Design of Maximally Flat Filters for Signal Processing Applications

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Abstract: - Existing window based methods used for the design of finite impulse response filters suffer from possessing characteristics like the minimum sidelobe energy and the maximally flatness within the passband. The proposed method solves these problems. This paper presents a generalized window based approach to the design of maximally flat finite impulse response filters. The basis functions, which very closely approximates the prolate spheroidal wave functions, are explored. The novelty of this proposal is the use of these basis functions for designing the filter. Further, an additional parameter '*a*' is incorporated to control the filter specifications. An explicit formula for computation of the frequency response is derived, which is a new contribution. It is shown that the impulse response coefficients of the maximally flat filter can be obtained directly from the frequency response. Frequency domain characterization is made. Further contribution is to meet the given filter specifications with lower order generalized prolate type window functions. Statistical analysis is performed to validate the proposed method. The proposed maximally flat filter outperforms the state-of-the-art methods. Finally, it is concluded that the proposed filter exhibits better error convergence and tracking performance, which may be useful for precision filtering of biomedical signals because of their low passband and stopband errors.

Key-Words: - Digital filter design, maximally flat filters, prolate spheroidal wave functions, t-Test.

1 Introduction

Digital filters play a significant role in signal processing. Due to the advent of digital signal processing (DSP) techniques, there is a strong need to design and develop efficient digital filters. In this context, a large number of research papers appear each year with continuous perfection in the digital filter design methods. Digital filter is an important fragment of the digital signal processing system. Filters are classified into two types; i.e. FIR and IIR depending on the form of filter equations and the structure of the implementation [1, 2]. Window based technique and frequency sampling technique is two mostly used methods of FIR filter design. Requirements of minimum ripples in the passband and the stopband, stopband attenuation [3,4] and transition width decides a design criterion. In the past, researchers have proposed numerous procedures for the design of digital filters. It is noteworthy to say that FIR filter offers numerous benefits than IIR filter. Recently, researchers ponder the design of FIR filters as a demanding yet challenging problem.

The signal processing researchers have shown tremendous effort to improve accuracy, sidelobe reduction, processing speed and ease in implementation of digital filters. During the past few decades, different window functions have been developed for the design of FIR filters [5,6]. The Kaiser [7,8] window based FIR filters is useful for approximating the minimum sidelobe energy in the magnitude response. Dolph–Chebyshev [9] window based design approximates the minimum peak sidelobe ripple. In this connection, it is pertinent to mention here that B-spline window functions are found to be more widespread, because they very closely follow the looked-for frequency response [10-13]. Rectangular, Hanning and Hamming window based FIR filter performances are analyzed in [14]. These window functions exhibit sidelobes, which is not desirable for high precision filtering applications. Analysis of the Blackman window and Flat top window based FIR lowpass filter is presented in [15]. The Blackman window based filter has shown higher sidelobe roll-off rate than the Flat top window based design. From the analysis, it is seen that the Flat top window based design provides a better filtering effect compared to the Blackman window based method [15].

In [16], the authors have suggested an adjustable window for the design of FIR filter. They have combined a fixed window, tan hyperbolic function and a weighted cosine series. The weighted cosine series is simply multiplied by a variable. They have compared the performance of their window with the hamming and Kaiser window. Authors in [17] presented an adjustable window combining Blackman and Lanczos windows. The performance is compared with Gaussian, Lanczos and Kaiser windows. All these three windows have some control mechanism. Design of a lowpass FIR filter using Nuttall, Taylor and Tukey windows has been presented in [18]. Their behavior with adjustable window based design has not been compared. However, the above mentioned methods have more or less concentrated on the reduction of sidelobe roll-off rate. These windows based design methods exhibit better results when the order of the window is very high. They have never focused on the design issue of maximally flat filters. They are also silent about the direct implementation of FIR filters.

Nevertheless, recently the direct implementation of the FIR filters has shown much promise for gaining the maximally flat (MF) characteristics. These methods also provide us minimum energy in the sidelobe of the magnitude response. These direct implementation methods have shown definite progress above the window based design methods for implementation of the digital FIR filters. The reason is that the maximum sidelobe ripple of a particular window is fixed. The stopband attenuation is also fixed. Usually one encounters two critical issues while designing filters – i) control over the passband ripple, ii) control over the stopband ripple. These are a few flaws of the window based design methods. Note that for a specified attenuation specification, it is very important to search for an effective window function. We know that the impulse response (for an FIR) for a given desired $H(\omega)$ is given by

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega)e^{jwn}d\omega \qquad (1)$$

For discrete case the filtered output is given as

$$y[k] = \sum_{n=-\infty}^{\infty} x[k-n]\,h[n] \qquad (2)$$

The Gibb's phenomenon of truncating the above convolution series (Eq. (2)) is well known from the literature. Hence, the ideal filters are not implementable in practice. Several problems arise with the design and implementation of ideal filters. It is noteworthy to mention here that the impulse response of an ideal filter $h_d(n)$ decreases as 'n' increases. This is one of the major problems in implementing ideal filters in reality. This limits the stability of a digital filter and, thus, in turn, it is never feasible to implement a stable FIR filter. Hence, the truncation of the convolution series is required. The causal FIR filter obtained by simply truncating the convolution series shown in Eq. (2) exhibits an oscillatory behavior, which is known as the Gibb's phenomenon. The oscillatory nature or the number of ripples in the filter response increases with the increase in the order of the filter.

Owing to the above-mentioned reasons, the FIR digital filter designers concentrate on the alternative approaches to obtain the impulse response coefficients $h[n]$ or $H(\omega)$ that approximates the desired response. The best alternative is the direct design of FIR filter coefficients. The FIR filter coefficients are directly obtained from the frequency response equation displayed in Eq. (1), where $H(\omega)$ is the desired response. The direct design method of FIR filter coefficients has been implemented since 1999 [10]. The improvement in the response of filters based on the direct design reported in [10] has been found superior to other filter design methods. However, results were obtained based on more or less heuristics. They [10] have chosen fractional order (for example $\rho = 3.139$) heuristically to get the desired specifications instead of considering the order of filters to be pure integer numbers. B-spline interpolation techniques are popular because of certain properties of B-spline polynomials. Generalized B-spline interpolation schemes are more popular, because of additional smoothing. A brief idea about B-spline interpolation is discussed in [11]. Note that the sidelobe energy (of the magnitude response of the filter under consideration) can be greatly minimized by using the generalized B-spline basis functions. The sidelobe ripples are also significantly reduced by using these window functions.

It is noteworthy to mention here that the investigation of a signal, which is both time and band-limited, is very challenging. Solving the unsolved problems is given more significance nowadays. In this connection, exploring the beauty of prolate spheroidal wave functions (PSWF) is a worthwhile subject of study. Prolate meaning is – having a polar diameter of larger length than the equatorial diameter. Note that the PSWF are a set of functions derived from time-limiting and low passing followed by a second time-limiting operation. The PSWF are the time-limited functions. Slepian, Landau and Pollak [19] performed pioneering work in this area. It is important to claim the fact that the PSWF play an important role in solving different engineering problems. These

PSWF also play significant role in designing more ideal lowpass filters for signal processing applications. This is the motivation behind this proposal. In this work, we explore a set of basis functions, which very closely approximates the PSWF (Ref. Appendix A).

This has motivated us to design MF filters using spheroidal type generalized B-spline basis functions. Further motivation is the direct design. This paper presents a more generalized approach to design maximally flat filters with an additional parameter which is an improvement over the earlier methods [10,14-18]. The additional parameter is useful to control the filter response. The frequency response of the proposed MF filters has been compared with state-of-the-art methods. From the results given in the paper, it is observed that the suggested MF filter shows better error convergence and MF characteristics. It is believed that the proposed filters are very useful for signal processing applications.

The organization of the paper is as follows: A brief introduction to direct design of FIR filters has been given in the introduction section. Section 2 deals with the proposed method. Section 3 presents the results and discussions. The paper concludes in Section 4.

2 Proposed Method

The proposed method is an attempt to approximate the rectangular shape frequency response of an ideal low pass filter. It is noteworthy to point out the fact here that the direct method smooths out the sharp transition curve. Direct design approaches to implement FIR filters are becoming popular and use trapezoidal filters. In this section, we propose a direct approach to design a digital FIR filter using a generalized B-spline window function [12]. In this approach, a second order ($\rho=2$) B-spline function has been proposed. The window function is constructed by convolving a pulse (rectangular shape) of width $2\alpha\omega_c/\rho$ and height $\pi/\alpha\omega_c$ with itself. The focus is to obtain a smooth curve in the transition band. Additional smoothing in the transition band is achieved by repeatedly convolving two rectangular shaped window functions (width of each pulse is half and the height of each pulse is twice with respect to the original one). This approach is repeated and the ρ^{th} order window function is generated. The newly proposed window function is then convolved with the ideal filter frequency response (rectangular shape) to substitute the sharp transition curves.

The desired frequency response can be obtained by using the following relation:

$$\begin{aligned} H(\omega) &= H_I(\omega) * \phi(\omega) * \phi(\omega) * \cdots * \phi(\omega) \\ &= H_I(\omega) * \psi(\omega) \end{aligned} \tag{3}$$

where * denotes the convolution, $H_I(\omega)$ is the ideal low pass filter response and $\psi(\omega)$ is the required convolving function. Note that α and ρ are the controlling parameters. As per the above theoretical discussions, it is clear that ρ is an integer and depends on the number of poles of the transfer function of the filter. This impose a limitation on further improvement of the filter response. This has motivated the author to develop a more generalized approach to develop digital filters. The use of two sided Laplace transform is very common in signal processing. The main objective of this work is to introduce another controlling parameter 'a' in the design process. With a proper choice of the tuning parameter 'a', the desired response can be obtained without disturbing the order (ρ) of the filter to a nearby fractional number.

In this section, we propose a generalized window function by compelling an exponentially decreasing function instead of a rectangular pulse as displayed in Fig.1.

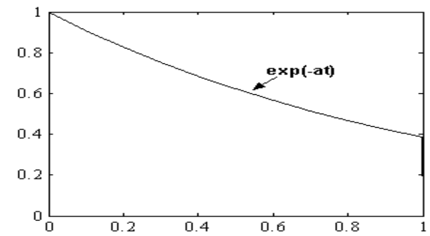

Fig.1. The window function.

Here, a group of generalized window functions is constructed following the basic idea that the generating function used to construct the window function of degree 'n' has (n-1) continuous derivatives. Note that these functions also satisfy a linear differential equation of order ($n+1$). A generalized window function can be defined as [12]. The polynomial $B^{n1}(x)$ is a particular case of these generalized windows. The following relation of $B^{nr}(x)$ is obtained.

$$B^{nr}(x) = N\sum_{j=0}^{n+1} w_j \, g^{nr}\left(x-x_j\right) u\left(x-x_j\right) \tag{4}$$

where n is the degree, r is the type, N is the constant of normalization, and $u(x)$ refers to a unit step function. Note that $g^{nr}(x)$ is the creating function, w_j are coefficients of multiplication in the j^{th} slice of the window function. The number of generalized window functions are more for a degree $n>3$ [12]. For a given degree n, many types r of generalized windows can be generated.

$$N = n+1 \quad (5)$$

$$w_k = \frac{1}{\prod_{\substack{j=0 \\ j\neq k}}^{n+1} (x - x_j)} \quad (6)$$

$$g(x) = x^n \quad (7)$$

The generalized window considered here is real. The generating function considered for the filter design is a solution of a differential equation satisfying continuity up to $(\rho-2)th$ order. The generating function also satisfies required initial conditions. Note that initial conditions in two sided Laplace transform play important role in analysis of system functions. The poles of the generalized B-spline window $(B(\omega))$ must be symmetrical about the real axis. To make the proposed window symmetrical about the center, the ensuing mirror symmetry condition is enacted.

$$g(x) = |g(-x)| \quad (8)$$

In order to satisfy Eq. (8), $(B(\omega))$ is symmetrical function. It possesses symmetry about the imaginary axis. Thus, for the proposed window, the poles of $(B(\omega))$ possesses four quadrant symmetry.

The proposed window function of any order can be calculated from a zero-degree window shown in Fig.1. Using the convolution properties discussed in [12], various types of windows are generated. The principled feature of the proposed functions is that these functions give compact support. One can generate many such types of windows for a degree more than 3. The generating function, which is considered as the seed, is $g^{01}(x) = \mathrm{e}^{(-a*x)}$, where '$a$' is the scale factor. The proposed idea is different from the design proposed in [10], where the generating function is a rectangular function. Whereas in this work, an exponentially decreasing window is used to generate the first generating function in the series. An additional parameter 'a' called scale factor is hosted. Interestingly, this factor guides the user for approximating the filter bandwidth competently.

The proposed window of zero degree is

$$B^{01}(x) = \frac{a}{(1-e^{-a})}\left[e^{-ax}u(x) - e^{-a}e^{-a(x-1)}u(x-1)\right] \quad (9)$$

It is interesting enough to note here that when $a \to 0$, this approaches to a zero order window proposed in [12]. This is same as that of a centered rectangular pulse. The first order generalized B-spline window can be generated by recognizing the fact that the poles of $(B(\omega))$ have four quadrant symmetry. The conditions are $s^2 = 0$, $s = \pm a$, $s = \pm ja$. In this filter development, we consider the third case, i.e. $s = \pm ja$.

Similarly, a large number of windows are constructed. A particular window is chosen from a set of functions for designing the MF filter. The frequency response closely approximates the desired response by increasing the degree of the window functions. In this method of filter design, the proposed window is used as the convolving function $\phi(\omega)$ of Eq. (3). The convolving function for the window is expressed as

$$B(\omega) = N\rho\frac{\pi}{\alpha\omega_c}\sum_{k=0}^{\rho} w_k\, g\left(\frac{\rho}{2}\left(\frac{\omega}{\alpha\omega_c}+1\right)-k\right) \quad (10)$$

$B(\omega)$ is a polynomial well-defined between the interval [-$\alpha\omega_c$ to $\alpha\omega_c$]. For other interval values, it is zero. Note that $H(\omega)$ (Eq. (3)) is obtained by repeated convolution. The function used is displayed in Fig. 2. Here, $N = \frac{a}{(1-e^{-a})}$. The corresponding ω_k values are displayed in [12].

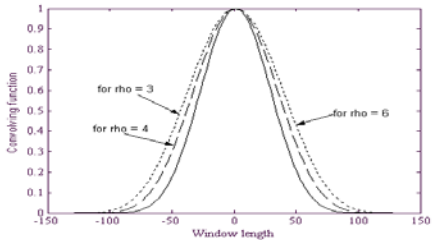

Fig. 2. The convolving function.

The filter response (lowpass) achieved by smearing the process $H_I(\omega) * B(\omega)$ is written as

$$H(\omega) = 1 - \frac{N}{\rho!}\sum_{k=0}^{\rho}(-1)^k w_k \binom{\rho}{k}\left[\frac{\rho}{2}\left(\frac{|\omega|-\omega_c}{\alpha\omega_c}+1\right)-k\right] \quad (11)$$

The discrete time impulse response is achieved by evaluating the inverse Fourier transform of Eq. (11). The discrete time impulse response of the lowpass filter is

$$h[n] = \frac{\omega_c}{\pi}\frac{\sin(n\omega_c)}{(n\omega_c)}\left(\frac{a^2}{1-\cos a}\right)\frac{\left(\cos(n/(M+1)) - \cos a\right)}{a^2 - \left(n/(M+1)\right)^2}\left(\mathrm{sinc}\right)^{\rho} \quad (12)$$

where sinc = $\sin(pi*x)\big/(pi*x)$ and $x = \left(\frac{n}{M+1}\right)$

The magnitude persists unity value all through the passband and zero all through the stopband frequencies. Hence, these filters are called maximally flat filters. We use an ideal filter, which is applied before $B(\omega)$. This is vital to escape the sharp transition. In this work, a lowpass filter is designed. Similarly, the discrete time impulse response for an ideal highpass, bandpass, and bandstop filter can also be designed. This kind of direct approach may be useful for many mean square signal-processing applications.

In this paper, standard definitions for cut off frequency ω_c, stopband attenuation A_s, and transition width $\Delta\omega$ are considered [6].Note that the stopband attenuation $A_s = -20\log_{10}(\delta_s)$ dB and the cutoff frequency $\omega_c = (\omega_s + \omega_p)/2$.

It is important to design MF filters for mean square signal processing applications. A real-world methodology is to multiply the ideal impulse response $h_d(n)$ by an appropriate window $w(n)$ with finite duration. As a result, the decay is faster. The impulse response reduces to a zero value quickly. The middle part of the impulse response can be used for designing a linear phase FIR filter. For an increased filter length, the ripples never vanish when we use a rectangular window. However, a non-rectangular window lessens its magnitude smoothly. One can see many single parameter window functions in the literature [14-18]. Examples include – Hanning, Hamming, Blackman, etc. These window functions have the control over the transition width $\Delta\omega$ of the filter mapped to a value of M (filter length). With an increased value of M only better results are expected. Unfortunately, the researcher, while using these filters, has no control over other given specifications like – stopband attenuation, flatness, the shape of the magnitude response etc.

Kaiser [8] investigated a window that used an extra control parameter. The designer can easily choose this parameter (β) to facilitate control over the stopband attenuation A_s. It is seen from the literature that the Kaiser window based filter design is a popular choice. Interestingly, the design method is very simplified and the magnitude response is better than other methods. This may be the reason why the Kaiser window based filter design is popular. However, the Kaiser window based filter offers very little control over the magnitude response. Subsequently, several researchers attempted to find improved design schemes. In this connection, window functions comprising additional parameters ware investigated. The main thrust was to accomplish a fitted transition width together with a higher stopband attenuation while holding the former filter features unaffected. Authors in [10] presented two extra control factors for achieving good filter results. The transition width and the stopband were controlled using those two additional parameters. The number of convolution operations were controlled using the filter order 'ρ'. As a result, the shape of the filter response is controlled.

In this context, a new method for the design of MF filter with improvement mechanism is explored. The key to the improvements claimed is the inclusion of an additional control parameter 'a'. The error in the stop band is minimized drastically while preserving the other specifications of the filter under consideration. The control parameter 'a' proves better to control the flatness of the desired response of the filter. This proposal has certain advantages over other methods.

2.1 Windows of MF filter

The window for the newly proposed MF filter investigated in this work is derived using Eq. (12):

$$W(n) = \left(\frac{a^2}{1-\cos a}\right)\frac{\left(\cos(n/(M+1)) - \cos a\right)}{a^2 - \left(n/(M+1)\right)^2}\left(sinc\right)^{\rho} \quad (13)$$

The control factors 'a', 'ρ', and 'M' are deployed to improve the filter response. For a comparison, the Kaiser window and the proposed window functions are displayed in Fig. 3, 4. Note that the length and the order are same for these window functions.

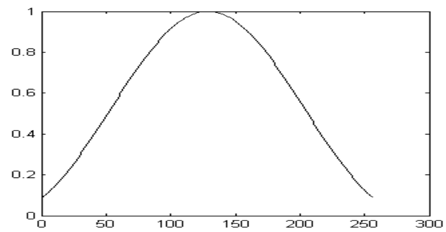

Fig. 3. Kaiser window ($\beta = 4$).

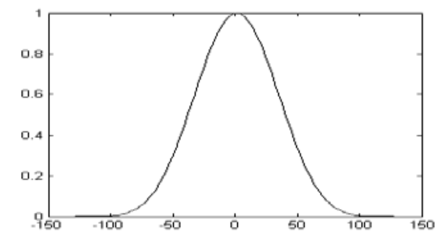

Fig. 4. The proposed window ($\rho = 4$ and $a=1$).

It is noteworthy to reiterate the fact that the proposed window offers compact support with finite spectral information. When the order is ρ-1, the window is flat. With higher order, the window is abridged smoothly to bind the energy of polynomial ($B(\omega)$) defined in Eq. (10). This is the reason why the passband and stopband error is minimized. Through extensive simulation, it is observed that the proposed window can be truncated smoothly for optimal value of 'a'. The proposed window developed in this work is different from the idea proposed in [10]. The additional factor 'a' is included along with other factors 'α' and 'ρ'. The frequency responses of the spline window and the proposed window for diverse 'a' and 'ρ' values are displayed in Fig. 5, 6(a) and 6(b).

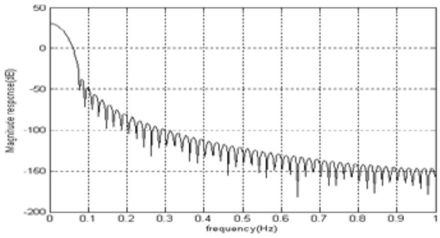

Fig. 5. B-Spline window, $\rho = 4$.

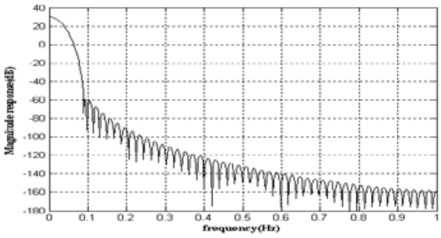

Fig. 6. (a) Proposed window, $\rho = 4$ and $a = 0.2$.

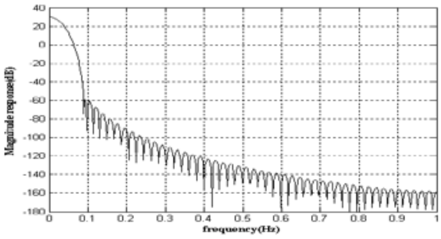

Fig. 6. (b) Proposed window, $\rho = 4$ and $a = 1.55$.

It is evident from Fig. 5 and 6 that the proposed window function has shown improved results. Tuning of the extra factor 'a' is crucial. When $a = 1.55$, the result is still better. Therefore, it is claimed that a significant improvement is achieved in the filter frequency response. Hence, the window for the MF filter can be defined as in Eq. (13). In this work, the order of the filter can assume only positive integer numbers.

2.2 Parameters used in the Design

The objective function mainly used for filter design is the error function. Note that the error in the passband of the filter response is defined as

$$E_p^2(\omega) = \frac{1}{\omega_1}\int_0^{\omega_1}\left[1-\hat{H}(\omega)\right]^2 W_A(\omega)d\omega \qquad (14)$$

where ω_1 represents the buildup interval. In general, the buildup interval is considered as the passband frequency ω_p. $W_A(\omega)$ denotes the weighting function. Here we consider the unity value for $W_A(\omega)$. An alternative measure for the passband error ω_m is [10].

$$\omega_m = \frac{\int_0^{\omega_1} \omega E_p(\omega)d\omega}{\int_0^{\omega_1} E_p(\omega)d\omega}, \qquad \text{for } 0 < \omega_1 \le \omega_p \qquad (15)$$

Note that ω_m is called the mean *rms* error frequency, ω_p provide us the centered error frequency within passband frequencies $0 < \omega_1 \le \omega_p$.

Extensive simulation work is carried out to design the MF filter using the proposed window function. The impulse response defined in Eq. (12) is used to determine the frequency responses. Here, the different filter parameters are varied over the ranges: 0< ρ <15, 0.001< α <0.8, 4<=M<=250, and 0.05 π <= ω_c <=0.5 π and 0.001< a <2.00. Stopband attenuation, passband error, and the transition width are measured for each designed filter. Interestingly, variations found in the filter specifications are negligible. Stopband attenuation, passband error, and the transition width are found very close to the actual values. The frequency responses are displayed in Fig. 7, 8. These plots are obtained for different values of filter parameters discussed above. A performance comparison is made in terms of frequency response. The improved version of the results is obtained by using the proposed idea of MF filter design.

3 Results and Discussions

A maximally flat filter is developed to meet the following specifications: A_s = 80 dB, ω_c = 0.4 π and M = 128. For a comparison, the Roark method [10] of maximally flat filter is also designed. Using the maximally flat design formula given in [10] (for these filter specifications), the value of 'ρ' needed to achieve 80 dB attenuation in the stopband is 3.545. Whereas the same specifications are achieved in our case (the proposed method) with $\rho = 3$ and $a = 1.728$. All other parameters are chosen same for the design of maximally flat filter. When the responses are obtained for the two windows, it is seen that the proposed window provides a small error in the stopband attenuation than the B-spline window proposed in [10].

The filter magnitude responses for the MF filter [10] and the proposed MF filter $|H(\omega)|$ are plotted in Fig. 7 and 8, respectively. As it is seen, the proposed filter maintains the superior flat passband features of MF filter over the Kaiser filter. Our proposed MF filter achieves the higher attenuation in the stopband. The filter discussed here also achieves a tighter transition width. It is observed that higher values of the scale factor '*a*' may lead to overshoots in the passband. Further, it may have significant gain beyond the filter passband, which is not desirable. With a good choice of the scale factor, the optimum response is accomplished. Usually it is expected that the mean frequency error ω_m should be as large as possible for a lowpass filter. This is due to the fact that errors are focused near the filter transition band. The proposed window based MF filter exhibits less error near $\omega = \pi$ than the other filters. A comparison can also be made to the other windows used for design of FIR filters [14-18]. All the comparison shown here are for the same filter parameters ω_c = 0.4π, M = 128, and ρ = 4 and $\alpha = 0.99$.

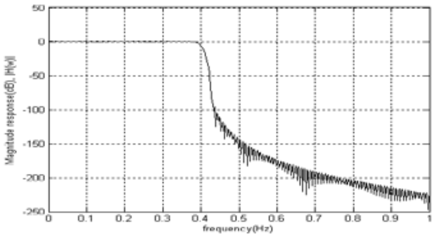

Fig. 7. Maximally flat filter response, ω_c =0.4π, M=128, ρ = 4 and $\alpha = 0.99$.

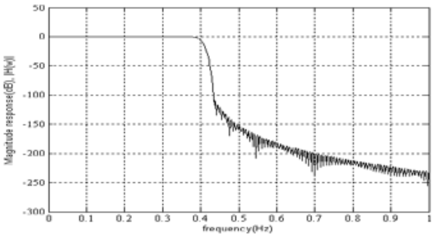

Fig. 8. Proposed filter response for ω_c =0.4π, M=128, ρ = 4, $\alpha = 0.99$ and $a = 1.752$.

The proposed filter response is seen to have more attenuation in the stopband. This paper presents an efficient filter with better passband and stopband cutoff rate. The proposed filtering delivers a correction over the aliasing inaccuracies. It also shows the noise reduction.

In case of Hamming window, Blackman window and Hanning window, the stopband attenuation is fixed, which depends on the type of a window. The transition width is controlled by the order of the filter i.e. 2*M*. It is discussed that additional stopband attenuation can be achieved at the overhead of transition width. In this proposed filter, the attenuation in the stopband is achieved by increasing the factor 'ρ' i.e. the number of convolution operations for obtaining $B(\omega)$. The stopband attenuation also depends upon the scale parameter '*a*'. But should not be more than a specified limit. If '*a*' is increased beyond a certain limit, one may find an overshoot in the frequency response near the cutoff frequency. For a comparison, Hamming window, Blackman window and Hanning window based FIR filters are also designed with the same parameters as reported in [14]. Blackman and Lanczos window based adjustable window is also designed by considering the same parameters suggested in [17]. Our method is compared with five different existing methods. Statistical analysis is performed and results are presented in this section to validate our method. Tables 1-8 summarize outcome of various performance parameters calculated by means of 5 design methods for FIR filters of order 20 and 30.

Table 1 Statistical analysis for the passband ripple (order = 20)

Design Method	Normalized passband ripple			
	Max.	Mean	Var.	Stdv.
Gupta [14]	1.1486	1.0506	0.0090	0.0940
Karmaker et al. 17]	1.1348	1.0398	0.0074	0.0854
Kaiser [8]	1.1260	1.0310	0.0052	0.0713
Roark et al. [10]	1.0721	1.0202	0.0037	0.0604
Proposed	1.0319	1.0088	0.0030	0.0545

Table 2 Statistical analysis for the passband ripple (order=30)

Design Method	Normalized passband ripple			
	Max.	Mean	Var.	Stdv.
Gupta	1.0172	0.9967	0.0085	0.0913
Karmaker et al.	1.0045	0.9810	0.0070	0.0825
Kaiser	1.0016	0.9780	0.0050	0.0700
Roark et al.	0.9986	0.9741	0.0033	0.0571
Proposed	0.9967	0.9781	0.0027	0.0518

Tables 1 and 2 display values of the normalized passband ripple for 20 and 30 order filters, respectively. The passband ripple obtained by our proposed method is the lowest in terms of mean, maximum (Max.), variance (Var.) and standard deviation (Stdv.) as compared to others. This shows that the proposed method is very efficient in reduction of passband ripple with the least deviation, which is a desirable property for an effective digital filter. On comparing tables 1 and 2, it can be easily verified that the ripples are less in 30 order filter.

Table 3 Statistical analysis for the stopband ripple (order= 20)

Design Method	Normalized stopband ripple		Transition
	Max.	Mean	Width
Gupta	0.1457	0.1150	0.0717
Karmaker et al.	0.1341	0.1085	0.0749
Kaiser	0.1193	0.1043	0.0779
Roark et al.	0.1057	0.0766	0.0838
Proposed	0.0956	0.0612	0.0926

Table 4 Statistical analysis for the stopband ripple (order= 30)

Design Method	Normalized Max.	stopband ripple Mean	Transition Width
Gupta	0.1073	0.0520	0.0593
Karmaker et al.	0.0901	0.0443	0.0655
Kaiser	0.0864	0.0403	0.0659
Roark et al.	0.0762	0.0353	0.0668
Proposed	0.0697	0.0316	0.0673

The statistical analysis for the stopband ripple are listed in tables 3 and 4. From table 3, it can be easily verified that the maximum normalized stopband ripple value of the proposed method is 0.0956, i.e. minimum as compared to other methods. Similar conclusions can be drawn from table 4 for the filter of order 30. With an increased order, the stopband ripple is reduced significantly. There is an improvement of about 9% over the second contestant, i.e. the method presented in [10].

Table 5 Statistical analysis for stopband attenuation (order=20)

Design Method	Stopband attenuation (dB)		
	Mean	Var.	Stdv.
Gupta	-18.9936	-50.8949	-25.4582
Karmaker et al.	-19.5038	-52.7345	-26.3829
Kaiser	-19.7611	-55.9924	-28.0017
Roark et al.	-22.5830	-57.6077	-28.8106
Proposed	-24.6650	-59.6305	-29.7923

Table 6 Statistical analysis for stopband attenuation (order=30)

Design Method	Stopband attenuation (dB)		
	Mean	Var.	Stdv.
Gupta	-25.9988	-51.1762	-25.6005
Karmaker et al.	-27.4343	-53.4436	-26.7381
Kaiser	-28.2848	-56.4988	-28.2560
Roark et al.	-29.4472	-58.2189	-29.1134
Proposed	-30.5313	-60.3555	-30.1955

Stop band attenuation plays an essential role in the filter design. More is the stopband attenuation, better is the performance of the filter. The stopband attenuation for different orders are computed and presented in tables 5 and 6. It is observed that the proposed design method offers an average stopband attenuation of -24.6650 dB for order 20 while -30.5313 dB for order 30. This is the best result. The variance and standard deviation provided by the proposed design method is also minimum i.e. -59.6305 dB and -29.7923 dB for order 20. Similar range of values are seen for order 30. The best part of the proposed design method is that the variance and standard deviation is the lowest. This emphasizes the accuracy and precision of the result. The minimum values of variance and standard deviation indicates consistent and precise results. Qualitative analysis of the filters is presented in tables 7 and 8 for order 20 and 30, respectively. Our approach offers a minimum stopband attenuation of -20.6754 dB and a maximum average passband attenuation of 0.3558 dB, which is the best result for order 20. An improvement of about 40% over Roark method [10] is observed while considering the average passband attenuation. Similarly, an improvement of about 12% is seen over Roark method [10] while considering the average passband attenuation for order 30. Therefore, it is claimed that our method is better than others for lower order filter design. The reason may be due to the incorporation of additional tuning parameter in the basis function itself.

Table 7 Qualitative analysis (order 20)

Design Method	Min stopband attenuation (dB)	Max average passband attenuation (dB)
Gupta	-16.9320	1.0960
Karmaker et al.	-18.0438	0.9937
Kaiser	-17.6256	0.8631
Roark et al.	-19.7268	0.5923
Proposed	-20.6754	0.3558

Table 8 Qualitative analysis (order 30)

Design Method	Min stopband attenuation (dB)	Max average passband attenuation (dB)
Gupta	-19.5534	0.2738
Karmaker et al.	-21.1446	0.1884
Kaiser	-21.3792	0.1673
Roark et al.	-22.6236	0.1273
Proposed	-23.4294	0.1124

The filter developed here can be used for general applications, mostly in speech processing. The filter can be used to reduce the noise encountered, mostly in voice communication. The filter can also be used in the biomedical applications. In biomedical applications, the low passband and stopband error is mostly required while there is little effect of transition width. The extra control parameter used here can be expected to change adaptively to reduce the noise or hybrid echo that arise in digital cellular systems. The filter developed here is expected to be more efficient than earlier methods due to a smooth window function with the compact support together with an extra control parameter, which is required to provide more control over the filter response without affecting other specifications.

Many filtering applications require flatness in the passband and stopband. The flatness feature is given more importance than other filter specifications like transition width, stopband attenuation etc. In medical science applications, the stopband attenuation is also considered as an important specification. In digital cellular network applications, the energy in the side lobe must be suppressed. All these requirements are met with the proposed generalized prolate type window based MF filters. It is pertinent to mention here that our method is distinctive from other methods in the sense that the designer can meet the given filter specifications using lower order proposed generalized window functions. Whereas, other design methods require higher order window functions to achieve the same. The earlier methods need more computations because of an increase in the order of the window functions. On the other hand, we suggested new prolate type window function which is very useful to meet the required filter specifications with lower order.

Recently, the t-Test is recommended for a statistical analysis. Computation of the means of two groups, which are statistically different from each other, is facilitated by using Eq. (16).

$$t = \frac{m_o - m_a}{\sqrt{\left(\sigma_o^2 / n_o\right) + \left(\sigma_a^2 / n_a\right)}} \tag{16}$$

The standard error of difference (St_{error}) is calculated using Eq. (17).

$$St_{error} = S_p \left(\frac{1}{n_o} + \frac{1}{n_a} \right)^{0.5} \tag{17}$$

Table 9 t-Test result (order= 30)

Design Method	Passband ripple		Stopband ripple	
	t-value	St_{error}	t-value	St_{error}
Gupta	0.8386	0.0331	1.0152	0.0200
Karmaker et al.	0.4014	0.0311	0.6979	0.0182
Kaiser	0.3487	0.0275	0.5395	0.0164
Roark et al.	0.2361	0.0243	0.2541	0.0154

In this work, the t-Test is performed by independently calculating statistical parameters for other design methods with respect to the proposed technique. The high positive *t* values indicate the superiority of the proposed technique over other methods. These results validate our proposed method of designing the MF filter for signal processing applications. The idea of using non-rectangular generalized smooth window seems to be very promising for designing MF filters of different order. This indicates that the B-spline approach [10] is better than other three methods [14], [17], [8]. This evokes the significance of using a smoothed window rather than using a rectangular type of window. Further improvements are seen in our case. Here, a more generalized window with additional control parameter is investigated. From Table 9, it is evident that the proposed method has shown better performance than all other methods in the design of MF filter with order 30. The *t* value and the standard error of difference of [10] is minimum, which indicates its dominance over other three techniques [14], [17], [8]. The proposed design idea offers a better error convergence and tracking performance, which is marked from the results presented.

4 Conclusion

Among the window based design methods, Kaiser window is known to be popular for controlling the magnitude response, i.e. the transition width of the magnitude response for specific stopband attenuation. The order of the filter has to be increased for the perfect design. The Kaiser window based filter and the MF filter designed by Roark have to compromise for some filter parameters to get the desired specifications. On the other hand, this paper has suggested an efficient MF filter that has an additional control parameter so that the filter designer can get the filter parameters without much disturbing the desired specifications. There are some applications where passband error and stopband error are equally important than errors occurring near the cutoff frequencies. Examples of such applications include – communication, acoustic, biomedical, physiological etc. The proposed MF filters may be useful for such application areas.

The proposed MF filters have shown less passband error and greater stopband attenuation than the other filter designing methods used before. The other specifications of these MF filters are also satisfactory for many signal-processing applications. These filters provide us another advantage that we have an explicit equation for $H(\omega)$ or $B(\omega)$ by which one can easily compute the frequency response of the filter and the window function, respectively. The FIR filters designed using other windows have no such explicit equations. Further, an additional tuning parameter has been incorporated in the proposed MF filters for improving the filter characteristics.

Certain improvements may be possible with the proposed MF filter based on prolate spheroidal wave design idea. The idea may be extended to the design of fractional order MF filters.

Appendix—A:

Link with the prolate spheroidal wave functions

Let *g(x)* with unit norm and band limited to ω_σ be the time limited function. The energy of the function *g(x)*, is given by $\chi = \int_{-T}^{+T} |g(x)|^2 \, dx$, for

$0 \le T < +\infty$ [19]. The solution $g(x)$ is the eigenvector corresponding to the largest eigenvalue satisfying the following relation

$$\int_{-T}^{+T} g(\tau)\frac{\sin \omega_\sigma (x-\tau)}{\pi(x-\tau)} d\tau = \begin{cases} \lambda g(x), & for \ |x| < T, \\ 0, & otherwise. \end{cases} \quad \text{(A.1)}$$

Let us denote $g_k(x)$ as the *k*th eigenvector associated with Eq. (A.1) corresponding to the eigenvalue λ_k. These eigenvectors $g_k(x)$ have the following properties: a) $g_k(x)$ can be uniquely generated with an appropriate scale factor, b) $g_i(x)$ and $g_j(x)$ are orthogonal when $i \neq j$, and c) performing a suitable normalization, one can generate a set of orthonormal basis functions bandlimited to $[-\omega_\sigma, \omega_\sigma]$. The orthonormal functions, thus formed, are commonly known as *prolate spheroidal wave functions*. Thus, the generalized B-spline interpolation matrix $A^{(n)}$ is positive definite and $1 > \lambda_0 > \lambda_1 > \lambda_2 > \cdots > \lambda_k > 0$. Hence, the proposed generating function is close to the *prolate spheroidal wave functions.*

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