Modeling and $\mu$ Synthesis Control of Vehicle Active Suspension with Motor Actuator

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Abstract: This paper presents the modeling and $\mu$ synthesis for a vehicle active suspension system with a motor actuator, and analyses the performances of the active suspension with $\mu$ synthesis controller. Firstly, considering parameter uncertainties and unmodeled dynamics of suspension and motor actuator, an uncertain model has been built by linear fraction transformation. Secondly, a $\mu$ synthesis controller is designed for this uncertain system. Finally, frequency and time responses of passive and active suspensions have been obtained by simulation. For comparison purpose, an $H_\infty$ synthesis controller is proposed as well. The results show that the designed $\mu$ synthesis controller can improve ride comfort significantly compared with passive suspension, and is slightly worse than the $H_\infty$ synthesis controller in the sensitive frequency range for human. Both robust stability and robust performance of the perturbed system with the $\mu$ synthesis controller have been verified by $\mu$ analysis method, indicating significantly superiority to the $H_\infty$ synthesis controller. Additionally, the sensitiveness of uncertainties is demonstrated by illustrations of perturbed systems with parameter variations.

Key-Words: Active Vehicle Suspension, Motor Actuator, $\mu$ Synthesis, Robust Stability, Robust Performance

1 Introduction

The aims of vehicle active suspension are to improve ride comfort by suppressing the vibrations from road roughness and provide good handling performance. Since random road excitation is actually a disturbance with limited energy band, the active suspension is designed to attenuate the road disturbance and achieve a trade-off between the conflicting suspension performance requirements. Additionally, the uncertainties of actual suspension system usually including parameter variations of suspension and unmodeled dynamics of actuator affect the suspension performances and even result in the instability of suspension system.

Several controllers have been proposed to solve these problems of active suspension. The linear quadratic Gaussian (LQG) controller can attenuate road disturbance and measurement noise effectively [1-3], but it has a poor stability margin for uncertain system, i.e., system may be unstable in the presence of the uncertainties. Then, $H_\infty$ synthesis is used to deal with the uncertainties in an unstructured way and provides an optimal controller to ensure the robustness and required performances of suspension system [4-9]. In spite of this, the controller is too conservative [10]. In addition, another useful strategy, i.e., linear parameter varying (LPV) control combining $H_\infty$ synthesis with gain-scheduling approach [11-14], is applied to active suspension. LPV controller is gain-scheduled by the measured or estimated information of variable parameters to adapt to the variation of parameters. But it cannot deal with unmodeled dynamics of actuator. In a word, the above three controllers are undesirable for the perturbed system with the mixed uncertainties including parameter variations and unmodeled dynamics.

In order to concern the mixed uncertainties and reduce the conservativeness of controller, the mixed $\mu$ synthesis controller is designed for active suspension [15, 16]. In the design process, $\mu$ synthesis controller applies the structured uncertainties, providing more detailed information of uncertainty characteristics, so that it can ensure robust stability and robust performance of the perturbed system.

In this study, the research work is carried out on the developed suspension system which is featured by a type of DC motor actuator produced by SJTU [17-19]. Due to the special structure of ball screw-nut mechanism, the moments of inertia of rotor and nut can affect directly the performances of controlled suspension. Meanwhile, the unmodeled high-order dynamics of the motor’s control circuit
will also influence the system robustness significantly. Therefore, these two elements should be considered in the actuator model. In addition, the practical suspension system always varies with operation conditions. For instance, the sprung mass varies in a large range with the number of passengers, which may degrade the suspension performances. Other variable parameters, such as the stiffness of suspension spring and tire, the damping of actuator and motor constant, are also important to ensure the effectiveness of the control. Hence, the multiple parameter uncertainties should be taken into account in the suspension model.

Based on the literature review of control strategies for vehicle active suspensions and the uncertain model with the mixed uncertainties including parameter uncertainties and unmodel dynamics, μ synthesis method is adopted to design the controller for the active suspension with motor actuator.

In general, this paper focuses on the modeling of a vehicle active suspension system with a DC motor actuator and proposes a robust control with μ synthesis method to deal with the mixed uncertainties. For comparison purpose, an H∞ synthesis controller is also designed.

The structure of the paper is as follows. Section 2 presents the modeling of suspension system. Section 3 discusses the robust control design based on μ synthesis and H∞ synthesis. And Section 4 shows the benefits of μ synthesis controller through performance comparison with passive suspension and H∞ synthesis controller.

2 MODELING OF SUSPENSION AND MOTOR SYSTEM

2.1 Quarter-vehicle Suspension Model

The suspension model usually includes quarter-vehicle model, half vehicle model and full vehicle model [20]. Although the quarter-vehicle model with two degrees of freedom (2DOF) [1] is simple, it is effective to have a good understanding of suspension performances, such as ride comfort and road holding. The half vehicle model (4DOF) is used to evaluate pitch and bounce of vehicle body, especially with wheelbase preview control [2], while the full vehicle model (7DOF) [4] is more complicated and accurate to evaluate the attitude of vehicle body, i.e., pitch, roll and bounce. The results of the controllers for the quarter-vehicle model always coincide with that of the half/full vehicle models to a great extent, although it is accepted that further important practical problems appear when the control strategy is implemented on the half/full vehicle models. Hence, the quarter-vehicle suspension model is used as shown in Fig.1. $C_f$ represents the equivalent damper coefficient of motor actuator. The active force $F_o$ is provided by motor actuator. The dynamic equations of the suspension system are as follows:

$$m_b \dot{x}_b = -K_s (x_b - x_w) - C_f (\dot{x}_b - \dot{x}_w) + F_o$$  \hspace{1cm} (1)

$$m_w \dot{x}_w = K_s (x_b - x_w) + C_f (\dot{x}_b - \dot{x}_w) - K_f (x_w - x_g) - F_o$$  \hspace{1cm} (2)

Fig.1 Quarter-vehicle suspension model

2.2 Motor Dynamic Model

The developed prototype of motor actuator is composed of a three-phase permanent-magnet brushless DC motor, a ball screw and a nut, as shown in Fig.2. The rotor angular motion is converted to the ball screw linear motion by the nut. Thus, the motor output torque can be transferred to the active vertical force. Due to the high rotor speed during the suspension operation, the high frequency effects of the moment of inertia of rotor and nut should be considered. The dynamic model for the rotor and the nut is demonstrated in Fig.3.
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Because of the non-linearity of suspension spring and actuator friction, the variations of spring stiffness and damping coefficient also need to be considered. Due to the change of temperature and pressure, tire stiffness is assumed to be uncertain. In addition, the non-linearity of motor model cannot be neglected, e.g. the motor constant may be inaccurate as a result of the measurement error, etc. Hence, five main uncertain parameters are selected in a range of practical variations around their nominal values, respectively,

\[ m_b = \bar{m}_b (1 + d_{mb}\delta_{mb}) \]
\[ K_s = \bar{K}_s (1 + d_{ks}\delta_{ks}) \]
\[ K_l = \bar{K}_l (1 + d_{kl}\delta_{kl}) \]
\[ C_f = \bar{C}_f (1 + d_{cf}\delta_{cf}) \]
\[ \Phi = \bar{\Phi} (1 + d_{\Phi}\delta_{\Phi}) \]

where \( d_{mb} = 0.2 \), \( d_{ks} = 0.2 \), \( d_{kl} = 0.1 \), \( d_{cf} = 0.2 \), \( d_{\Phi} = 0.2 \), \( \|\delta_{mb}\|, \|\delta_{ks}\|, \|\delta_{kl}\|, \|\delta_{cf}\|, \|\delta_{\Phi}\| \leq 1 \). The scalars, denoted as \( d \), represent the percentages of variation around the nominal values and indicate the degree of parameter uncertainty. And \( \|\| \) represents the norm of the actual parameter deviation. The symbols \( m_b, K_s, K_l, C_f, \Phi \), respectively refer to the above 5 uncertain parameters and \( \bar{m}_b, \bar{K}_s, \bar{K}_l, \bar{C}_f, \bar{\Phi} \) are defined as corresponding nominal parameters.

All uncertainty parameters can be written in the form of lower Linear Fractional Transformation (LFT), which is necessary for structure uncertainty analysis. The LFT representations are expressed as follows:

\[
\frac{1}{m_b(m_u + I_d) + m_u I_d} = F_i \begin{bmatrix} \beta & -\alpha \end{bmatrix} \delta_{\text{ub}} \left[ 1 - (\beta - \alpha) \delta_{\text{ub}} \right]^{-1} \beta
\]

\[
= F_i \begin{bmatrix} \beta & -\alpha \end{bmatrix} \delta_{\text{ub}} = F_i (M_{ub}, \delta_{ub})
\]

\[
K_s = \bar{K}_s (1 + d_{ks}\delta_{ks})
\]

\[
= F_i \begin{bmatrix} \bar{K}_s & d_{ks} \end{bmatrix} \delta_{ks} = F_i (M_{ks}, \delta_{ks})
\]

\[
K_l = \bar{K}_l (1 + d_{kl}\delta_{kl})
\]

\[
= F_i \begin{bmatrix} \bar{K}_l & d_{kl} \end{bmatrix} \delta_{kl} = F_i (M_{kl}, \delta_{kl})
\]
$$C_f = \tilde{C}_f (1 + d_{\delta, \delta_f})$$

$$= F_1 \left( \begin{bmatrix} \tilde{C}_f & d_{\delta} \\ \tilde{C}_f & 0 \end{bmatrix}, \delta_f \right) = F_1 (M_{\delta_f}, \delta_f)$$ (13)

$$\Phi = \bar{\Phi} (1 + d_{\delta_0} \delta_0)$$

$$= F_1 \left( \begin{bmatrix} \bar{\Phi} & d_{\Phi} \\ \bar{\Phi} & 0 \end{bmatrix}, \delta_0 \right) = F_1 (M_{\Phi}, \delta_0)$$ (14)

in which $$\alpha = \frac{m_{\delta} d_{\delta_b}}{m_b + \gamma_1 \gamma_2}$$, $$\beta = \frac{\gamma_1}{m_b + \gamma_1 \gamma_2}$$.

$$\delta_j (j \in \{ mb, ks, kt, cf, \Phi \})$$ represents the uncertainty which can be extracted from the differential equations by the LFT representation.

Define that $$p = \begin{bmatrix} p_{mb1} & p_{mb2} & p_{ks} & p_{cf} & p_{\delta_0} \end{bmatrix}$$ and $$q = \begin{bmatrix} q_{mb1} & q_{mb2} & q_{ks} & q_{cf} & q_{\delta_0} \end{bmatrix}$$ are the inputs and outputs of $$\delta_j$$ blocks, respectively.

According to equations (8)–(14), the system block diagram with uncertain parameters can be obtained in Fig. 4.

Three necessary suspension performances are chosen, i.e. body acceleration (BA), suspension working space (SWS) and dynamic tire deflection (DTD), as the performance outputs $$z_p = [z_1, z_2, z_3] = \begin{bmatrix} \text{BA} & \text{SWS} & \text{DTD} \end{bmatrix}^T$$. The measured outputs are obtained by two linear potentiometers, whose voltages are given by $$y = \begin{bmatrix} y_1, y_2 \end{bmatrix} = \begin{bmatrix} \text{BA}_m & \text{SWS}_m \end{bmatrix}$$, $$\alpha_1$$ and $$\alpha_2$$ are the gains of accelerometer and displacement meter, respectively. By defining the state vector and input vector respectively as $$X = \begin{bmatrix} x_0 & \dot{x}_0 & x_c & \dot{x}_c \end{bmatrix}^T$$ and $$u = \begin{bmatrix} x_p & n^T \end{bmatrix}^T$$, where $$n = \begin{bmatrix} n_1 & n_2 \end{bmatrix}$$ are two noises of $$y$$, respectively, the corresponding differential equations of the above uncertain model can be expressed as below:

$$\dot{x}_b = -\beta m_a \bar{K}_a \left( x_0 - x_c \right) - \beta m_a \tilde{C}_f \left( \dot{x}_b - \dot{x}_c \right)$$

$$+ \beta m_a \tilde{d}_i - \beta I_d \bar{K}_a \left( x_0 - x_c \right) - \alpha p_{mb1}$$

$$- \beta m_a d_{\delta_0} p_{ks} - \beta I_d d_{\delta_0} p_{\delta_0} - \beta m_a d_{\delta_f} p_{\delta_f}$$

$$+ \beta m_a d_{\delta_0} p_{\delta_0}$$

$$\tilde{x}_w = \left( \gamma_1 - \beta \gamma_2 \gamma_2 \right) \left[ \bar{K}_a \left( x_0 - x_c \right) + \tilde{C}_f \left( \dot{x}_b - \dot{x}_c \right) \right]$$

$$+ \left( \gamma_1 + \beta \gamma_2 \gamma_2 - \beta I_d \right) \bar{K}_a \left( x_0 - x_c \right)$$

$$- \alpha p_{mb2} + \left( \gamma_1 - \beta \gamma_2 \gamma_2 \right) d_{\delta_0} p_{ks}$$

$$+ \left( \gamma_1 + \beta \gamma_2 \gamma_2 - \beta I_d \right) d_{\delta_0} p_{\delta_0}$$

$$+ \left( \gamma_1 - \beta \gamma_2 \gamma_2 \right) d_{\delta_f} p_{\delta_f}$$

$$+ \left( \gamma_1 + \beta \gamma_2 \gamma_2 \right) d_{\delta_0} p_{\delta_0}$$
\[ q_{nb1} = -\beta m_w \bar{K}_s (x_b - x_w) - \beta m_w \bar{C}_f (\dot{x}_b - \dot{x}_w) + \beta m_a \bar{D} i - \beta I_d \bar{K}_s (x_b - x_g) - \alpha p_{nb1} \\
- \beta m_a d_{ls} p_{ls} - \beta I_d d_{li} p_{li} - \beta m_a d_{pf} p_{pf} + \beta m_a d_{df} p_{df} \]

(17)

\[ q_{nb2} = -\beta \gamma \gamma_2 [\bar{K}_s (x_b - x_w) + \bar{C}_f (\dot{x}_b - \dot{x}_w)] \\
+ \Phi_i + \beta (\gamma \gamma_2 - I_d) \bar{K}_s (x_b - x_g) - \alpha p_{nb2} \]

(18)

\[ q_{ib} = \bar{K}_i (x_b - x_a) \]

(19)

\[ q_{ib} = \bar{K} i (x_b - x_a) \]

(20)

\[ q_{cf} = \bar{C}_f (\dot{x}_b - \dot{x}_w) \]

(21)

\[ q_{cf} = \bar{C} i \]

(22)

\[ z_1 = \dot{x}_b = -\beta m_w \bar{K}_s (x_b - x_w) - \beta m_w \bar{C}_f (\dot{x}_b - \dot{x}_w) + \beta m_a \bar{D} i - \beta I_d \bar{K}_s (x_b - x_g) - \alpha p_{nb1} \]

\[ - \beta m_a d_{ls} p_{ls} - \beta I_d d_{li} p_{li} - \beta m_a d_{pf} p_{pf} + \beta m_a d_{df} p_{df} \]

(23)

\[ z_2 = x_b - x_w \]

(24)

\[ y_1 = \alpha_1 \dot{x}_b \]

(25)

\[ y_2 = \alpha_2 (x_b - x_w) \]

(26)

Thus, the perturbed suspension model with measurement noises is summarized as

\[
\begin{bmatrix}
X \\
q \\
z_p \\
y
\end{bmatrix} = \begin{bmatrix}
A & B_p & B_1 & B_2 \\
C_1 & D_{g} & D_{d1} & D_{d2} \\
C_1 & D_{g} & D_{d1} & D_{d2} \\
C_1 & D_{g} & D_{d1} & D_{d2}
\end{bmatrix}
\begin{bmatrix}
X \\
p \\
q \\
u_1
\end{bmatrix}
= G_{suss}(s)
\]

(28)

Fig.5 Perturbed suspension model with measurement noise

2.3 Motor Electrical Model

The electrical model of the three-phase permanent-magnet brushless DC motor can be simplified as a classic RL-circuit \[21\] as depicted in Fig.6.

\[ V_{act} = V_{bat} - V_{bemf} = L \frac{di}{dt} + Ri \]

(31)

Fig.6 Simplified circuit diagram of DC motor

The back electromotive force \( V_{bemf} \) of motor is proportional to the angular velocity as follows:

\[ V_{bemf} = K_p \omega = \frac{2\pi K_p}{P} \]

(30)

in which \( K_p \) is voltage constant of motor.

The battery voltage \( V_{bat} \) can be adjusted to realize the necessary control current \( i \) and the voltage on the equivalent resistance and equivalent inductance \( V_{act} \) can be obtained as follows:

\[ V_{act} = V_{bat} - V_{bemf} = L \frac{di}{dt} + Ri \]

The nominal transfer function of electrical motor model is considered as a first-order, phase-lag model, i.e.,

\[ G_m(s) = \frac{i(s)}{V_{act}(s)} = \frac{1}{Ls + R} = \frac{1}{T_{mc}s + 1} \]

(32)

where \( K_{mc} = 1/R \) and \( T_{mc} = L/R \).

Assume that the gain coefficient \( K_{mc} \) varies with relative error 10% around its nominal value and the time constant \( T_{mc} \) with relative error 20%. In order to account for unmodeled dynamics of motor, the uncertainty is approximated by the input multiplicative uncertainty caused by the perturbed transfer functions as below:

\[ G_m(s) = (1 + W_m \delta_m) \bar{G}_m(s) \]

(33)

where \( \delta_m \in \mathbb{C}, |\delta_m| \leq 1 \) and the uncertainty weighting function \( W_m \) is chosen to satisfy the following inequality:

\[ \left| \frac{G_m(j\omega) - \bar{G}_m(j\omega)}{\bar{G}_m(j\omega)} \right| < W_m(j\omega) \]

(34)

Considering the influence of the variations of \( K_{mc} \) and \( T_{mc} \), \( W_m \) is obtained as follows:

\[ W_m = \frac{0.381s + 36.6466}{s + 359.4713} \]

Due to writing simplicity, \( V_{act} \) is substituted by
u, the block diagram of electrical model with the input multiplicative uncertainty is described by equation (35) as shown in Fig.7.

\[ i = F_u (G_{mp}, \delta_w) u \]  

\[ (35) \]

Fig.7 Block diagram of perturbed motor electrical model

2.4 Augmented System Model
Combining suspension model with motor model, the block diagram of entire system can be obtained as shown in Fig.8, in which \( \delta_m \) is a complex uncertainty and \( \Delta_p \) are real uncertainties. In order to limit the control voltage within a required range, \( u \) is added to the output vector as a performance index. The blocks in the dashed-line frame can be integrated into \( G_{sys} \), and the entire perturbed suspension system can be described by equation (36), as shown in Fig.9,

\[ \begin{bmatrix} z \\ y \end{bmatrix} = F_u \left( G_{sys}, \Delta \right) \begin{bmatrix} x_g \\ n \\ u \end{bmatrix} \]  

\[ (36) \]

where \( z = [z_p \ z_u]^T \) represents the performance outputs including suspension performances \( z_p \) and control input \( z_u = u \) and the diagonal uncertainty matrix becomes \( \Delta = \text{diag}[\Delta_p, \delta_w] \).

Fig.8 Block diagram of entire system with uncertainties

\[ \begin{bmatrix} p \\ q \end{bmatrix} = G_{sys} \begin{bmatrix} \delta_r \\ \delta_m \end{bmatrix} \]

where \( p = [p_p \ p_w]^T \) and \( q = [q_p \ q_w]^T \) are the inputs of perturbation. The control aim is to reduce \( \infty \) norm of the transfer function matrices from \( d \) to \( z \), for all possible uncertainty matrices \( \Delta : \| \Delta \|_\infty < 1 \). The weighting functions \( W_i \) and \( W_n \) represent frequency domain models of disturbance \( \tilde{w} \) and noises \( \tilde{n} \), respectively, and the weighting functions \( W_p \) and \( W_u \) are the performance outputs. The dashed-line rectangle represents the augmented model \( P \).

Fig.9 LFT representation of entire system with uncertainties

The closed-loop augmented model, which includes the entire system, the feedback structure, the weighting functions and the controller, as well as the model uncertainties and the performance objectives, is depicted in Fig.10. In the diagram, \( \tilde{d} = [\tilde{w} \ \tilde{n}]^T \) are the inputs of perturbation. The control aim is to reduce \( \infty \) norm of the transfer function matrices from \( d \) to \( z \), for all possible uncertainty matrices \( \Delta : \| \Delta \|_\infty < 1 \). The weighting functions \( W_i \) and \( W_n \) represent frequency domain models of disturbance \( \tilde{w} \) and noises \( \tilde{n} \), respectively, and the weighting functions \( W_p \) and \( W_u \) are the performance outputs. The dashed-line rectangle represents the augmented model \( P \).

Fig.10 Closed-loop augmented system

3 \( \mu \) synthesis of augmented system
The closed-loop augmented system in Fig.10 is a standard robust design problem. \( P \) and \( K \) are interconnected by lower LFT to form a matrix as below:

\[ M = F_i (P, K) = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \]  

\[ (37) \]

with respect to the uncertainty set \( \Delta \) and a fictitious uncertainty block \( \Delta_f \), which is called the performance uncertainty block with \( \| \Delta_f \|_\infty \leq 1 \) as shown in Fig.11.
The nominal performance transfer function can be derived as \( \bar{\mathcal{P}} = M \Delta \), so the nominal performance requirement can be obtained as:
\[
\left\| M \Delta \right\| < 1 \quad (38)
\]

Necessary and sufficient conditions for robust stability and robust performance can be formulated in terms of the structured singular value denoted as \( \mu \). In order to guarantee the robust stability of the perturbed system, it is required that
\[
\mu_s(\Delta_1(s)) < 1 \quad (39)
\]
in which \( \mu_s(\Delta_1(s)) = \sup \mu_s(M_{11}(j\omega)) \).

The perturbed performance transfer function can be derived that
\[
\bar{\mathcal{P}} = [M_{22} + M_{21}\Delta(I - M_{11}\Delta)M_{12}] \bar{d} = F_\Delta(M, \Delta) \bar{d}
\]
Thus, the perturbed performance requirement can be set as
\[
\left\| F_\Delta(M, \Delta) \right\| < 1 \quad (40)
\]
The inequality (40) is equivalent to the system loop in Fig.11 to be robustly stable with respect to \( \Delta_f \). Hence, robust performance can be equivalently considered as a robust stabilization problem with respect to a structured uncertainty \( \bar{\Delta} = \text{diag}[\Delta, \Delta_f] \), \( \left\| \bar{\Delta} \right\| \leq 1 \). The robust performance can be guaranteed by the necessary and sufficient condition as
\[
\mu_s(M(s)) < 1 \quad (41)
\]
where \( \mu_s(M(s)) = \sup \mu_s(M(j\omega)) \).

Based on the above \( \mu \) analysis, the \( \mu \) synthesis problem for the augmented model is to find a linear, output controller \( K \) to generate an output feedback to ensure the inequalities (38), (39) and (41). Or, equivalently, the objective is solve for \( K \)
\[
\sup \mu_s(M(P, K)(j\omega)) < 1 \quad (42)
\]

The \( D-K \) iteration \( \mu \) synthesis method [10] is used to solve (42). The method is based on solving the following optimization problem, for a stabilizing controller \( K \) and a diagonal constant scaling matrix \( D \),
\[
\inf_{\kappa \in \mathbb{R}} \sup_{\omega \in [0, \infty]} \sigma\left[DMD^{-1}(j\omega)\right] \quad (43)
\]
where the scaling matrix set \( D \) is defined as
\[
D = \text{diag}[D_1, D_2, D_3, D_4, D_5, d_1, I_1, I_2]:
\]
\[
D_1 \in \mathbb{C}^{2\times 2}, \quad D_2 \in \mathbb{C}^{1\times 1}, \quad D_3 \in \mathbb{C}^{1\times 1}, \quad D_4 \in \mathbb{C}^{1\times 1},
\]
\[
D_5 \in \mathbb{C}^{1\times 1}, \quad D_1 = D^*, \quad d_i > 0, \quad d_1 > 0, \quad I_1 = I^{1\times 1}, \quad I_2 = I^{3\times 4}
\]

Corresponding to inequality (42), the stabilizing controller is to be found such that
\[
\sup_{\omega \in [0, \infty]} \sigma\left[DMD^{-1}(j\omega)\right] < 1 \quad (44)
\]
The \( D-K \) iteration method is to reduce the left-hand-side value of (44), for \( K \) and \( D \) in turn while keeping the other one fixed.

The \( D-K \) iteration algorithm is as follows:
Step 1: start with an initial guess for \( D \), usually set \( D=I \).
Step 2: fix \( D \) and solve the \( H_\infty \) optimization for \( K \),
\[
K = \arg \inf_{\kappa \in \mathbb{R}} \left\| F\left(DF(P, K) D^{-1}, K\right) \right\|
\]
Step 3: fix \( K \) and solve the following convex optimization problem for \( D \) at each frequency over a selected frequency range,
\[
D(j\omega) = \arg \inf_{\omega \in [0, \infty]} \sigma\left[DF(P, K) D^{-1}(j\omega)\right]
\]
Step 4: curve fit \( D(j\omega) \) to get a stable , minimum-phase \( D(s) \); go to Step 2 and repeat, until inequality (44) is satisfied.

In this study, an \( H_\infty \) synthesis controller is also designed concerning the nominal augmented system to compare with the \( \mu \) synthesis controller. According to the ISO 2631-3, human sensitivity to vibrations is frequency-dependent. The weighting functions are chosen to improve the suspension performances in this frequency range. The weighting functions of both \( \mu \) synthesis and \( H_\infty \) synthesis are determined as follows:
\[
W_r = 0.1 , \quad W_s = \text{diag}[10^3, 10^3] , \quad W_w = 0.0012 , \quad W_p = \text{diag}[W_{r1}, W_{r2}, W_{r3}]
\]
where
\[
W_{r1} = 5.3 \times 10^{-5} s^2 + 99.4 s + 14431.9 , \quad s^2 + 3364.8 s + 720.9
\]
\[
W_{r2} = \frac{s^2 + 471.6 s + 7245.1}{s^2 + 699.2 s + 1854.3} , \quad W_{r3} = \frac{2 s^2 + 247.7 s + 19411.2}{1.1 s^2 + 157.7 s + 293.8}
\]
4 Simulation results
The simulations for the designed $\mu$ synthesis controller and $H_\infty$ synthesis controller have been carried out in MATLAB. The nominal parameters of suspension and motor are chosen based on the practical case for a single real wheel of a small saloon car and shown in Table 1 below. The damping coefficient $C_p$ of passive suspension is $2083\text{N}\cdot\text{s/m}$ and the other parameters of passive suspension are the same as those of active suspension.

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</table>

4.1 Frequency responses of the closed-loop nominal model
The frequency responses of the closed-loop nominal suspension model are illustrated in Fig.12. Based on ISO2361 of human response to vibration, the sensitive frequency range, i.e., 4-8 Hz (25-50 rad/s), is chosen to evaluate the designed suspension with both $\mu$ and $H_\infty$ synthesis controllers. From Fig.12, it is obvious that BA has been decreased greatly for two active suspensions, which implies that ride comfort can be improved significantly compared with passive suspension. Furthermore, $H_\infty$ synthesis controller achieves slightly less BA than $\mu$ synthesis controller in the sensitive frequency range. The price of better ride comfort is that two controllers has a high gain of SWS in the wheel-hop frequency and low frequency under 10 rad/s.

Fig.13 shows the nominal performance of two controllers, which can be quantified by $\infty$-norm of $\bar{\varphi}/\bar{d}$. It is clear that $\infty$-norm of the $H_\infty$ synthesis controller is a bit lower, which implies that its nominal performance is a little better than that of the $\mu$ synthesis controller. The reason of this conclusion is that the $H_\infty$ synthesis controller is designed for the nominal system without perturbations, while the design of $\mu$ synthesis controller takes robust performance of the perturbed system into account, consequently, leading to the cost of nominal performance.

4.2 Time responses of the closed-loop nominal model
In order to test the validity of the designed controllers, a set of simulations in time domain is conducted on random road profile based on the closed-loop nominal model. The road condition is that vehicle is driven on the C-class road whose roughness coefficient is $63256\times10^{-10}\text{m/cycle}$ at the cruising speed of 20m/s. Fig.14 compares the root-mean-square (RMS) values of suspension performances of both passive and active suspensions.
It can be seen that BA has been reduced 39.81% and 44.46% by $\mu$ synthesis controller and $H_\infty$ synthesis controller respectively, compared with passive suspension. Since the random road input is subject to Gaussian distribution, the performance outputs obey a Gaussian distribution for the linear system, which implies that the target RMS values are limited in one third of the maximum values. SWS is designed for the limitation of $\pm 0.053$ m, indicating that SWS$_{rms}$ value cannot exceed 0.0177 m. Although SWS of active suspension is increased up to 37.5% compared with passive suspension as shown in Fig.14(b), SWS$_{rms}$ is limited still in the required range.

![Fig.14 RMS values of suspension performances on random road profile](image)

**4.3 $\mu$ analysis of robust stability and robust performance**

Based on inequality (39) and (41), the structured singular value $\mu$ of two controllers are calculated to evaluate robust stability and robust performance of the perturbed system. The upper and lower bounds of $\mu_\Delta (M_{II})$ and $\mu_\Delta (M)$ with $H_\infty$ synthesis controller are shown in Fig.15. Since the maximum values of $\mu_\Delta (M_{II})$ and $\mu_\Delta (M)$ are larger than 1, it can be concluded that robust stability and robust performance of $H_\infty$ synthesis controller cannot be guaranteed.

Fig.16 illustrates the upper and lower bounds of $\mu_\Delta (M_{II})$ and $\mu_\Delta (M)$ with $\mu$ synthesis controller. The maximum value of $\mu_\Delta (M_{II})$ is only 0.33, indicating that robust stability is achieved, i.e., the system stability is preserved for $\| \Delta \|_\infty < 1/0.33$. The maximum value of $\mu_\Delta (M)$ in the robust performance analysis is 0.97, which can ensure the robust performance.

Overall, $H_\infty$ synthesis controller simply focuses on the nominal model, causing the price of robust performance, while $\mu$ synthesis controller can meet the requirement of robust performance.
4.4 Analysis of sensitive uncertainties

To further illustrate the advantages of $\mu$ synthesis controller in robust properties, it is necessary to investigate the effects of five uncertain parameters and unmodeled dynamics on the frequency responses of the perturbed systems. Fig.17 shows the frequency responses of the perturbed systems with $m_b$ variations. Similarly, Fig.18–Fig.22 shows the frequency responses with $K_c$, $K_r$, $C_f$, $\Phi$ variations and unmodeled dynamics, respectively. Note that these frequency responses are obtained by assuming that one parameter varies while other variable parameters are frozen.

It can be seen that the curves of frequency responses of BA fluctuate more violently in Fig.17 and Fig.22, which implies that $m_b$ and the unmodeled dynamics are the sensitive factors of ride comfort. Moreover, the curves of SWS in low frequency fluctuate more violently in Fig.21 and Fig.22, indicating that SWS is sensitive to the changes of $\Phi$ and the unmodeled dynamics. Hence, it is obvious that $m_b$, $\Phi$ and unmodeled dynamics are the sensitive factors which can influence significantly on the suspension performances. Especially, since the unmodeled dynamics have a direct impact on the active control force, it could be the most important factors in these three sensitive factors.

Additionally, since the curve clusters of the $\mu$ synthesis controller are more intensive than those of the $H_{\infty}$ synthesis controller, it is concluded that the $\mu$ synthesis controller has better robustness than the $H_{\infty}$ synthesis controller, which also verify better robust performance of the $\mu$ synthesis controller.
5 Conclusions
This paper builds an uncertain model for an active electrical vehicle suspension system taking parameter uncertainties and high-order unmodeled dynamics of suspension and motor actuator into consideration, and then proposes a mixed $\mu$ synthesis controller based on $D-K$ iteration method to meet the requirement of robust stability and robust performance. In order to examine the feasibility and effectiveness of the proposed controller, simulations are carried out. For comparison purpose, an $H_\infty$ synthesis controller is designed as well. The frequency and time responses show that comparing with passive suspension, the designed $\mu$ synthesis controller achieves the significant improvement of ride comfort, while it is slightly worse than the $H_\infty$ synthesis controller in the sensitive frequency range. Furthermore, robust stability and robust performance of the perturbed system are tested by $\mu$ analysis. The results indicate that the $\mu$ synthesis controller is superior to the $H_\infty$ synthesis controller in robust performance. Finally, the sensitiveness of uncertainties is demonstrated,
showing that $m_0$, $\Phi$ and unmodeled dynamics are sensitive factors compared with $K_r$, $K_f$, and $C_f$.

Future research work will be conducted on the performance comparisons between motor actuator and other suspension actuators, e.g., electrohydraulic actuator.

Acknowledgments
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References
## Appendix

### Notation

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(BA)</td>
<td>Body acceleration, m/s²</td>
<td>(m_w)</td>
<td>Wheel mass, kg</td>
</tr>
<tr>
<td>(C_f)</td>
<td>Equivalent damping coefficient, N·s/m</td>
<td>(P_l)</td>
<td>Lead of ball screw, m</td>
</tr>
<tr>
<td>(C_p)</td>
<td>Passive damping coefficient, N·s/m</td>
<td>(R)</td>
<td>Resistance, Ω</td>
</tr>
<tr>
<td>DTD</td>
<td>Dynamic tire displacement, m</td>
<td>SWS</td>
<td>Suspension working space, m</td>
</tr>
<tr>
<td>(i)</td>
<td>Motor current, A</td>
<td>(T_w)</td>
<td>Motor output torque, N·m</td>
</tr>
<tr>
<td>(I_d)</td>
<td>Equivalent inertia, kg</td>
<td>(T_{me})</td>
<td>Time constant of motor</td>
</tr>
<tr>
<td>(J_n)</td>
<td>Moment of inertia of nut, kg·cm²</td>
<td>(T_o)</td>
<td>Actual total output torque, N·m</td>
</tr>
<tr>
<td>(J_r)</td>
<td>Moment of inertia of rotor, kg·cm²</td>
<td>(V_{bomf})</td>
<td>Back electromotive force, V</td>
</tr>
<tr>
<td>(K_E)</td>
<td>Voltage constant of motor, V·s/rad</td>
<td>(V_{bat})</td>
<td>Battery voltage, V</td>
</tr>
<tr>
<td>(K_{acc})</td>
<td>Gain coefficient of motor</td>
<td>(\Phi)</td>
<td>Motor constant, N/A</td>
</tr>
<tr>
<td>(K_s)</td>
<td>Suspension stiffness, N/m</td>
<td>(\alpha_1)</td>
<td>Gain of accelerometer, V·s²/m</td>
</tr>
<tr>
<td>(K_t)</td>
<td>Tire stiffness, N/m</td>
<td>(\alpha_2)</td>
<td>Gain of displacement meter, V/m</td>
</tr>
<tr>
<td>(K_T)</td>
<td>Torque constant of motor, N·m/A</td>
<td>(\theta)</td>
<td>Rotation angle, rad</td>
</tr>
<tr>
<td>(L)</td>
<td>Inductance, H</td>
<td>(\omega)</td>
<td>Angular velocity, rad/s</td>
</tr>
<tr>
<td>(m_b)</td>
<td>Body mass, kg</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\beta m_s \bar{K}_s & -\beta m_s \bar{C}_f & \beta (m_s \bar{K}_s - I_d \bar{K}_i) & \beta m_s \bar{C}_f \\ \gamma_1 (1 - \beta \gamma_2) \bar{K}_s & \gamma_1 (1 - \beta \gamma_2) \bar{C}_f & \gamma_1 (\beta \gamma_2 - 1) (\bar{K}_s + \bar{K}_i) - \beta I_d \bar{K}_i & \gamma_1 (\beta \gamma_2 - 1) \bar{C}_f \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

\[
B_p = \begin{bmatrix} -\alpha & 0 & -\beta m_s d_{is} & -\beta I_d d_{is} & -\beta m_s d_{of} & \beta m_s d_{of} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

\[
B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \beta I_d \bar{K}_i & 0 & 0 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & \beta m_s \bar{\Phi} \\ 0 & 0 \end{bmatrix}
\]

\[
\gamma_1 (1 - \beta \gamma_2) d_{is} & \gamma_1 (\beta \gamma_2 - 1) d_{of} & \gamma_1 (1 - \beta \gamma_2) d_{of} & \gamma_1 (\beta \gamma_2 - 1) d_{of} \\
\end{bmatrix}
\]

\[
C_q = \begin{bmatrix} -\beta m_s \bar{K}_s & -\beta m_s \bar{C}_f & \beta (m_s \bar{K}_s - I_d \bar{K}_i) & \beta m_s \bar{C}_f \\ -\beta \gamma_1 \gamma_2 \bar{K}_s & -\beta \gamma_1 \gamma_2 \bar{C}_f & \beta \gamma_1 \gamma_2 (\bar{K}_s + \bar{K}_i) - \beta I_d \bar{K}_i & \beta \gamma_1 \gamma_2 \bar{C}_f \\ \bar{K}_s & \bar{C}_f & 0 & \bar{C}_f \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

\[
C_z = \begin{bmatrix} -\beta m_s \bar{K}_s & -\beta m_s \bar{C}_f & \beta (m_s \bar{K}_s - I_d \bar{K}_i) & \beta m_s \bar{C}_f \\ 0 & 0 & 1 & 0 \end{bmatrix}
\]
\[
C_y = \begin{bmatrix}
-\alpha \beta m_w \bar{K}_i & -\alpha \beta m_w \bar{C}_f & \alpha \beta (m_w \bar{K}_s - I_d \bar{K}_i) & \alpha \beta m_w \bar{C}_f \\
\alpha & 0 & -\alpha & 0 \\
0 & -\beta m_w d_{sv} & -\beta I_d d_{sv} & -\beta m_w d_{ef} & \beta m_w d_{ef} & 0 \\
0 & -\beta (\gamma \gamma_2 - I_d) d_{sv} & -\beta \gamma_2 d_{sv} & \beta \gamma_2 d_{ef} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad D_{q1} = \begin{bmatrix}
\beta I_d \bar{K}_i & 0 & 0 \\
\beta (I_d - \gamma \gamma_2) \bar{K}_i & 0 & 0 \\
0 & 0 & 0 \\
-\bar{K}_i & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \\
D_{q2} = \begin{bmatrix}
\beta m_w \bar{\Phi} \\
\beta \gamma \gamma_2 \bar{\Phi} \\
0 \\
0 \\
0 \\
-\bar{\Phi}
\end{bmatrix}, \quad D_{q} = \begin{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{bmatrix}
\]

\[
D_{s2} = \begin{bmatrix}
-\alpha \alpha & 0 & -\alpha \beta m_w d_{sv} & -\alpha \beta I_d d_{sv} & -\alpha \beta m_w d_{ef} & \alpha \beta m_w d_{ef} \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad D_{s} = \begin{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{bmatrix}
\]

\[
D_{r1} = \begin{bmatrix}
\beta I_d \bar{K}_i & 0 & 0 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{bmatrix}, \quad D_{r2} = \begin{bmatrix}
\beta m_w \bar{\Phi} \\
0 \\
0
\end{bmatrix}, \quad D_{r} = \begin{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{bmatrix}
\]

\[
D_{t1} = \begin{bmatrix}
\alpha \beta m_w \bar{C}_f & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}, \quad D_{t2} = \begin{bmatrix}
\alpha \beta m_w \bar{\Phi} \\
0 \\
0
\end{bmatrix}
\]