ij}(t))\big|, \\
\big|G_j(y_j^\Delta(t-\beta_{ij}(t)))\big| &\le l_j\big|y_j^\Delta(t-\beta_{ij}(t))\big|.
\end{aligned}$$

The initial condition of (7) is

$$\psi_i(t) = \varphi_i(t) - \bar{\varphi}_i(t),\ t \in [-\tau, 0]_{\mathbb{T}}, i = 1, 2, \dots, n.$$

Let H_i and K_i be defined by

$$\begin{aligned}
&H_i(\omega) \\
&= \delta_i^- - \omega - \exp(\omega \sup_{s\in\mathbb{T}} \mu(s))\Big(\delta_i^+\tau_i^+ \exp(\omega\tau_i^+)
\end{aligned}$$

$$+\sum_{j=1}^{n} a_{ij}^{+} L_j \exp(\omega\alpha_{ij}^{+}) + \sum_{j=1}^{n} b_{ij}^{+} l_j \exp(\omega\beta_{ij}^{+})\Big),$$
$$i = 1, 2, \ldots, n$$

and

$$\begin{aligned} & k_i(\omega) \\ = \ & \delta_i^- - \omega - \big(\delta_i^+ \exp(\omega \sup_{s\in\mathbb{T}} \mu(s)) \\ & +\delta_i^- - \omega\big)\Big(\delta_i^+ \tau_i^+ \exp(\omega\tau_i^+) \\ & + \sum_{j=1}^{n} a_{ij}^{+} L_j \exp(\omega\alpha_{ij}^{+}) + \sum_{j=1}^{n} b_{ij}^{+} l_j \exp(\omega\beta_{ij}^{+})\Big), \\ & i = 1, 2, \ldots, n. \end{aligned}$$

By (H_3), we can get

$$\begin{aligned} H_i(0) = \ & \delta_i^- - \Big(\delta_i^+ \tau_i^+ + \sum_{j=1}^{n} a_{ij}^{+} L_j + \sum_{j=1}^{n} b_{ij}^{+} l_j\Big) \\ & > 0, i = 1, 2, \ldots, n \end{aligned}$$

and

$$\begin{aligned} K_i(0) = \ & \delta_i^- - (\delta_i^+ + \delta_i^-)\Big(\delta_i^+ \tau_i^+ + \sum_{j=1}^{n} a_{ij}^{+} L_j \\ & + \sum_{j=1}^{n} b_{ij}^{+} l_j\Big) > 0, \ i = 1, 2, \ldots, n. \end{aligned}$$

Since H_i, K_i are continuous on $[0, +\infty)$ and $H_i(\omega), H_i(\omega) \to -\infty$ as $\omega \to +\infty$, so there exist $\omega_i, \omega_i^* > 0$ such that $H_i(\omega_i) = K_i(\omega_i^*) = 0$ and $H_i(\omega) > 0$ for $\omega \in (0, \omega_i)$, $K_i(\omega) > 0$ for $\omega \in (0, \omega_i^*), i = 1, 2, \ldots, n$.

Let $a = \min\{\omega_1, \omega_2, \ldots, \omega_n, \omega_1^*, \omega_2^*, \ldots, \omega_n^*\}$, we have $H_i(a) \geq 0, K_i(a) \geq 0, i = 1, 2, \ldots, n$. Then, there exists a positive constant $\lambda \in (0, \min\{a, \min_{1\leq i\leq n}\{\delta_i^-\}\})$ such that

$$H_i(\lambda) > 0, \ K_i(\lambda) > 0, \ i = 1, 2, \ldots, n,$$

which implies that

$$\begin{aligned} & \frac{\exp(\lambda \sup_{s\in\mathbb{T}} \mu(s))}{\delta_i^- - \lambda}\Big(\delta_i^+ \tau_i^+ \exp(\lambda\tau_i^+) \\ & + \sum_{j=1}^{n} a_{ij}^{+} L_j \exp(\lambda\alpha_{ij}^{+}) + \sum_{j=1}^{n} b_{ij}^{+} l_j \exp(\lambda\beta_{ij}^{+})\Big) \\ & < 1 \end{aligned} \tag{8}$$

and

$$\begin{aligned} & \Big(1 + \frac{\delta_i^+ \exp(\lambda \sup_{s\in\mathbb{T}} \mu(s))}{\delta_i^- - \lambda}\Big)\Big(\delta_i^+ \tau_i^+ \exp(\lambda\tau_i^+) \\ & + \sum_{j=1}^{n} a_{ij}^{+} L_j \exp(\lambda\alpha_{ij}^{+}) \\ & + \sum_{j=1}^{n} b_{ij}^{+} l_j \exp(\lambda\beta_{ij}^{+})\Big) < 1, \\ & i = 1, 2, \ldots, n. \end{aligned} \tag{9}$$

Let

$$M = \max_{1\leq i\leq n}\Bigg\{\frac{\delta_i^-}{\delta_i^+ \tau_i^+ + \sum_{j=1}^{n} a_{ij}^{+} L_j + \sum_{j=1}^{n} b_{ij}^{+} l_j}\Bigg\},$$

by (H_3) we have $M > 1$. Thus

$$\begin{aligned} \frac{1}{M} < \ & \frac{\exp(\lambda \sup_{s\in\mathbb{T}} \mu(s))}{\delta_i^- - \lambda}\Big(\delta_i^+ \tau_i^+ \exp(\lambda\tau_i^+) \\ & + \sum_{j=1}^{n} a_{ij}^{+} L_j \exp(\lambda\alpha_{ij}^{+}) + \sum_{j=1}^{n} b_{ij}^{+} l_j \exp(\lambda\beta_{ij}^{+})\Big). \end{aligned}$$

Rewrite (7) in the form

$$\begin{aligned} & y_i^{\Delta}(t) + \delta_i(t) y_i(t) \\ = \ & \delta_i(t) \int_{t-\tau_i(t)}^{t} y_i^{\Delta}(\theta)\Delta\theta \\ & + \sum_{j=1}^{n} a_{ij}(t) F_j(y_j(t - \alpha_{ij}(t))) \\ & + \sum_{j=1}^{n} b_{ij}(t) G_j(y_j^{\Delta}(t - \beta_{ij}(t))), i = 1, 2, \ldots, n. \end{aligned} \tag{10}$$

Multiplying the both sides of (10) by $e_{-\delta_i}(t, \sigma(s))$ and integrating over $[0, t]_{\mathbb{T}}$, we can get

$$\begin{aligned} & y_i(t) \\ = \ & y_i(0) e_{-\delta_i}(t, 0) \\ & + \int_0^t e_{-\delta_i}(t, \sigma(s))\Big\{\delta_i(s) \int_{s-\tau_i(s)}^{s} y_i^{\Delta}(\theta)\Delta\theta \\ & + \sum_{j=1}^{n} a_{ij}(s) F_j(y_j(s - \alpha_{ij}(s))) \\ & + \sum_{j=1}^{n} b_{ij}(s) G_j(y_j^{\Delta}(s - \beta_{ij}(s)))\Big\}\Delta s, \\ & i = 1, 2, \ldots, n. \end{aligned} \tag{11}$$

It is easy to see that

$$\|y(t)\|_X = \|\psi\|_X \leq M e_{\ominus\lambda}(t, 0)\|\psi\|_X, \forall t \in [-\tau, 0]_{\mathbb{T}}.$$

Now, we claim that

$$\|y(t)\|_X \leq M e_{\ominus\lambda}(t, 0)\|\psi\|_X, \forall t \in (0, +\infty)_{\mathbb{T}}. \tag{12}$$

To prove (12), we first show that for any $p > 1$, the following inequality holds

$$\|y(t)\|_X < pMe_{\ominus\lambda}(t,0)\|\psi\|_X,\ \forall t \in (0,+\infty)_{\mathbb{T}}. \quad (13)$$

If (13) is not true, then there exist a $t_1 \in (0,+\infty)_{\mathbb{T}}$ and $i_1, i_2 \in \{1,2,\dots,n\}$ such that

$$\begin{aligned} |y(t_1)\|_X &= \max\{\|y(t_1)\|_0, \|y^{\Delta}(t_1)\|_0\} \\ &= \max\{|y_{i_1}(t_1)|, |y^{\Delta}_{i_2}(t_1)|\} \\ &\geq pMe_{\ominus\lambda}(t_1,0)\|\psi\|_X \end{aligned}$$

and

$$\|y(t)\|_X \leq pMe_{\ominus\lambda}(t,0)\|\psi\|_X,\ t \in [-\tau, t_1]_{\mathbb{T}}.$$

Therefore, there must exist a constant $c \geq 1$ such that

$$\begin{aligned} |y(t_1)\|_X &= \max\{\|y(t_1)\|_0, \|y^{\Delta}(t_1)\|_0\} \\ &= \max\{|y_{i_1}(t_1)|, |y^{\Delta}_{i_2}(t_1)|\} \\ &= cpMe_{\ominus\lambda}(t_1,0)\|\psi\|_X \end{aligned} \quad (14)$$

and

$$\|y(t)\|_X \leq cpMe_{\ominus\lambda}(t,0)\|\psi\|_X, t \in [-\tau, t_1]_{\mathbb{T}}. \quad (15)$$

By (11), (14), (15) and (H_1)-(H_3), we can obtain

$$\begin{aligned} & |y_{i_1}(t_1)| \\ \leq\ & \|\psi\|_X e_{-\delta_{i_1}}(t_1,0) + cpMe_{\ominus\lambda}(t_1,0)\|\psi\|_X \\ & \times \int_0^{t_1} e_{-\delta_{i_1}}(t_1,\sigma(s)) e_{\lambda}(t_1,\sigma(s)) \\ & \times \Big\{ \delta^+_{i_1} \int_{s-\tau_{i_1}(s)}^{s} e_{\lambda}(\sigma(s),\theta)\Delta\theta \\ & + \sum_{j=1}^n a^+_{i_1 j} L_j e_{\lambda}(\sigma(s), s-\alpha_{i_1 j}(s)) \\ & + \sum_{j=1}^n b^+_{i_1 j} l_j e_{\lambda}(\sigma(s), s-\beta_{i_1 j}(s)) \Big\} \Delta s \\ \leq\ & \|\psi\|_X e_{-\delta_{i_1}}(t_1,0) + cpMe_{\ominus\lambda}(t_1,0)\|\psi\|_X \\ & \times \int_0^{t_1} e_{-\delta_{i_1}\oplus\lambda}(t_1,\sigma(s)) \\ & \times \Big\{ \delta^+_{i_1}\tau^+_{i_1} e_{\lambda}(\sigma(s), s-\tau_{i_1}(s)) \\ & + \sum_{j=1}^n a^+_{i_1 j} L_j e_{\lambda}(\sigma(s), s-\alpha_{i_1 j}(s)) \\ & + \sum_{j=1}^n b^+_{i_1 j} l_j e_{\lambda}(\sigma(s), s-\beta_{i_1 j}(s)) \Big\} \Delta s \\ \leq\ & \|\psi\|_X e_{-\delta_{i_1}}(t_1,0) + cpMe_{\ominus\lambda}(t_1,0)\|\psi\|_X \\ & \times \int_0^{t_1} e_{-\delta_{i_1}\oplus\lambda}(t_1,\sigma(s)) \\ & \times \Big\{ \delta^+_{i_1}\tau^+_{i_1} \exp\big[\lambda(\tau^+_{i_1} + \sup_{s\in\mathbb{T}}\mu(s))\big] \\ & + \sum_{j=1}^n a^+_{i_1 j} L_j \exp\big[\lambda(\alpha^+_{i_1 j} + \sup_{s\in\mathbb{T}}\mu(s))\big] \\ & + \sum_{j=1}^n b^+_{i_1 j} l_j \exp\big[\lambda(\beta^+_{i_1 j} + \sup_{s\in\mathbb{T}}\mu(s))\big] \Big\} \Delta s \\ =\ & cpMe_{\ominus\lambda}(t_1,0)\|\psi\|_X \Big\{ \frac{1}{pM} e_{-\delta_{i_1}\oplus\lambda}(t_1,0) \\ & + \exp\big(\lambda \sup_{s\in\mathbb{T}}\mu(s)\big) \Big[\delta^+_{i_1}\tau^+_{i_1}\exp(\lambda\tau^+_{i_1}) \\ & + \sum_{j=1}^n a^+_{i_1 j} L_j \exp(\lambda\alpha^+_{i_1 j}) + \sum_{j=1}^n b^+_{i_1 j} l_j \exp(\lambda\beta^+_{i_1 j}) \Big] \\ & \times \int_0^{t_1} e_{-\delta_{i_1}\oplus\lambda}(t_1,\sigma(s))\Delta s \Big\} \\ <\ & cpMe_{\ominus\lambda}(t_1,0)\|\psi\|_X \Big\{ \frac{1}{M} e_{-(\delta^-_{i_1}-\lambda)}(t_1,0) \\ & + \exp\big(\lambda \sup_{s\in\mathbb{T}}\mu(s)\big) \Big[\delta^+_{i_1}\tau^+_{i_1}\exp(\lambda\tau^+_{i_1}) \\ & + \sum_{j=1}^n a^+_{i_1 j} L_j \exp(\lambda\alpha^+_{i_1 j}) + \sum_{j=1}^n b^+_{i_1 j} l_j \exp(\lambda\beta^+_{i_1 j}) \Big] \\ & \frac{1}{-(\delta^-_{i_1}-\lambda)} \int_0^{t_1} -(\delta^-_{i_1}-\lambda) e_{-(\delta^-_{i_1}-\lambda)}(t_1,\sigma(s))\Delta s \Big\} \\ =\ & cpMe_{\ominus\lambda}(t_1,0)\|\psi\|_X \Big\{ \Big[\frac{1}{M} - \frac{\exp\big(\lambda \sup_{s\in\mathbb{T}}\mu(s)\big)}{\delta^-_{i_1}-\lambda} \\ & \times \Big(\delta^+_{i_1}\tau^+_{i_1}\exp(\lambda\tau^+_{i_1}) + \sum_{j=1}^n a^+_{i_1 j} L_j \exp(\lambda\alpha^+_{i_1 j}) \\ & + \sum_{j=1}^n b^+_{i_1 j} l_j \exp(\lambda\beta^+_{i_1 j}) \Big) \Big] e_{-(\delta^-_{i_1}-\lambda)}(t_1,0) \\ & + \frac{\exp\big(\lambda \sup_{s\in\mathbb{T}}\mu(s)\big)}{\delta^-_{i_1}-\lambda} \Big(\delta^+_{i_1}\tau^+_{i_1}\exp(\lambda\tau^+_{i_1}) \\ & + \sum_{j=1}^n a^+_{i_1 j} L_j \exp(\lambda\alpha^+_{i_1 j}) + \sum_{j=1}^n b^+_{i_1 j} l_j \exp(\lambda\beta^+_{i_1 j}) \Big) \Big\} \\ <\ & cpMe_{\ominus\lambda}(t_1,0)\|\psi\|_X. \end{aligned} \quad (16)$$

It follows from Lemma 4 and (11) that for $i = 1,2,\dots,n$,

$$y^{\Delta}_i(t)$$

$$= -\delta_i(t)y_i(0)e_{-\delta_i}(t,0) + \bigg(\delta_i(t)\int_{t-\tau_i(t)}^{t} y_i^{\Delta}(\theta)\Delta\theta$$
$$+\sum_{j=1}^{n} a_{ij}(t)F_j(y_j(t-\alpha_{ij}(t)))$$
$$+\sum_{j=1}^{n} b_{ij}(t)G_j(y_j^{\Delta}(t-\beta_{ij}(t)))\bigg)$$
$$+\int_0^t -\delta_i(t)e_{-\delta_i}(t,\sigma(s))\bigg\{\delta_i(s)\int_{s-\tau_i(s)}^{s} y_i^{\Delta}(\theta)\Delta\theta$$
$$+\sum_{j=1}^{n} a_{ij}(s)F_j(y_j(s-\alpha_{ij}(s)))$$
$$+\sum_{j=1}^{n} b_{ij}(s)G_j(y_j^{\Delta}(s-\beta_{ij}(s)))\bigg\}\Delta s. \qquad (17)$$

Then, by (14), (15) and (17), we have

$$|y_{i_2}^{\Delta}(t_1)|$$
$$\le \delta_{i_2}^{+}e_{-\delta_{i_2}}(t_1,0)\|\psi\|_X + cpMe_{\ominus\lambda}(t_1,0)\|\psi\|_X$$
$$\times\bigg(\delta_{i_2}^{+}\int_{t_1-\tau_{i_2}(t_1)}^{t_1} e_{\lambda}(t_1,\theta)\Delta\theta$$
$$+\sum_{j=1}^{n} a_{i_2j}^{+}L_je_{\lambda}(t_1,t_1-\alpha_{i_2j}(t_1))$$
$$+\sum_{j=1}^{n} b_{i_2j}^{+}l_je_{\lambda}(t_1,t_1-\beta_{i_2j}(t_1))\bigg)$$
$$+\delta_{i_2}^{+}cpMe_{\ominus\lambda}(t_1,0)\|\psi\|_X\int_0^{t_1} e_{-\delta_{i_2}}(t_1,\sigma(s))$$
$$\times e_{\lambda}(t_1,\sigma(s))\bigg\{\delta_{i_2}^{+}\int_{s-\tau_{i_2}(s)}^{s} e_{\lambda}(\sigma(s),\theta)\Delta\theta$$
$$+\sum_{j=1}^{n} a_{i_2j}^{+}L_je_{\lambda}(\sigma(s),s-\alpha_{i_2j}(s))$$
$$+\sum_{j=1}^{n} b_{i_2j}^{+}l_je_{\lambda}(\sigma(s),s-\beta_{i_2j}(s))\bigg\}\Delta s$$
$$\le \delta_{i_2}^{+}e_{-\delta_{i_2}}(t_1,0)\|\psi\|_X + cpMe_{\ominus\lambda}(t_1,0)\|\psi\|_X$$
$$\times\bigg(\delta_{i_2}^{+}\tau_{i_2}^{+}e_{\lambda}(t_1,t_1-\tau_{i_2}(t_1))$$
$$+\sum_{j=1}^{n} a_{i_2j}^{+}L_je_{\lambda}(t_1,t_1-\alpha_{i_2j}(t_1))$$
$$+\sum_{j=1}^{n} b_{i_2j}^{+}l_je_{\lambda}(t_1,t_1-\beta_{i_2j}(t_1))\bigg)$$
$$+\delta_{i_2}^{+}cpMe_{\ominus\lambda}(t_1,0)\|\psi\|_X\int_0^{t_1} e_{-\delta_{i_2}\oplus\lambda}(t_1,\sigma(s))$$
$$\times\bigg\{\delta_{i_2}^{+}\tau_{i_2}^{+}e_{\lambda}(\sigma(s),s-\tau_{i_2}(s))$$
$$+\sum_{j=1}^{n} a_{i_2j}^{+}L_je_{\lambda}(\sigma(s),s-\alpha_{i_2j}(s))$$
$$+\sum_{j=1}^{n} b_{i_2j}^{+}l_je_{\lambda}(\sigma(s),s-\beta_{i_2j}(s))\bigg\}\Delta s$$
$$\le \delta_{i_2}^{+}e_{-\delta_{i_2}}(t_1,0)\|\psi\|_X + cpMe_{\ominus\lambda}(t_1,0)\|\psi\|_X$$
$$\times\bigg(\delta_{i_2}^{+}\tau_{i_2}^{+}\exp(\lambda\tau_{i_2}^{+}) + \sum_{j=1}^{n} a_{i_2j}^{+}L_j\exp(\lambda\alpha_{i_2j}^{+})$$
$$+\sum_{j=1}^{n} b_{i_2j}^{+}l_j\exp(\lambda\beta_{i_2j}^{+})\bigg)\bigg(1+\delta_{i_2}^{+}\exp(\lambda\sup_{s\in\mathbb{T}}\mu(s))$$
$$\times\int_0^{t_1} e_{-\delta_{i_2}\oplus\lambda}(t_1,\sigma(s))\Delta s\bigg)$$
$$< cpMe_{\ominus\lambda}(t_1,0)\|\psi\|_X\bigg\{\frac{\delta_{i_2}^{+}}{M}e_{-\delta_{i_2}\oplus\lambda}(t_1,0)$$
$$+\bigg(\delta_{i_2}^{+}\tau_{i_2}^{+}\exp(\lambda\tau_{i_2}^{+}) + \sum_{j=1}^{n} a_{i_2j}^{+}L_j\exp(\lambda\alpha_{i_2j}^{+})$$
$$+\sum_{j=1}^{n} b_{i_2j}^{+}l_j\exp(\lambda\beta_{i_2j}^{+})\bigg)\bigg(1+\delta_{i_2}^{+}\exp(\lambda\sup_{s\in\mathbb{T}}\mu(s))$$
$$\times\int_0^{t_1} e_{-\delta_{i_2}\oplus\lambda}(t_1,\sigma(s))\Delta s\bigg)\bigg\}$$
$$\le cpMe_{\ominus\lambda}(t_1,0)\|\psi\|_X\bigg\{\frac{\delta_{i_2}^{+}}{M}e_{-(\delta_{i_2}^{-}-\lambda)}(t_1,0)$$
$$+\bigg(\delta_{i_2}^{+}\tau_{i_2}^{+}\exp(\lambda\tau_{i_2}^{+}) + \sum_{j=1}^{n} a_{i_2j}^{+}L_j\exp(\lambda\alpha_{i_2j}^{+})$$
$$+\sum_{j=1}^{n} b_{i_2j}^{+}l_j\exp(\lambda\beta_{i_2j}^{+})\bigg)\bigg(1+\delta_{i_2}^{+}\exp(\lambda\sup_{s\in\mathbb{T}}\mu(s))$$
$$\times\frac{1}{-(\delta_{i_2}^{-}-\lambda)}\big(e_{-(\delta_{i_2}^{-}-\lambda)}(t_1,0)-1\big)\bigg)\bigg\}$$
$$\le cpMe_{\ominus\lambda}(t_1,0)\|\psi\|_X\bigg\{\bigg[\frac{1}{M} - \frac{\exp(\lambda\sup_{s\in\mathbb{T}}\mu(s))}{\delta_{i_2}^{-}-\lambda}$$
$$\times\bigg(\delta_{i_2}^{+}\tau_{i_2}^{+}\exp(\lambda\tau_{i_2}^{+}) + \sum_{j=1}^{n} a_{i_2j}^{+}L_j\exp(\lambda\alpha_{i_2j}^{+})$$
$$+\sum_{j=1}^{n} b_{i_2j}^{+}l_j\exp(\lambda\beta_{i_2j}^{+})\bigg)\bigg]\delta_{i_2}^{+}e_{-(\delta_{i_2}^{-}-\lambda)}(t_1,0)$$
$$+\bigg(1+\frac{\delta_{i_2}^{+}\exp(\lambda\sup_{s\in\mathbb{T}}\mu(s))}{\delta_{i_2}^{-}-\lambda}\bigg)$$
$$\times\bigg(\delta_{i_2}^{+}\tau_{i_2}^{+}\exp(\lambda\tau_{i_2}^{+}) + \sum_{j=1}^{n} a_{i_2j}^{+}L_j\exp(\lambda\alpha_{i_2j}^{+})$$

$$+\sum_{j=1}^{n} b_{i_2 j}^{+} l_j \exp(\lambda\beta_{i_2 j}^{+})\Big)\Big\}$$
$$< cpMe_{\ominus\lambda}(t_1, 0)\|\psi\|_X. \quad (18)$$

Together with (16) and (18), then

$$\|y(t_1)\|_X < cpMe_{\ominus\lambda}(t_1, 0)\|\psi\|_X,$$

which contradicts (14), so (13) holds. Let $p \to 1$, then (12) holds. Therefore, the almost periodic solution of system (1) is globally exponentially stable. This completes the proof. □

4 An example

Consider the following neutral delay neural network on time scale $\mathbb{T}$:

$$\begin{aligned} x_i^{\Delta}(t) &= -\delta_i(t)x_i(t-\tau_i(t)) \\ &\quad +\sum_{j=1}^{3} a_{ij}(t)f_j(x_j(t-\alpha_{ij}(t))) \quad (19) \\ &\quad +\sum_{j=1}^{3} b_{ij}(t)g_j(x_j^{\Delta}(t-\beta_{ij}(t)))+I_i(t), \end{aligned}$$

where $i = 1, 2, 3$, and

$$\delta_1(t) = 0.55+0.1|\sin t|, \delta_2(t) = 0.6+0.2|\cos\sqrt{2}t|,$$
$$\delta_3(t) = 0.7+0.25|\sin t|, \tau_1(t) = \frac{1.5+|\sin\sqrt{2}t|}{250},$$
$$\tau_2(t) = \frac{2+0.5|\cos\sqrt{2}t|}{150},\ \tau_3(t) = \frac{3+0.5|\sin 2t|}{250},$$
$$(a_{ij}(t))_{3\times3} =$$
$$\begin{pmatrix} 0.10|\sin t| & 0.15|\cos\sqrt{2}t| & 0.06|\cos t| \\ 0.15|\cos t| & 0.12|\cos\sqrt{3}t| & 0.05|\sin t| \\ 0.13|\sin t| & 0.08|\sin\sqrt{5}t| & 0.09|\sin t| \end{pmatrix},$$
$$(b_{ij}(t))_{3\times3} =$$
$$\begin{pmatrix} 0.15|\sin t| & 0.06|\sin\sqrt{2}t| & 0.05|\cos t| \\ 0.05|\sin t| & 0.03|\cos\sqrt{3}t| & 0.07|\sin t| \\ 0.15|\cos t| & 0.05|\sin\sqrt{5}t| & 0.08|\cos t| \end{pmatrix},$$

$$\begin{aligned} f_1(x) &= 0.3|x|,\ f_2(x) = 0.5|\sin x|,\ f_3(x) = |x|, \\ g_1(x) &= 0.6|\sin x|,\ g_2(x) = 1.5|x|, \\ g_3(x) &= 1.5|\cos x|. \end{aligned}$$

Let $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$. Take $\alpha_{ij} > 0, \beta_{ij} > 0, I_i(i, j = 1, 2, 3)$ to be arbitrary almost periodic functions. By a direct calculation, we can get $r = 0.7492 < 1$ and $-\delta_i \in \mathcal{R}^+$. By Theorem 6 and Theorem 7, system (19) has a unique almost periodic solution which is globally exponentially stable.

5 Conclusion

This paper is concerned with a neutral-type neural networks with time-varying delay in the leakage term on time scales (system (1)), based on the theory of calculus on time scales, by using the exponential dichotomy of linear dynamic equations and Banach's fixed point theorem as well as some mathematical methods, some sufficient conditions are obtained for the existence and exponential stability of almost periodic solution of system (1). From Theorem 6 and Theorem 7, we can see that the existence and exponential stability of almost periodic solutions for system (1) only depends on time delays τ_i (the delays in the leakage term) and does not depend on time delays α_{ij} and β_{ij}. These results have important leading significance in designs and applications of such neural networks.

The results obtained in this paper can be applied to the analysis of the periodic (and almost periodic) dynamical regimes into the dynamical systems with strange attractors [20], and to non-autonomous solutions' analysis of non-autonomous gyrostats' systems [21]. Also, one may consider many other systems, see [22-25].

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