A Geometric Method for Analyzing of Second-Order Autonomous Systems and Its Applications

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Abstract: - In this paper, the behavior of a second-order dynamical system around its equilibrium point will be analyzed based on the behavior of some appropriate equipotential curves which will be considered around the same equilibrium point. In fact two sets of equipotential curves are considered so that a set of the equipotential curves has a role as the upper band of the system trajectory and another set plays a role as the lower band. It will be shown that stability of the system around its equilibrium point can be assessed using the behavior of these two set of equipotential curves. As will be shown, asymptotically stability and instability analysis of the system only need the analysis of the upper band set of the equipotential curves, but oscillation behavior analysis of the system need to analyze both the lower band set of the equipotential curves and the upper band set. The method can even detect a stable limit cycle appearing in the oscillation systems. The proposed method is geometric and has some applications such as designing of oscillators. Finally, some examples, practical designing of oscillators and simulation results will be presented to verify and validate the presented method.

Key-Words: - Equilibrium point, stability, instability, autonomous system.

1 Introduction

As we know, two classic methods are essentially used to analyze the stability of a nonlinear system [1], [2]. The first method is stability analysis using energy function and the second method is based on the linearization of the system around its equilibrium point. Sometimes the first method is called “Direct Method” and the second method is called “Indirect Method”. In fact, the first method was presented by the mathematician called Lyapunov. The complexity of the first method is to find the appropriate energy functions to assess the stability of a nonlinear system. The second method has two major weaknesses. Firstly, if the eiginvalues of the coefficient matrix in the linear system resulted of linearization have real parts that are equal zero, the original nonlinear system may be stable or instable around the equilibrium point. Secondly, the method can only assess stability as the form of local stability around the equilibrium point. The proposed method mentioned in this paper is a geometric method which is suitable to analyze the stability of a second-order autonomous systems. This kind of system is very important because it models the behavior of some devices such as oscillators [3].

Also two basic methods are essentially used to analyze the behavior of limit cycles appearing in nonlinear system [1], [2]. The first method is to draw the trajectories of the system using the softwares such as Matlab in order to detect the limit cycles of the system. It is clear that the limit cycles that is detected by this approach can be recognized as stable, unstable or semi-stable [1], [2]. The second method is based on the linearization of the system around its equilibrium point or points. The second method has two major weaknesses. Firstly, the equilibrium point of the system, which the system is linearized around it, must be on the limit cycle or in a small neighborhood of it otherwise the method can not detect the real behavior of the nonlinear system. Secondly, the method can only assess the behavior of the system around the limit cycle as the form of point to point if and only if these points all locate on the limit cycle [1]. Also there are several classic methods to design oscillators. The most general and important method is based on using the equation, which the open loop transfer function of the system or electronic circuit equals minus one [3]. It is clear that the design is done in Laplace domain. As we will see, the proposed geometric method in this paper is much easier than above method to design electronic oscillator [3], [4]. There are some researches presenting some geometric and numerical methods to analyze the stability and behavior of some special systems such as Josephson junction system [5-19]. These methods can not be applied for analyzing of general second or higher order systems.

2 Equipotential Curves

Consider the second-order autonomous system described by the following equation

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases} \tag{1}$$

Definition 1: A set is said "compact" if it is bounded and closed [1], [4].

Definition 2: Consider the set called P so that $P \subset R^2$, P is said "invariant set" if the trajectories of the system beginning in the P remain in it as $t \to \infty$ [1], [20], [21].

Definition 3: Suppose that the $X = 0$ is the equilibrium point of the second-order autonomous system described by the equation (1) and suppose that the compact set called M includes the equilibrium point (the origin). The closed curves belonging the M, which is described by $u(x_1, x_2) = C$ so that $C \in R$ and enclosing the equilibrium point, are called equipotential curves because for each value of C there is a closed curve with the potential of C, so all points locating on the $u(x_1, x_2) = C$ have the equal potential the numerical quantity of which is C.

3 Stability Analysis

Theorem 1: The second-order autonomous system described by (1) is local asymptotic stable around the equilibrium point ($X = 0$) if there are equipotential curves $u(x_1, x_2) = C$ with clockwise direction, enclosing the equilibrium point and further on the trajectories of the system (1)

$$\frac{du(x_1, x_2)}{dt} < 0 . \tag{2}$$

Proof: From $u(x_1, x_2) = C$, we have

$$\frac{\partial u(x_1, x_2)}{\partial x_1} \dot{x}^*_1 + \frac{\partial u(x_1, x_2)}{\partial x_2} \dot{x}^*_2 = 0 \tag{3}$$

and as a result, the dynamic of $u(x_1, x_2) = C$ can be expressed as

$$\begin{cases} \dot{x}^*_1 = \dfrac{\partial u(x_1, x_2)}{\partial x_2} \\ \dot{x}^*_2 = -\dfrac{\partial u(x_1, x_2)}{\partial x_1} \end{cases} \tag{4}$$

where $\dot{x}^*_1$ and $\dot{x}^*_2$ are the state variables of the dynamic of $u(x_1, x_2) = C$. The velocity vector on the $u(x_1, x_2) = C$ symbolized by $\vec{V}_u$ is define as $\vec{V}_u = \dot{x}^*_1 \ \vec{u}_{x_1} + \dot{x}^*_2 \ \vec{u}_{x_2}$, so from (4) we found that

$$\vec{V}_u = \frac{\partial u(x_1, x_2)}{\partial x_2} \vec{u}_{x_1} + \left(-\frac{\partial u(x_1, x_2)}{\partial x_1}\right) \vec{u}_{x_2} \tag{5}$$

where $\vec{u}_{x_1}$ and $\vec{u}_{x_2}$ are respectively the unity vectors of the x_1 axis and x_2 axis. Also, the velocity vector of the system (1) is defined as

$\vec{\dot{X}} = \dot{x}_1 \vec{u}_{x_1} + \dot{x}_2 \vec{u}_{x_2}$,so it can be written as

$$\vec{\dot{X}} = f_1(x_1,x_2)\ \vec{u}_{x_1} + f_2(x_1,x_2)\ \vec{u}_{x_2}. \quad (6)$$

The derivative $\frac{du(x_1,x_2)}{dt}$ on the trajectories of the system (1) can be expressed as

$$\frac{du(x_1,x_2)}{dt} = \frac{\partial u(x_1,x_2)}{\partial x_1}\dot{x}_1 + \frac{\partial u(x_1,x_2)}{\partial x_2}\dot{x}_2 \quad (7)$$

or

$$\frac{du(x_1,x_2)}{dt} = \frac{\partial u(x_1,x_2)}{\partial x_1} f_1(x_1,x_2) + \frac{\partial u(x_1,x_2)}{\partial x_2} f_2(x_1,x_2) \quad (8)$$

$$= \overline{\vec{V}_u \times \vec{\dot{X}}}$$

where $\overline{\vec{V}_u \times \vec{\dot{X}}} = |\vec{V}_u|.|\vec{\dot{X}}|\sin(\alpha)$ and α is the angle between $\vec{V}_u$ and $\vec{\dot{X}}$. So the inequality (2) can be written as

$$\overline{\vec{V}_u \times \vec{\dot{X}}} < 0 \quad (9)$$

and this means that the direction of the trajectories of the system (1) are to inside of the equipotential curves $u(x_1,x_2) = C$ as shown in Fig. 1, on the other hand $C \in R$ and C can be changed so that closed curves $u(x_1,x_2) = C$ enclosing the equilibrium point could tend to be smaller and smaller and finally approach to the equilibrium point (the origin). This means that the direction of the trajectory of the system (1) will tend to the origin, so the system (1) is asymptotically stable.

Theorem 2: The second-order autonomous system described by (1) is unstable around the equilibrium point ($X = 0$) if there are equipotential curves $u(x_1,x_2) = C$ with clockwise direction, enclosing the equilibrium point and further on the trajectories of the system (1)

$$\frac{du(x_1,x_2)}{dt} > 0. \quad (10)$$

Proof: It follows from the proof of the theorem 1 that inequality (10) means that

$$\overline{\vec{V}_u \times \vec{\dot{X}}} > 0 \quad (11)$$

and in the similar manner with the proof of the theorem 1, the direction of the trajectories of the system (1) are to outside of the equipotential curves $u(x_1,x_2) = C$ as shown in Fig. 2, so the trajectories tend to infinity (far and farther of the equilibrium point) and this means that the system around the equilibrium point ($X = 0$) is unstable.

Definition 4: A limit cycle is said asymptotic stable if all trajectories in vicinity of the limit cycle converge to it as $t \to \infty$. Otherwise the limit is semi-stable or unstable[2].

Theorem3: Consider the second-order autonomous system (1), suppose that no equilibrium point belongs to the compact set M which encloses the origin ($X = 0$). There are equipotential curves $u_1(x_1,x_2) = C_1$ and $u_2(x_1,x_2) = C_2$ with clockwise directions that belong to M, does not intersect one with another, enclose the origin and satisfy the following inequalities on the trajectories of the system (1)

$$\frac{du_1(x_1,x_2)}{dt} \geq 0 \quad (12)$$

$$\frac{du_2(x_1,x_2)}{dt} \leq 0 \quad (13)$$

if and only if there exists an asymptotic stable limit cycle L so that

$$L \subset \operatorname{int}\Omega \quad (14)$$

where $\Omega(C_1,C_2)$ is the region located between $u_1(x_1,x_2) = C_1$ and $u_2(x_1,x_2) = C_2$.

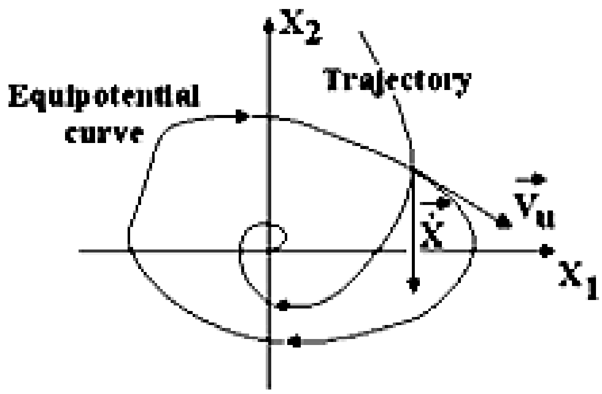

Fig. 1.The system is asymptotic stable.

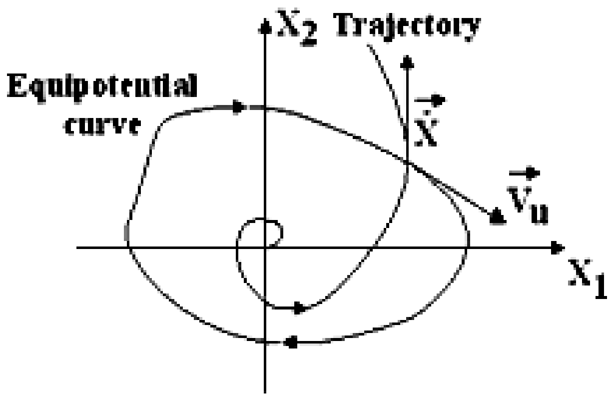

Fig. 2. The system is unstable.

Proof: ($\Rightarrow$) Similar to the theorem 1, the inequality (12) can be rewritten as

$$\overline{\vec{V}_{u_1} \times \vec{\dot{X}}} \geq 0. \tag{15}$$

This means that the direction of the trajectories of the system (1) is to the outside of the equipotential curves $u_1(x_1, x_2) = C_1$ as shown in Fig. 3. In the similar manner the inequality (13) can be expressed as

$$\overline{\vec{V}_{u_2} \times \vec{\dot{X}}} \leq 0 \tag{16}$$

where $\vec{V}_{u_2}$ is the velocity vector on the $u_2(x_1, x_2) = C_2$ and this means that the direction of the trajectories of the system (1) is to the inside of the equipotential curves $u_1(x_1, x_2) = C_1$ as shown in Fig. 3. On the other hand there is no equilibrium points belonging to M and consequently to $\Omega(C_1, C_2)$, so there is an asymptotic stable limit cycle L so that $L \subset \operatorname{int}\Omega$.

($\Leftarrow$) The necessary condition can similarly be proofed using above geometric concepts.

Remark1: If on the trajectories of the system (1) $\frac{du_1(x_1, x_2)}{dt} = 0$ or $\frac{du_2(x_1, x_2)}{dt} = 0$, the equipotential curve $u_1(x_1, x_2) = C_1$ or $u_2(x_1, x_2) = C_2$ itself is the limit cycle respectively.

Example 1: Consider the following system

$$\begin{cases} \dot{x}_1 = x_1({x_1}^2 + {x_2}^2 - 2) - 4x_1{x_2}^2 \\ \dot{x}_2 = 4{x_1}^2 x_2 + x_2({x_1}^2 + {x_2}^2 - 2) \end{cases}.$$

By choosing $u(x_1, x_2) = \frac{1}{2}({x_1}^2 + {x_2}^2) = C$, for $C < 1$, it can been seen that not only the equipotential curves $u(x_1, x_2) = C$ are closed but also on the trajectories of the system $\frac{du(x_1, x_2)}{dt} < 0$, so the system is asymptotic stable.

Example 2: Consider the following system

$$\begin{cases} \dot{x}_1 = e^{x_1}\sin(x_2) - 4{x_1}^3 \\ \dot{x}_2 = e^{x_2}\sin(x_1) + {x_2}^7 \end{cases}.$$

By choosing $u(x_1, x_2) = -\frac{1}{2}{x_1}^{-2} - \frac{2}{3}{x_2}^{-6} = C$, for all value of the C, it can been seen that not only the equipotential curves $u(x_1, x_2) = C$ are closed but also on the trajectories of the system $\frac{du(x_1, x_2)}{dt} > 0$, so the system is unstable.

Example 3: Consider the following system

$$\begin{cases} \dot{x}_1 = x_2 - {x_1}^7({x_1}^4 + 2{x_2}^2 - 10) \\ \dot{x}_2 = -{x_1}^3 - 3{x_2}^2({x_1}^4 + 2{x_2}^2 - 10) \end{cases}.$$

By choosing

$u_1(x_1, x_2) = \frac{1}{4}{x_1}^4 + \frac{1}{2}{x_2}^2 = C_1$,for

$0 < C_1 < 2.5$, it can been seen that not only the equipotential curves $u_1(x_1, x_2) = C_1$ are closed but also we have

$\frac{du_1(x_1, x_2)}{dt} > 0$. Also by choosing

$u_2(x_1, x_2) = \frac{1}{4}{x_1}^4 + \frac{1}{2}{x_2}^2 = C_2$, for $2.5 < C_2$, it can been seen that not only the equipotential curves $u_2(x_1, x_2) = C_2$ are closed but also $\frac{du_2(x_1, x_2)}{dt} < 0$, so there is an asymptotic stable limit cycle located between the $u_1(x_1, x_2) = C_1$ and $u_2(x_1, x_2) = C_2$. The area located between the $u_1(x_1, x_2) = C_1$ and

$u_2(x_1, x_2) = C_2$ is an invariant set as the following

$$\Omega(C_1, C_2) = \left\{ (x_1, x_2) \middle| C_1 < \frac{1}{4}x_1^{\ 4} + \frac{1}{2}x_2^{\ 2} < C_2 \right\} \quad (17)$$

where $0 < C_1 < 2.5 < C_2$. It is clear that the limit cycle can be estimated by varying the C_1 and C_2 in (17). In above set by increasing C_1 and decreasing C_2, the limit cycle can be found as

$$\frac{1}{4}x_1^{\ 4} + \frac{1}{2}x_2^{\ 2} = 2.5 .$$

4 Control of Limit Cycle Using State Feedback

Consider the following nonlinear autonomous system

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, u_1^*) \\ \dot{x}_2 = f_2(x_1, x_2, u_2^*) \end{cases} \quad (18)$$

where u_1^* and u_2^* are the control inputs as the form of state feedback presented by the following equations

$$\begin{cases} u_1^* = h_1(x_1, x_2) \\ u_2^* = h_2(x_1, x_2) \end{cases} . \quad (19)$$

Now, the question is that how $h_1(x_1, x_2)$ and $h_2(x_1, x_2)$ must be chosen so that an asymptotic stable limit cycle can be added to the system (18)? The condition $\frac{du_1(x_1, x_2)}{dt} \geq 0$ on the trajectories of the system (18) in the theorem 3 can be rewritten as following inequality

$$\begin{aligned} &\frac{du_1(x_1, x_2)}{dx_1} f_1(x_1, x_2, u_1^*) + \\ &\frac{du_1(x_1, x_2)}{dx_2} f_2(x_1, x_2, u_2^*) \geq 0 \end{aligned} \quad (20)$$

and in the similar manner the $\frac{du_2(x_1, x_2)}{dt} \leq 0$ appeared in the theorem 3, can be expressed as

$$\begin{aligned} &\frac{du_2(x_1, x_2)}{dx_1} f_1(x_1, x_2, u_1^*) + \\ &\frac{du_2(x_1, x_2)}{dx_2} f_2(x_1, x_2, u_2^*) \leq 0 \end{aligned} . \quad (21)$$

The inequalities (20) and (21) give the conditions which have to be satisfied by u_1^*, u_2^*, $u_1(x_1, x_2)$ and $u_2(x_1, x_2)$ in order to appear an asymptotic limit cycle in the system (18).

Example 4: Consider the following system

$$\begin{cases} \dot{x}_1 = x_2^{\ 7} - x_1^{\ 3} + u_1^* \\ \dot{x}_2 = -x_1 - x_1^{\ 2}x_2 + u_2^* \end{cases} \quad (22)$$

It is clear that the equilibrium point at the origin is asymptotic stable. Now, the state feedback lows (u_1^* and u_2^*) have to be determined so that an asymptotic stable limit cycle can be added to the resulted closed loop system. By choosing equipotential curves as

$$u_1(x_1, x_2) = 4x_1^{\ 2} + x_2^{\ 8} = C_1 ; 0 < C_1 < 12 \quad (23)$$

$$u_2(x_1, x_2) = 4x_1^{\ 2} + x_2^{\ 8} = C_2 ; 14 < C_2$$

(24)

and replacing (23) and (24) in (20) and (21), respectively the following inequalities are found

$$8x_1(x_2^{\ 7} - x_1^{\ 3} + u_1^*) + 8x_2^{\ 7}(-x_1 - x_1^{\ 2}x_2 + u_2^*) \geq 0 ;$$

$$\text{for } 0 < 4x_1^{\ 2} + x_2^{\ 8} < 12 \quad (25)$$

and

$$8x_1(x_2^{\ 7} - x_1^{\ 3} + u_1^*) + 8x_2^{\ 7}(-x_1 - x_1^{\ 2}x_2 + u_2^*) \leq 0 ;$$

$$\text{for } 14 < 4x_1^{\ 2} + x_2^{\ 8} . \quad (26)$$

It can be derived from (25) and (26) that

$$-8x_1^{\ 4} - 8x_1^{\ 2}x_2^{\ 8} + 8x_1u_1^* + 8x_2^{\ 7}u_2^* \geq 0 ;$$

$$\text{for } 0 < 4x_1^{\ 2} + x_2^{\ 8} < 12 \quad (27)$$

and

$$-8x_1^{\ 4} - 8x_1^{\ 2}x_2^{\ 8} + 8x_1u_1^* + 8x_2^{\ 7}u_2^* \leq 0 ;$$

$$\text{for } 14 < 4x_1^{\ 2} + x_2^{\ 8} . \quad (28)$$

By choosing the state feedback laws as the following forms

$$\begin{cases} u_1^* = h_1(x_1, x_2) = x_1(\beta + \frac{3}{4}x_2^{\ 8}) \\ u_2^* = h_2(x_1, x_2) = 0 \end{cases} \quad (29)$$

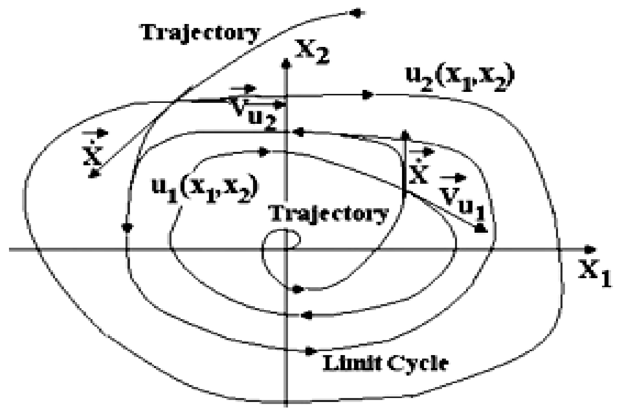

Fig. 3.The system has an asymptotic stable limit cycle.

and by replacing (29) in (27) and (28), the following inequalities are found as the conditions to appear an asymptotic limit cycle in the system

$$-2x_1^2(4x_1^2+x_2^8-4\beta)\geq 0\,;$$
$$\text{for}\,0<4x_1^2+x_2^8<12 \tag{30}$$

and

$$-2x_1^2(4x_1^2+x_2^8-4\beta)\leq 0\,;$$
$$\text{for}\,14<4x_1^2+x_2^8\,. \tag{31}$$

The inequalities (30) and (31) both are satisfied, when

$$3\leq\beta\leq\frac{14}{3}\,. \tag{32}$$

It also follows from the theorem 3 that the asymptotic stable limit cycle L, which is added to the system (18) using state feedback, appears in the following region

$$L\subset\left\{(x_1,x_2)\,\middle|\,12\leq 4x_1^2+x_2^8\leq 14\right\}. \tag{33}$$

5 Design of Oscillation in Electronic Circuits

Consider the basic model of an oscillator shown in Fig. 4 including LC tank and dependent current source. The state equations of the circuit can be written as the following autonomous system

$$\begin{cases}\dot{x}_1=-\dfrac{1}{C}x_2+\dfrac{1}{C}u_1^*\\ \dot{x}_2=\dfrac{1}{L}x_1\end{cases} \tag{34}$$

where x_1 and x_2 are defined as the voltage which appears across the capacitor and the current of the inductor respectively. As we see the dependent current source plays the role of the control input. By considering the equation (19) and defining the state feedback as the following equation

$$u_1^*=f(\|x_1\|_\infty)x_1 \tag{35}$$

, the equation (34) can be rewritten as

$$\begin{cases}\dot{x}_1=-\dfrac{1}{C}x_2+\dfrac{1}{C}f(\|x_1\|_\infty)x_1\\ \dot{x}_2=\dfrac{1}{L}x_1\end{cases} \tag{36}$$

where $\|\,.\,\|_\infty$ is infinite norm and $f(\,.\,)$ is determined by the electronic elements such as BJT, MOSFET and… used to design the oscillator. Now, the equipotential curves $u_1(x_1,x_2)=C_1$ and $u_2(x_1,x_2)=C_2$ are considered as the circles surrounding the origin and described by the following equations

$$u_1(x_1,x_2)=\frac{1}{L}x_1^2+\frac{1}{C}x_2^2=C_1 \tag{37}$$

$$u_2(x_1,x_2)=\frac{1}{L}x_1^2+\frac{1}{C}x_2^2=C_2 \tag{38}$$

where $C_1\leq C_2$. By checking (12) and (13) of the theorem 3, we have

$$\frac{du_1(x_1,x_2)}{dt}=\frac{2}{LC}f(\|x_1\|_\infty)x_1^2\geq 0 \tag{39}$$

$$\frac{du_2(x_1,x_2)}{dt}=\frac{2}{LC}f(\|x_1\|_\infty)x_1^2\leq 0\,. \tag{40}$$

It is clear that above inequalities can not both be satisfied unless

$$f(\|x_1\|_\infty)=0\,. \tag{41}$$

This is the necessary and sufficient condition to appear oscillation in the circuit.

Suppose that the oscillation appearing in the circuit is as the form of sinusoidal wave, in other word the voltage appearing across the capacitor of the tank is expressed by the following equation

$$x_1=V_m\cos(\omega_s t) \tag{42}$$

It is clear that

$$\omega_s=\frac{1}{\sqrt{LC}} \tag{43}$$

and

$$\|x_1\|_\infty=\sup\left|V_m\cos(\omega_s t)\right|=V_m \tag{44}$$

, so the equation (32) can be rewritten as

$$f(\|x_1\|_\infty) = f(V_m) = 0\,. \quad (45)$$

Example 5: Consider the Hartly and Colpitts oscillators shown in Fig. 5 and Fig. 6 and their equivalent circuit shown in Fig. 7. The equivalent circuit can be summarized and drawn again as Fig. 8. By comparing Fig. 8 and Fig. 4 and using equation (45), the necessary and sufficient condition to appear sinusoidal oscillation on the output of the Hartly and Colpitts oscillators can be found as the following equation

$$f(V_m) = G_m(nV_m)n - G_L - n^2(G_E + \frac{G_m(nV_m)}{\alpha}) = 0$$

where $G_m(\ .\ )$ is the large signal transconductance of the BJT used in the oscillator. It derives from the above equation that

$$G_m(nV_m) = \frac{G_L + n^2 G_E}{n(1-\frac{n}{\alpha})}\,. \quad (46)$$

From equation (46), the necessary and sufficient condition to appear oscillation in the Colpitts oscillator can be rewritten as

$$\frac{G_m(nV_m)}{g_m} = \frac{G_L + n^2 G_E}{g_m n(1-\frac{n}{\alpha})} \quad (47)$$

where g_m is the small signal transconductance of the BJT evaluated at the quiescent point. By defining $x = \frac{nV_m}{(kT/q)}$ where $k = 1.37\times10^{-23}\ \frac{J}{K}$ is boltzmann constant and $q = 1.6\times10^{-19} C$ is the charge of an electron, the equation (47) can be rewritten as

$$\frac{G_m(x)}{g_m} = \frac{G_L + n^2 G_E}{g_m n(1-\frac{n}{\alpha})}\,. \quad (48)$$

On the other hand, we have [3]

$$\frac{G_m(x)}{g_m} = \frac{2I_1(x)}{xI_0(x)}[1 + \frac{\ln(I_0(x))}{(qV_\lambda/kT)}], \quad (49)$$

where V_λ is the sum of the quiescent voltages appearing across the total resistance between base and emitter and $I_n(x)$ is a modified Bessel function of order n and is defined as

$$I_n(x) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{x\cos\theta}\cos(n\theta)d\theta\,. \quad (50)$$

It can be derived from equations (48) and (49) that

$$\frac{2I_1(x)}{xI_0(x)}[1 + \frac{\ln(I_0(x))}{(qV_\lambda/kT)}] = \frac{G_L + n^2 G_E}{g_m n(1-\frac{n}{\alpha})}\,. \quad (51)$$

The equation (51) can be used to earn the amplitude of the oscillation (V_m).

Example 6: Consider the Colpitts oscillator shown in Fig. 9 so that $R_L = 10\ k\Omega$, $C_1 = 1\ nF$, $C_2 = 79\ nF$, $R_E = 20\ k\Omega$, $L = 10\ \mu H$, $V_{CC} = -10\ V$ and $\alpha = 0.99 \approx 1$.

We have

$$C = \frac{C_1 C_2}{C_1 + C_2} \approx 1\ nF\ , n = \frac{C_1}{C_1 + C_2} = \frac{1}{80}\,,$$

$$V_\lambda = 9.3\ V \text{ and } I_C \approx I_E = \frac{9.3\ V}{20\ k\Omega} = 0.465\ mA\,.$$

The frequency can be found as

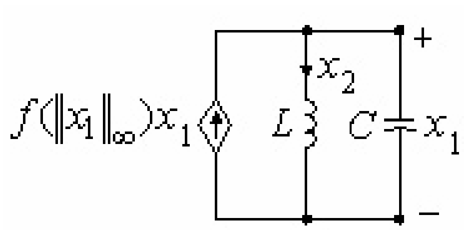

Fig. 4. The basic model of an oscillator.

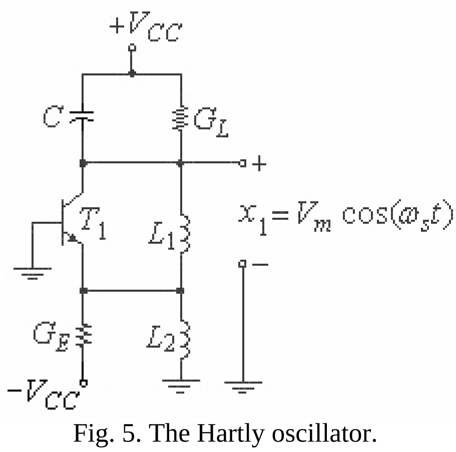

Fig. 5. The Hartly oscillator.

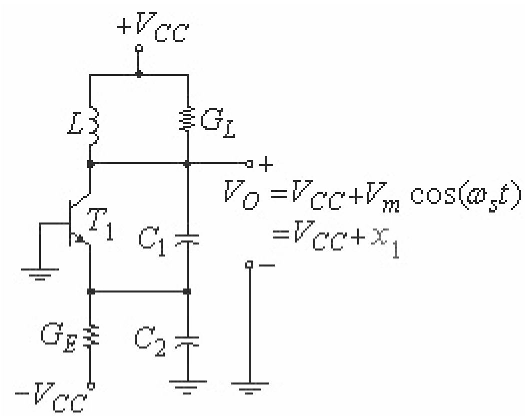

Fig. 6. The Colpitts oscillator.

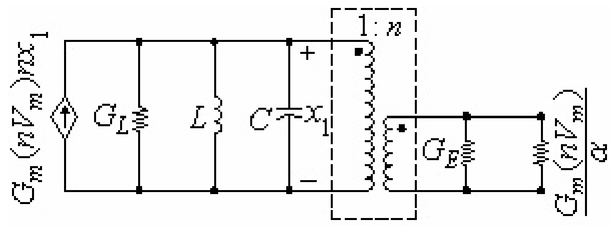

Fig. 7. The equivalent circuit of the oscillators.

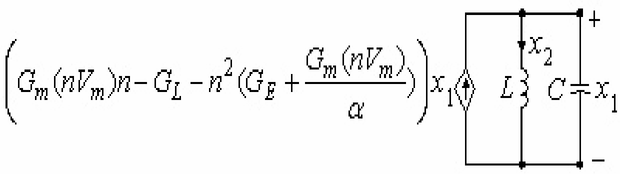

Fig. 8. The summarized equivalent circuit of the oscillators.

$$\omega_s = \frac{1}{\sqrt{LC}} = 10^7 \ \ rad/\sec$$

and

$$g_m = \frac{qI_C}{kT} = \frac{1}{56} \ \ \Omega^{-1}.$$

Replacing g_m, n, G_L, α and G_E in equation (44) results that

$$\frac{2I_1(x)}{xI_0(x)}[1+\frac{\ln(I_0(x))}{(9.3/0.026)}] = 0.448 \tag{52}$$

We obtain $x \approx 3.5$ and the numerical solution of the equation (52) and replacing in $x = \frac{nV_m}{(kT/q)}$

gives $V_m = 7.9 \ V$, and finally

$$V_O(t) = 10 + 7.9\cos(10^7 t) \ [v]. \tag{53}$$

The circuit was simulated by PROTEUS-6 software and again repeated by PSPICE-8 software, the results were the same. As we see, the result of the simulations shown in Fig. 10, validates the proposed geometric method and the numerical result expressed by eq. (54).

Example 7: Consider the Hartly oscillator shown in Fig. 11 so that $R_L = 10 \ k\Omega$, $C = 10 \ nF$, $R_E = 20 \ k\Omega$, $L_1 = 1\mu H$, $L_2 = \frac{1}{80}\mu H$, $V_{CC} = -10 \ V$ and $\alpha = 0.99 \approx 1$.

We have

$$L = L_1 + L_2 \approx 1\mu H \ , n = \frac{L_2}{L_1 + L_2} \approx \frac{1}{80},$$

$$V_\lambda = 9.3 \ V \ \text{ and } I_C \approx I_E = \frac{9.3 \ V}{20 \ k\Omega} = 0.465 \ mA.$$

The frequency can be earn as

$$\omega_s = \frac{1}{\sqrt{LC}} = 10^7 \ \ rad/\sec$$

and

$$g_m = \frac{qI_C}{kT} = \frac{1}{56} \ \ \Omega^{-1}.$$

Replacing g_m, n, G_L, α and G_E in equation (51) results that

$$\frac{2I_1(x)}{xI_0(x)}[1+\frac{\ln(I_0(x))}{(9.3/0.026)}] = 0.448 \tag{54}$$

Again we obtain $x \approx 3.5$ and the numerical solution of the equation (46) and replacing in $x = \frac{nV_m}{(kT/q)}$ results that $V_m = 7.9 \ V$. So we have

$$V_O(t) = 7.9\cos(10^7 t) \ [v]. \tag{55}$$

Again the simulation result shown in Fig. 12 validates the proposed geometric method and the numerical result expressed by eq. (55).

6 Conclusion

In this paper, the behavior of a second-order dynamical system around its equilibrium point was analyzed based on the behavior of some appropriate equipotential curves which were considered around the same equilibrium point. In fact two sets of equipotential curves were considered so that a set of the equipotential curves had a role as the upper band of the system trajectory and another set played a role as the lower band. It was shown that stability of the system around its equilibrium point can be assessed using the behavior of these two set of equipotential curves. It was shown that the proposed geometric method can detect a stable limit cycle appearing in the systems and also the method has some useful applications such as designing of oscillators. Some examples and practical design of oscillators were presented and simulation results validated the presented method. For future works the method can be extended for analyzing of third order dynamical systems.

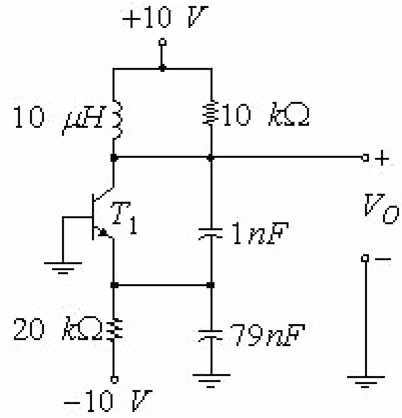

Fig. 9. The simulated Colpitts oscillator.

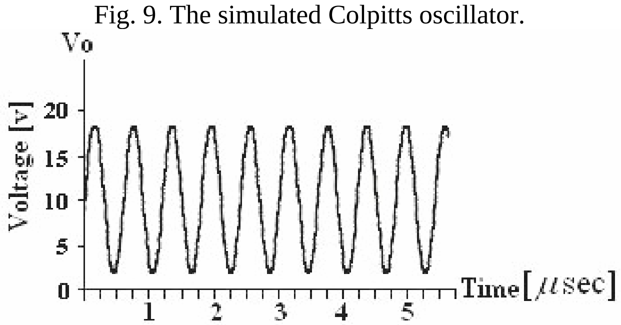

Fig. 10. The result of the simulation.

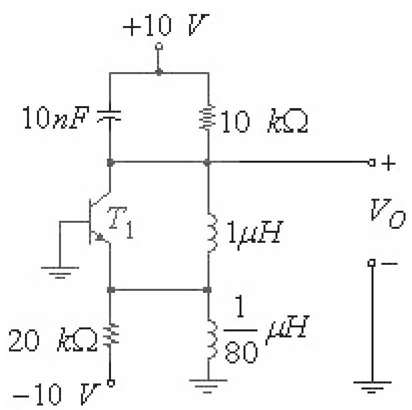

Fig. 11. The simulated Hartly oscillator.

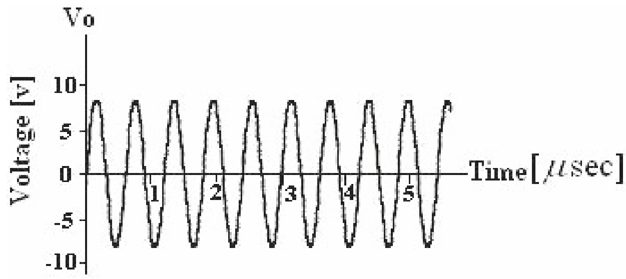

Fig. 12. The result of the simulation.

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