Low Thrust Sub-Optimal Transfer Trajectories to the Moon

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Abstract: - In the present research the problem of transferring a spacecraft from the Earth to the Moon with minimum fuel consumption is considered. The Two-Body model is assumed to be a valid mathematical representation for the dynamics during the transfer. The spacecraft starts in a Low Earth Orbit and then goes to a Polar Orbit around the Moon. In previous publications, impulsive and optimal low thrust maneuvers were used to perform this transfer. In the present paper this research is extended to consider two types of sub-optimal maneuvers: the first one using a linear and the second one using a quadratic form for the direction of the thrust. To obtain those low thrust sub-optimal maneuvers, the Euler-Lagrange equations are also used here, as done before in the optimal approach. They give a set of differential equations that can be used to solve numerically the problem. The results obtained here show that the extra fuel expenditure caused by using both sub-optimal approaches for the control are small, so this is an interesting approach if a simple implementation for the hardware is desired. Two types of missions are studied, one with a single spacecraft and another one with a set of two spacecrafts.

Key Words: Orbital Maneuvers, Astrodynamics, Space Trajectories, Low Thrust, Sub-optimal Control.

1 Introduction

Missions to the Moon are one of the most interesting activities in aerospace engineering. The construction of a manned lunar base is one of the next steps in space. In this context, missions to the Moon to explore its surface are very important. In particular, a polar orbit is very interesting, because it can make measurements in parts of the Moon that is not yet known.

So, the Lunar Polar Orbiter mission is an interesting idea to be considered. It is made by one or two spacecrafts that are in a polar orbit around the Moon. The goal is to make measurements in the surface of the Moon and in its neighborhood. The data collected is then used for several tasks, like the site selection of a lunar base; improvements of trajectory calculations for orbits around the Moon; study of mineral exploitation, etc.

This paper makes a study of trajectories that can be used to go to the Moon. It is assumed that the vehicle starts its trajectory in a Low Earth Orbit that is circular with semi-major axis of 6570 km and in the plane of the motion of the Moon around the Earth. The mathematical model given by the Two-Body problem is valid for each phase of the mission.

The main goal is to look in more details the differences, in terms of fuel consumption, between the two options of engines: impulsive and low thrust. In particular, to complement results available in the literature, as shown in the next section, to study a new approach to this transfer: the low thrust maneuver using sub-optimal control, where the direction of the thrust is assumed to be a linear or a quadratic function of the range angle, that is a variable that replaces the time in the equations of motion.

For the general problem of orbital maneuvers, many alternatives are studied in the literature, considering different conditions. An important field of research considers the so called low thrust maneuver. In this model, a force with low magnitude is applied during a finite time. To find the trajectory of the spacecraft is necessary to integrate the equations of motion. There are many results in the literature considering this model, beginning with the works of Lawden ([1], [2]). Many other more recent researches are available dealing with this model, like references [3] to [14].

A second approach uses the idea of an impulsive thrust, which is the case where the thrust is assumed to have an infinity magnitude. Several papers used this approach, like references [15] to [35].

Later, the idea of gravitational capture has been considered. In this situation the perturbation of a third-body [36] can be used to decrease the fuel consumption of an orbital maneuver. References [37] to [42] explain this idea in some detail.

Another approach that appeared in the literature, to find alternatives to reduce fuel expenditure in space missions, is the swing by maneuvers. References [43] to [58] show more details, as well as missions using this technique.

Looking in more detail for researches directly involved in transfers to the Moon, to have an idea of the state of the art of this particular problem, it is possible to find several different approaches to solve it. Belbruno and Miller [59] shows a direct application of the gravitational capture concept mentioned before in transfers to the Moon, with the objective of saving fuel in the mission by getting some energy from the gravity of the Sun and the Earth. Using the same idea of low thrust propulsion explored in the present paper, but dividing the mission in three stages, Pierson and Kluever [60] shows some options for this type of trajectory. After that, Betts and Erb [61] used an approach that combines a discretization of the trajectory with a nonlinear programming algorithm to obtain the Earth-Moon transfers. Song et al. [62] design a technique for a sub-optimal transfer from the Earth to the Moon using a continuous low thrust with variable magnitude. They combine analytical and numerical methods to formulate the optimization problem. It shows to be a good choice, if this more complicated propulsion system can be used. Then, Assadian and Pourtakdoust [63] studied optimal trajectories for an Earth-Moon trip using impulsive maneuvers combined with genetic algorithms. The transfer time and the total increment of velocity are the objective functions to be minimized. Another interesting research in the one made by Fazelzadeh and Varzandian [64], that studied the minimum-time Earth–Moon and Moon–Earth transfers using a continuous thrust. The trajectories are obtained by the time-domain finite element method. The formulation of the problem uses the restricted three-body problem. The performance index is the minimum-time problem with free final time. There are also some papers considering the invariant manifolds theory to obtain the Earth-Moon transfers. References [65] to [67] show some aspects of this approach.

In particular, reference [68] to [70] shows studies of those trajectories considering the optimal maneuver and some other aspects of the transfers to the Moon shown in the present paper.

An important aspect of the specific mission studied here is shown in reference [69], where the two scenarios considered here are described and explained in detail:

1) A single mission, with the spacecraft in a circular orbit around the Moon at an altitude of 100 km and 90 degrees of inclination;

2) A double mission, with the two spacecrafts in different orbits: the main one in a circular orbit around the Moon at an altitude of 100 km and 90 degrees of inclination and a sub-satellite (without propulsion) in a elliptical orbit around the Moon with semi-major axis of 3000 km, eccentricity of 0.37, argument of perigee of 0.25 degrees West and inclination of 90 degrees.

So, the present research extends previous studies mentioned above to cover the sub-optimal cases of linear and quadratic control. The sub-optimal solutions consume more fuel, but have an easier implementation of the hardware and are more reliable in terms of implementation. Those advantages comes from the fact that the engines need to follow a direction that has its motion restricted to a specified rule (linear or quadratic) and are not required to make fast and unpredictable motions. A propulsion system like this is less subjected to fails and requires less efforts from the attitude control system. An integrated study of orbit and attitude control is a good suggestion for future work, but is considered out of the scope of the present paper. In this way, the question of how much more fuel needs to be used to compensate these simplifications is very important.

2 Impulsive Maneuvers

The first step to study the Earth-Moon transfers using the sub-optimal low thrust maneuvers is to take a look at the more basic impulsive maneuvers, where it is assumed that the thrust is strong enough to provide an almost instantaneous change of the velocity. This is necessary in order to find the best transfer trajectories that will then be achieved using the low thrust propulsion. This study was already made and presented in a WSEAS conference in 2008 (see reference [68]). In order to compare the different strategies for the transfer, those results are required, so the main points are repeated here.

As explained before, the mission starts with the spacecraft leaving the Earth from a parking orbit that is circular with an altitude of 200 km and in the plane of the motion of the Moon around the Earth. The goal is to calculate the velocity increment, transfer time and the mass of fuel necessary for several trajectories, to be able to choose one for a more detailed analysis.

The present research considers two scenarios for the mission: having only one satellite or considering the existence of a sub-satellite, so constituting a mission with two spacecrafts. Both solutions are analyzed in more details here. To be compatible with reference [68], a final mass of 100 kg around the Moon is also used here for the single mission and a final mass of 120 kg is used for the double mission. So, the present paper adds the possibility of sub-optimal solutions, including a linear and a quadratic form for the control. Then, the savings that can be obtained by these less complex forms of the control over the impulsive maneuver, as well as the increasing in fuel expenditure that these implementations have with respect to the results obtained by the optimal maneuver calculated in reference [68], are shown.

Using the Two-Body model for the system composed by the Earth and the spacecraft, we can obtain the transfer time for different trajectories [68]. The orbits are assumed to be elliptical with perigee of 6570 km. Table 1 shows the more interesting trajectories found and Fig. 1 shows a sketch of those trajectories, that are all planar, since the size of the Moon is very small when compared to the Earth-Moon distance and the fact that the final orbit is polar does not require the transfers to be non planar.

Table 1 - Orbital parameters and transfer time for trajectories to the Moon [68].

Orbit	Semi-major axis (km)	Eccentricity	Transfer Time in hr. (days)
1	500000	0.986	58.5 (2.43)
2	400000	0.983	61.0 (2.54)
3	300000	0.978	67.0 (2.80)
4	250000	0.974	73.9 (3.08)
5	230000	0.971	77.0 (3.21)
6	220000	0.970	83.2 (3.47)
7	200000	0.967	100.0 (4.17)
8	195485	0.960	119.6 (4.98)

Assuming now that the propulsion system is impulsive, the magnitude of the total velocity increment necessary for the maneuvers can be obtained. Reference [68] shows the detailed calculations. The same calculations are repeated for the double mission, but the results are omitted here because they are very similar.

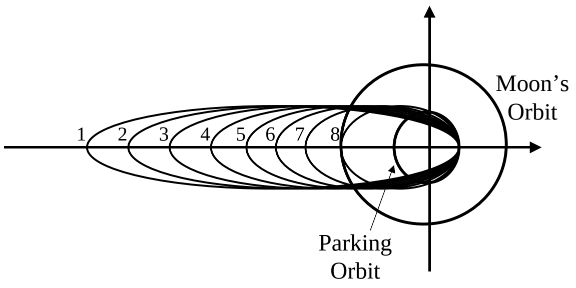

Fig. 1 – Sketch of the orbits.

To follow the goals of the present paper and to make a comparison of the impulsive approach with the low thrust idea, maneuver 6 was chosen because it has a good balance between consumption and time required for the maneuver. The ΔV for this maneuver is 4.05 km/s and the fuel consumed is 237 kg. To obtain the fuel consumption we assumed that the specific impulse of the fuel is 340 s. The equation that relates those two quantities is:

$$\Delta V = g_0 I_{sp} \ln\left(\frac{m_0}{m_f}\right) \tag{1}$$

where g_0 is the acceleration of gravity at the surface of the Earth, I_{sp} is the specific impulse of the engine, m_0 is the initial mass and m_f is the final mass of the spacecraft.

3 Low-Thrust Trajectory

The study of those maneuvers is now extended to consider the low thrust case, under different hypothesis for the control law. In this situation, a continuous force is applied to the spacecraft in order to reach the final orbit desired.

The spacecraft is assumed to travel in a planar Keplerian motion perturbed only by the thrust. This thrust has the following characteristics:

i) Constant magnitude;
ii) Fixed Ejection Velocity;
iii) Angular motions that can be controlled;
iv) Operation in on-off mode.

Then we search for the direction of the force to be applied at every instant of time, as well as the fuel consumed. The duration of the transfer is also obtained in this phase of the study.

This is an optimal control problem, formulated as:

Objective Function: M_f,

where M_f is the final mass of the spacecraft, and it has to be maximized by the control vector **u**(.);

Constraints: Equations of motion, initial and final orbits, and limits in the direction of the force;

Given: gravitational force field, initial and final orbits of the satellite, etc.

To avoid singularities in the equations of motion, the following set of variables are used [11], [12], [68], [70], [71]:

$$X_1 = [a(1-e^2)/\mu]^{1/2} \quad (2)$$

$$X_2 = e\cos(\omega-\phi) \quad (3)$$

$$X_3 = e\sin(\omega-\phi) \quad (4)$$

$$X_4 = (\text{Fuel Consumed})/m_0 \quad (5)$$

$$X_5 = t \quad (6)$$

$$X_6 = \cos(i/2)\cos((\Omega+\phi)/2) \quad (7)$$

$$X_7 = \sin(i/2)\cos((\Omega-\phi)/2) \quad (8)$$

$$X_8 = \sin(i/2)\sin((\Omega-\phi)/2) \quad (9)$$

$$X_9 = \cos(i/2)\sin((\Omega+\phi)/2) \quad (10)$$

$$\phi = \nu + \omega - s \quad (11)$$

where:

s = range angle of the spacecraft

a = semi-major axis

e = eccentricity

i = inclination

ω = argument of periapsis

ν = true anomaly of the spacecraft

Ω = argument of the ascending node

The equations of motion in those variables are [11], [12], [68], [70], [71]:

$$dX_1/ds = f_1 = SiX_1F_1 \quad (12)$$

$$dX_2/ds = f_2 = Si\{[(Ga+1)\cos(s)+X_2]F_1+GaF_2\sin(s) \quad (13)$$

$$dX_3/ds = f_3 = Si\{[(Ga+1)\sin(s)+X_3]F_1-GaF_2\cos(s) \quad (14)$$

$$dX_4/ds = f_4 = SiGaF(1-X_4)/(X_1W) \quad (15)$$

$$dX_5/ds = f_5 = SiGa(1-X_4)m_0/X_1 \quad (16)$$

$$dX_6/ds = f_6 = -SiF_3[X_7\cos(s)+X_8\sin(s)]/2 \quad (17)$$

$$dX_7/ds = f_7 = SiF_3[X_6\cos(s)-X_9\sin(s)]/2 \quad (18)$$

$$dX_8/ds = f_8 = SiF_3[X_9\cos(s)+X_6\sin(s)]/2 \quad (19)$$

$$dX_9/ds = f_9 = SiF_3[X_7\sin(s)-X_8\cos(s)]/2 \quad (20)$$

where:

$$Ga = 1 + X_2\cos(s) + X_3\sin(s) \quad (21)$$

$$Si = (\mu X_1^4)/[Ga^3m_0(1-X_4)] \quad (22)$$

$$F_1 = F\cos(\alpha)\cos(\beta) \quad (23)$$

$$F_2 = F\sin(\alpha)\cos(\beta) \quad (24)$$

$$F_3 = F\sin(\beta) \quad (25)$$

where:

F = magnitude of the force

W = ejection velocity of the gases

ν = true anomaly of the spacecraft.

The equations for the Lagrange multipliers are given below.

$$\frac{dp_1}{ds} = -\frac{4\sum_{j=1}^{9} p_j f_j + p_1 f_1 - p_4 f_4 - p_5 f_5}{X_1} \quad (26)$$

$$\frac{dp_2}{ds} = \frac{\cos(s)}{Ga}\left[3\sum_{j=1}^{9} p_j f_j - p_4 f_4 - p_5 f_5\right] - Sip_2F_1 - Sicos^2(s)(p_2F_1 - p_3F_2) - Sicos(s)sin(s)(p_2F_2 + p_3F_1) \quad (27)$$

$$\frac{dp_3}{ds} = \frac{\sin(s)}{Ga}\left[3\sum_{j=1}^{9} p_j f_j - p_4 f_4 - p_5 f_5\right] - Sip_3F_1 - Sicos(s)sin(s)(p_2F_1 - p_3F_2) - Sisin^2(s)(p_2F_2 + p_3F_3) \quad (28)$$

$$\frac{dp_4}{ds} = -\left[\frac{\sum_{j=1}^{9} p_j f_j - p_4 f_4 - p_5 f_5}{m_0(1-X_4)}\right] \quad (29)$$

$$\frac{dp_5}{ds} = 0 \quad (30)$$

$$\frac{dp_6}{ds} = \frac{-SiF_3[p_7\cos(s) + p_8\sin(s)]}{2} \quad (31)$$

$$\frac{dp_7}{ds} = \frac{SiF_3[p_6\cos(s) - p_9\sin(s)]}{2} \quad (32)$$

$$\frac{dp_8}{ds} = \frac{SiF_3[p_6\sin(s) + p_9\cos(s)]}{2} \quad (33)$$

$$\frac{dp_9}{ds} = \frac{-SiF_3[p_8\cos(s) - p_7\sin(s)]}{2} \quad (34)$$

The number of the new state variables is nine and this value is larger than the minimum number required to specify the system. It means that they are not independent.

This system is subjected to constraints in state and some of the Keplerian elements of the initial and final orbits may be specified. All the parameters (gravitational force field, initial values of the satellite, etc...) are assumed to be known.

Considering now the control, the sub-optimal approach uses a specific rule for the direction of the force to be applied. The most usual one is the linear approximation, where the direction is assumed to follow a straight line dependence with respect to the position of the satellite. In the present paper, this approximation is shown and also it goes up to an equation of the second order. This means that a quadratic parametrization is used for the control law, that are specified by the angles of pitch (α), the in-plane angle, and yaw (β), the out-of-plane angle. The equations for the quadratic form that specify this rule are given by:

$$\alpha = \alpha_0 + \alpha' * (s - s_0) + \alpha'' * (s - s_0)^2 \quad (35)$$

$$\beta = \beta_0 + \beta' * (s - s_0) + \beta'' * (s - s_0)^2 \quad (36)$$

where α_0, β_0, α', β', α'', β'' are the parameters that needs to be determined, s is the instantaneous range angle that specify the position of the satellite and s_0 is the position where the engine is turned on. To obtain the linear case it is only necessary to make $\alpha'' = \beta'' = 0.0$, to obtain:

$$\alpha = \alpha_0 + \alpha' * (s - s_0) \quad (37)$$

$$\beta = \beta_0 + \beta' * (s - s_0) \quad (38)$$

It is also necessary to include constraints in this type of problem. They can be represented by the equations:

$$\mathbf{S} (.) \geq 0 \quad (39)$$

$$\frac{\left(a - a^*\right)}{\left|a_0 - a^*\right|} = 0 \quad (40)$$

$$\frac{\left[a(1+e) - a^*\left(1+e^*\right)\right]}{\left|a_0\left(1+e_0\right) - a^*\left(1+e^*\right)\right|} = 0 \quad (41)$$

$$\frac{\left(i - i^*\right)}{\left|i_0 - i^*\right|} = 0 \quad (42)$$

$$\frac{\left(\Omega - \Omega^*\right)}{\left|\Omega_0 - \Omega^*\right|} = 0 \quad (43)$$

$$\frac{\left(\omega - \omega^*\right)}{\left|\omega_0 - \omega^*\right|} = 0 \quad (44)$$

The first equation is a generic inequality constraint, while the other five represents the constraints of having the final orbit specified.

Using this technique, it is necessary to find a set of eight variables (six in the linear case) that optimizes the problem (start and end of the application of the thrust and the parameters α_0, β_0, α', β', α'', β'') for each thrusting arc". This number of arcs is an input of the problem.

To solve the nonlinear programming problem, the gradient projection method was used [72]. It means that, at the end of the numerical integration, in each iteration, two steps are taken:

i) Force the system to satisfy the constraints by updating the control function according to:

$$\boldsymbol{u}_{i+1} = \boldsymbol{u}_i - \nabla \mathbf{f}^{\mathbf{T}} . \left[\nabla \mathbf{f} . \nabla \mathbf{f}^{\mathbf{T}}\right]^{-1} \mathbf{f} \quad (45)$$

where **f** is the vector formed by the active constraints;

ii) After the constraints are satisfied, try to minimize the fuel consumed. This is done by making a step given by:

$$\boldsymbol{u}_{i+1} = \boldsymbol{u}_i + \overline{\alpha} \frac{\mathbf{d}}{|\mathbf{d}|} \quad (46)$$

where:

$$\overline{\alpha} = \gamma \frac{J(\boldsymbol{u})}{\nabla J(\boldsymbol{u}).\mathbf{d}} \quad (47)$$

$$\mathbf{d} = -\left(\mathbf{I} - \nabla \mathbf{f}^{\mathrm{T}} \left[\nabla \mathbf{f} . \nabla \mathbf{f}^{\mathrm{T}}\right]^{-1} \mathbf{f}\right) \nabla \mathrm{J}(\boldsymbol{u}) \quad (48)$$

and **I** is the identity matrix, **d** is the search direction, J is the function to be minimized (fuel consumed) and γ is a parameter determined by a trial and error technique. The possible singularities in equations (45) to (48) are avoided by choosing the error margins for tolerance in convergence large enough. This procedure continues until $|\boldsymbol{u}_{i+1} - \boldsymbol{u}_i| < \varepsilon$ in both equations (45) and (46), where ε is a specified tolerance.

4 Results

Having those equations, it is possible to perform several simulations to verify the costs related to the use of the optimal, sub-optimal linear and sub-optimal quadratic forms for the control. Results from reference [68] for the optimal and impulsive cases are used to compare the results. It is important to emphasize that the low thrust propulsion system was applied only for the Earth-Moon trajectory. The lunar insertion is still performed by impulsive maneuvers. The satellite leaves the Earth from an orbit that is circular and has semi-major axis of 6570 km. Then, it goes to an orbit that has eccentricity 0.97 and semi-major axis 220000 km. This transfer is planar. The value used for the final mass of the satellite is 150 kg after the low thrust maneuver for the single mission and 180 kg for the double mission. These values are compatible with a final mass of 100 and 120 kg in lunar orbit, respectively. The engine is assumed to have a specific impulse of 3500 s, with the thrust magnitude assuming the values 200 and 20 N. Table 2 shows the results.

Considering that there are two spacecrafts, it is necessary to perform a more complex maneuver to take into account that the sub-satellite has no engine. So, the insertion into Moon's orbit is performed with both spacecrafts together, lying in the orbit designed for the sub-satellite. Then, after the separation of the spacecrafts, the primary one will be maneuvered to its final orbit.

Assuming that the optimal maneuver (making the insertion at the periapsis of the elliptical orbit) is used and considering that, after separating from the sub-satellite, the main spacecraft is transferred to its final orbit using a bi-impulsive Hohmann Transfer, the results for the ΔV required can be obtained. More details are shown in reference [68].

Figs. 2 and 3 are taken from reference [68] and they show the direction of the thrust for the optimal maneuver as a function of the range angle for the Low Transfer Maneuvers 1 and 2, respectively. Note that only the in-plane angle α is shown, since this is a planar maneuver and it implies that β = 0. Those figures are repeated here because their forms are required for the analysis of the differences between the linear and the quadratic sub-optimal approaches.

Table 2 - Fuel consumed and duration of the propelled phase of the transfer [68].

Mission	E-M (kg)	Ins. (kg)	Total (kg)	Time (h)
Imp 1	205.78	31.40	237.18	0.00
Imp 2	247.69	38.16	285.85	0.00
L. T. 1	22.37	31.40	53.77	10.97
L. T. 2	14.75	31.40	46.15	0.68
L. T. 3	27.05	38.16	65.21	12.89
L. T. 4	18.02	38.16	56.18	0.86

Imp 1: Single mission using an engine with specific impulse of 340 s;
Imp 2: Double mission using an engine with specific impulse of 340 s;
L. T. 1: Single mission using an engine with 20 N, specific impulse of 3500 s, assuming a mass of 150 kg for the spacecraft after the Low-Thrust maneuver;
L. T. 2: Single mission using an engine with 200 N, specific impulse of 3500 s, assuming a mass of 150 kg for the spacecraft after the Low-Thrust maneuver;
L. T. 3: Double mission using an engine with 20 N and specific impulse of 3500 s, assuming a mass of 180 kg for the spacecraft after the Low-Thrust maneuver;
L. T. 4: Double mission using an engine with 200 N and specific impulse of 3500 s, assuming a mass of 180 kg for the spacecraft after the Low-Thrust maneuver;
E-M: Fuel required for the Earth-Moon trajectory;
Ins.: Fuel required for the lunar insertion for a specific impulse of 340 s;
Total: Total fuel required (Earth-Moon trajectory and lunar insertion).
Time: Duration of the propelled phase of the transfer, which is the time that the engine is turned on. So, this time is zero for the impulsive assumption.

The next step is to apply the linear sub-optimal control for the same maneuvers. The same idea used in reference [68] of using two different engines with 20 N and 200 N is used again. These values can be reached with a more powerful propeller or using a combination of several smaller propellers. The goal of the present research is to find alternatives for the trajectories, without the concern if such a propeller is available or not in current times. The maneuvers used the constraint of starting at the range angle equal to zero.

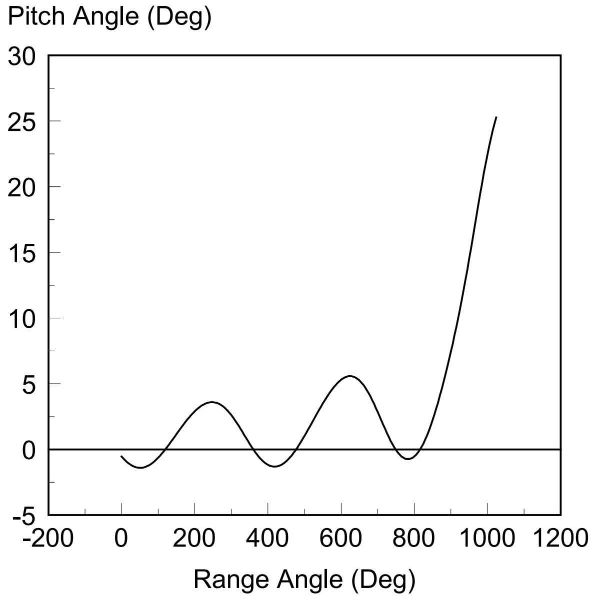

Fig. 2. – Pitch angle (deg) vs. Range angle (deg) for Low Thrust 1 maneuver.

Table 3 shows the results for the single mission and Table 4 shows the equivalent results for the double mission. In these Tables, maneuver 1 is the optimal one for the engine with 20 N level, maneuver 2 represents the situation where the thrust level is 20 N and the control is linear and maneuver 3 is the one with the same 20 N propeller, but the control is quadratic. Maneuver 4 is optimal with 200 N magnitude for the engine, maneuver 5 is realized with the thrust of 200 N and linear control and maneuver 6 has 200 N and use the quadratic control. Note that the transfers are not restricted to be completed in a single revolution, so values for S_f larger than 360 degrees means that the transfer took more than one revolution to be completed.

The results quantify some expected aspects, as shown below. When the thrust level is increased (in the simulations shown here this increase is from 20 N to 200 N) there is an almost proportional decrease in the time required for the transfer. Figs. 2 and 3 show the results in terms of the range angle, but when this quantity is converted to time this proportion is kept.

Regarding the sub-optimal approach, the results show that this is an interesting idea. The fuel consumption is increased when compared to the optimal solutions, as expected, but this increase is of the order of only a few percent.

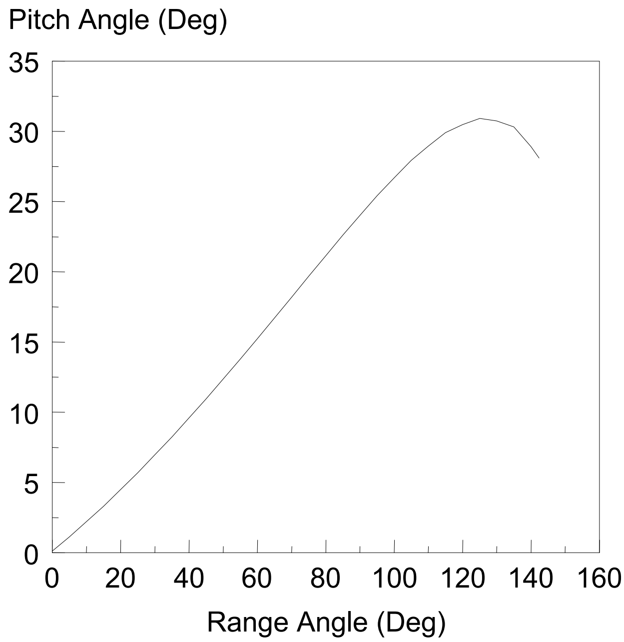

Fig. 3 – Pitch angle (deg) vs. Range angle (deg) for Low Thrust 2 maneuver.

So, considering the advantages of this type of solution, in general, it is a very interesting approach. The difference between the linear and the quadratic approach depends of each situation. The first two examples shown here exemplify very well this variation. The solution shown in Fig. 2 is very far from linear, so a quadratic form for the control increases the quality of the approximation very much. On the other side, Fig. 3 shows a solution that is very close to linear. So, in this situation, the linear approximation is almost as good as the quadratic one.

6 Conclusion

Quadratic and linear sub-optimal controls were used to determine Earth-Moon trajectories that has the goal of minimizing the fuel consumption.

Comparing the results obtained here for this problem with an optimal implementation available in the literature, it seems that this sub-optimal approach is adequate for the Earth-Moon transfer, since the extra fuel consumed is not very large.

The performance of the quadratic approximation is always better than the linear one, but this difference depends on the situation. In some situations, like in the trajectory shown in Fig. 3, the linear approximation is almost as good as the quadratic one. In some others, like the example shown in Fig. 2, this difference is larger.

The algorithms have good numerical behavior, but it is not fast enough to allow real time applications. Process time (CPU) is less than a minute for simple maneuvers, but takes more than an hour when several constraints and/or thrusting arcs are required.

Table 3
Sub-optimal transfers (linear and quadratic) for a single mission.

M	1	2	3	4	5	6
s_0 (deg)	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
S_f (deg)	1005.2	1055.7	1015.1	147.7	153.2	151.3
α_0 (deg)		9.21	7.33		-7.54	3.27
β_0 (deg)		0.0000	0.0000		0.0000	0.0000
α'		-0.160	-0.010		0.11	0.18
β'		0.0000	0.0000		0.0000	0.0000
α''			-0.008			-0.013
β''			0.0000			0.0000
Fuel (kg)	22.37	23.47	22.47	14.75	15.26	15.09

Table 4
Sub-optimal transfers (linear and quadratic) for a double mission.

M	1	2	3	4	5	6
s_0 (deg)	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
S_f (deg)	1203.1	1263.2	1215.1	162.3	170.4	172.8
α_0 (deg)		8.27	9.23		-4.23	7.37
β_0 (deg)		0.0000	0.0000		0.0000	0.0000
α'		-0.130	-0.012		0.09	0.19
β'		0.0000	0.0000		0.0000	0.0000
α''			-0.006			-0.015
β''			0.0000			0.0000
Fuel (kg)	27.05	28.42	27.33	18.02	18.93	18.75

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