Frequency domain identification of autoregressive models in the presence of additive noise

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Abstract: This paper describes a new approach for identifying autoregressive models from a finite number of measurements, in presence of additive and uncorrelated white noise. As a major novelty, the proposed approach deals with frequency domain data. In particular, two different frequency domain algorithms are proposed. The first algorithm is based on some theoretical results concerning the so–called dynamic Frisch Scheme. The second algorithm maps the AR identification problem into a quadratic eigenvalue problem. Both methods resemble in many aspects some other identification algorithms, originally developed in the time domain. The features of the proposed methods are compared each other and with those of other time domain algorithms by means of Monte Carlo simulations.

Key–Words: System identification; Autoregressive models; Frisch Scheme; Discrete Fourier Transform.

1 Introduction

Autoregressive (AR) models are commonly used in the field of spectral analysis and find applications in a wide range of engineering problems, like, for example, speech analysis, radar and sonar systems, vibration monitoring, astronomy, geophysics and seismology.

In spectral analysis literature [1, 2, 3] two different methodologies are usually described. The first methodology contains the classical nonparametric approaches, involving periodogram and correlogram methods. The second methodology contains the parametric approaches, also called model–based. These methods postulate a model for the data, that constitutes a means for parameterizing the spectrum. The spectral estimation problem is thus reduced to the estimation of the parameters of the model.

Both methodologies may offer advantages and disadvantages in order to obtain an accurate spectrum estimation with high resolution. They can be distinguished by the fact that nonparametric methods treat frequency domain data, while parametric methods are commonly developed in the time domain.

Within the class of parametric methods, AR models are widely used since they constitute the simplest description of a stochastic process and offer the possibility for simple and fast parameter identification procedures, based on least squares estimation schemes.

In many practical situations, however, simple AR models are not adequate since the signals are corrupted by noise. In these cases, classical AR identification methods give misleading results; in fact it can be proved that the estimated AR poles are biased toward the center of the unit circle, leading thus to a smoothed spectrum [4].

Several approaches have been developed to recover the AR parameters from noisy measurements. Since noisy AR processes admit an equivalent ARMA representation [5], the usual approaches for solving this problem consist in standard ARMA parameter estimators, like prediction errors methods [6].

Another classical approach for the identification of AR plus noise models consists in solving the so–called high–order Yule–Walker (HOYW) equations [7]. This method requires the knowledge of the autocorrelation function for high lags and is characterized by a poor estimate of the parameters. To compensate the estimation errors, an overdetermined set of HOYW equations is often considered [8]. In [9, 10] it has been shown that better results can be obtained by using both low and high order YW equations. Starting from this set up, in [11] a new method has been proposed, related to signal/noise subspace techniques. This approach uses a modified set of low and high order YW equations and maps the original problem into a quadratic eigenvalue problem. The estimates of

the AR parameters and of the noise variance are thus obtained by solving the associated generalized eigenvalue problem.

A different approach has been proposed in [12, 13, 14]. In these papers the AR plus noise identification problem is solved by using the theoretical results concerning the so–called dynamic Frisch Scheme [15, 16] which was originally developed for the identification of errors–in–variables systems.

In this work the identification of AR systems corrupted by additive white noise is addressed by using a frequency domain approach. In particular, two different frequency domain algorithms are proposed and their features are compared with each other and with those of other time domain methods.

In presence of non–periodic signals of finite length, leakage problems have been always considered to be the major drawback for frequency domain methods. In fact, leakage errors are present even in absence of disturbing noise. In this respect, an important result has been given in [17], where it has been proved that for a linear, discrete–time system, described by a rational transfer function of finite dimensions, the discrete Fourier transforms (DFTs) of the input–output signals are exactly linked by an extended model that includes also a polynomial term of finite order, that takes into account the leakage and transient effects.

From a theoretic point of view, this result has been formalized in [18], where the full equivalence between time and frequency domain identification methods has been proved, also for finite data records. However, from the practical point of view, the decision to implement a time or a frequency domain algorithm can strongly depend on the user choices and on the specific applications.

Frequency domain techniques for system identification are described in [19]. In most experimental situations the observations are collected as samples of time signals, so that a Fourier transformation is required before implementing a frequency domain algorithm. However, there exist occasions in which the data are more easily available as frequency samples. For example, in some experimental situations it may occur that the data are collected by a frequency analyzer which directly provides the Fourier transforms of the time signals. This situation is particularly common in vibrational analysis of mechanical systems [20].

Frequency domain approaches are characterized by some specific features that are not present in the time domain methods [21]. In particular, in the frequency domain the filtering operations are quite simple to implement, in fact they can be reduced to the selection of appropriate (weighted) frequencies in a limited band of the signal spectrum. As a consequence, frequency domain approaches allow to solve in a more direct and simple way all the problems where a trade–off between frequency resolution and noise level is present. This feature can be of great advantage in the identification of noisy AR processes with narrowband spectrum, as will be illustrated by a numerical example.

The organization of the paper is as follows. Section 2 defines the AR plus noise identification problem in the frequency domain, while Section 3 introduces a novel frequency domain description of the AR processes. In Section 4 the identification problem is reformulated as a Frisch Scheme problem and the search for the solution is analyzed within this context. Sections 5 describes a possible identification criterion, that can be directly formulated in the frequency domain. In particular, this criterion takes advantage of a set of equations similar to the HOYW equations. For this reason the method can be considered the frequency counterpart of the time domain approach proposed in [13, 14]. In Section 6, it is shown how the AR identification problem can be reformulated as a quadratic eigenvalue problem involving only the output noise variance. The obtained quadratic eigenvalue problem is thus solved by mapping it into a generalized eigenvalue problem. The method can be considered the frequency counterpart of the time domain approach proposed in [11]. In Section 7 the effectiveness of the proposed methods is verified by means of Monte Carlo simulations. It is shown that this new frequency domain methodology is characterized by high frequency resolution properties and is particularly suited for the identification of narrowband AR systems with close and sharp spectral peaks. Finally some concluding remarks are reported in Section 8.

2 Statement of the problem

Consider the following noisy AR model, of order n, described by the equations

$$x(t) = -\alpha_1\, x(t-1) - \cdots - \alpha_n\, x(t-n) + e(t) \tag{1}$$

$$y(t) = x(t) + w(t) \tag{2}$$

where $x(t)$ is the output of the noise–free AR model, driven by the white noise process $e(t)$. The available observation $y(t)$ is affected by the noise process $w(t)$.

The following assumptions are made.

A1. $e(t)$ is a zero–mean ergodic white process, with *unknown* variance σ_e^*.

A2. The order n of the AR model is assumed as *a priori* known.

A3. The additive noise $w(t)$ is a zero–mean ergodic white process, uncorrelated with $e(t)$, with *unknown* variance σ_w^*.

Let $\{y(t)\}_{t=0}^{N-1}$ be a set of noisy observations at N equidistant time instants. The corresponding Discrete Fourier Transform (DFT) is defined as

$$Y(\omega_k) = \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} y(t)\, \mathrm{e}^{-j\omega_k t}, \tag{3}$$

where $\omega_k = 2\pi k/N$ and $k = 0, \ldots, N-1$. In the frequency domain, the problem under investigation can be stated as follows.

Problem 1. Let $Y(\omega_k)$ be a set of noisy measurements generated by an AR plus noise system of type (1)–(2), under assumptions A1–A3, where $\omega_k = 2\pi k/N$ and $k = 0, \ldots, N-1$. Estimate the AR parameters α_i $(i = 1, \ldots, n)$ and the noise variances σ_e^*, σ_w^*.

3 A frequency domain setup

In this section a new frequency domain description for the noisy AR model (1)–(2) is introduced. This setup has been originally developed in [22, 23, 24] with reference to the identification of errors–in–variables systems.

Equation (1) can be rewritten as

$$A(z^{-1})x(t) = e(t) \tag{4}$$

where $A(z^{-1})$ is a polynomial in the backward shift operator z^{-1}

$$A(z^{-1}) = 1 + \alpha_1\, z^{-1} + \cdots + \alpha_n\, z^{-n}. \tag{5}$$

Similarly to equation (3), let $E(\omega_k)$, $X(\omega_k)$ be the DFTs of the signals $e(t)$, $x(t)$ appearing in equation (4). It is a well–known fact [17] that for finite N, even in absence of noise, the DFTs $E(\omega_k)$ and $X(\omega_k)$ exactly satisfy an extended model that includes also a transient term, i.e.

$$A(\mathrm{e}^{-j\omega_k})\, X(\omega_k) = E(\omega_k) + T(\mathrm{e}^{-j\omega_k}), \tag{6}$$

where $T(z^{-1})$ is a polynomial of order $n-1$

$$T(z^{-1}) = \tau_0 + \tau_1\, z^{-1} + \cdots + \tau_{n-1}\, z^{-n+1} \tag{7}$$

that takes into account the effects of the initial and final conditions of the experiment.

By considering the whole number of frequencies, eq. (6) can be rewritten in a matrix form. For this purpose, introduce the parameter vectors

$$\theta_\alpha = [1\, \alpha_1 \ldots \alpha_n]^T \tag{8}$$

$$\theta_\tau = [\tau_0 \ldots \tau_{n-1}]^T. \tag{9}$$

and define the following vector Θ, with dimension $p = 2n+1$, containing the whole number of parameters

$$\Theta = [\, \theta_\alpha^T \; -\theta_\tau^T \,]^T. \tag{10}$$

In absence of noise, the AR parameters can be recovered by means of the following procedure. Define the row vectors

$$Z_{n+1}(\omega_k) = [1 \;\; \mathrm{e}^{-j\omega_k} \;\; \ldots \;\; \mathrm{e}^{-j(n-1)\omega_k} \;\; \mathrm{e}^{-jn\omega_k}\,] \tag{11}$$

$$Z_n(\omega_k) = [1 \;\; \mathrm{e}^{-j\omega_k} \;\; \ldots \;\; \mathrm{e}^{-j(n-1)\omega_k}\,], \tag{12}$$

whose entries are constructed with multiple frequencies of ω_k, and construct the following matrices

$$\Pi = \begin{bmatrix} Z_{n+1}(\omega_0) \\ \vdots \\ Z_{n+1}(\omega_{N-1}) \end{bmatrix} \qquad \Psi = \begin{bmatrix} Z_n(\omega_0) \\ \vdots \\ Z_n(\omega_{N-1}) \end{bmatrix}. \tag{13}$$

of dimension $N \times (n+1)$ and $N \times n$, respectively. With the DFT samples $X(\omega_k)$ construct the following $N \times N$ diagonal matrix

$$V_X^{diag} = \mathrm{diag}\,[X(\omega_0),\, X(\omega_1), \ldots, X(\omega_{N-1})] \tag{14}$$

and compute the $N \times (n+1)$ matrix

$$\Pi_X = V_X^{diag}\, \Pi. \tag{15}$$

Then, construct the $N \times p$ matrix

$$\Phi_X = [\, \Pi_X \mid \Psi\,]. \tag{16}$$

Thus, eq. (6) for $k = 0, \ldots, N-1$ can be rewritten as

$$\Phi_X\, \Theta = V_E, \tag{17}$$

where $V_E = [E(\omega_0),\, E(\omega_1), \ldots, E(\omega_{N-1})]^T$.

Define now the $p \times p$ matrix

$$\Sigma_X = \frac{1}{N}(\Phi_X^H\, \Phi_X), \tag{18}$$

where $(\cdot)^H$ denotes the transpose and conjugate operation.

Because of assumption A1, when $N \to \infty$, from (17) it follows that

$$\bar{\Sigma}_X\, \Theta = 0, \tag{19}$$

where $\bar{\Sigma}_X$ is defined as

$$\bar{\Sigma}_X = \Sigma_X - \mathrm{diag}\,[\, \sigma_e^* \underbrace{0 \ldots 0}_{2n}\,]. \tag{20}$$

Remark 1. Since $x(t)$ is generated by the AR model (4), relation (6) cannot be satisfied by a polynomial $A(z^{-1})$ with order lower than n. Therefore, matrix $\bar{\Sigma}_X$ in (20) is positive semidefinite, with only one null eigenvalue, i.e.

$$\bar{\Sigma}_X \geq 0 \qquad \dim \ker \bar{\Sigma}_X = 1. \tag{21}$$

In presence of noise, the previous procedure can be modified as follows. With the DFT samples (3) construct the $N \times N$ diagonal matrix

$$V_Y^{diag} = \mathrm{diag}\,[Y(\omega_0),\, Y(\omega_1), \ldots, Y(\omega_{N-1})] \tag{22}$$

and compute the matrix

$$\Pi_Y = V_Y^{diag}\,\Pi\,. \tag{23}$$

Then, construct the $N \times p$ matrix

$$\Phi_Y = [\,\Pi_Y \mid \Psi\,]\,. \tag{24}$$

Because of Assumptions A3, when $N \to \infty$, we obtain the following $p \times p$ positive definite matrix

$$\Sigma_Y = \frac{1}{N}\left(\Phi_Y^H\,\Phi_Y\right) = \Sigma_X + \begin{bmatrix} \sigma_w^*\, I_{n+1} & 0 \\ 0 & 0_n \end{bmatrix}, \tag{25}$$

where 0_n is the null square matrix of dimension n.

By combining (20) and (25), it follows that

$$\Sigma_Y = \bar{\Sigma}_X + \tilde{\Sigma}^*, \tag{26}$$

where

$$\tilde{\Sigma}^* = \begin{bmatrix} \sigma_s^* & 0 & 0 \\ 0 & \sigma_w^*\, I_n & 0 \\ 0 & 0 & 0_n \end{bmatrix} \tag{27}$$

and

$$\sigma_s^* = \sigma_w^* + \sigma_e^*. \tag{28}$$

From (19) and (26), the parameter vector Θ, defined in (10), can be obtained as the kernel of

$$\left(\Sigma_Y - \tilde{\Sigma}^*\right)\Theta = 0, \tag{29}$$

after normalizing the first entry to 1.

Remark 2. It can be observed that for $k = 0, \ldots, \mathrm{floor}\left(\frac{N-1}{2}\right)$

$$Y(\omega_{N-1-k}) = \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} y(t)\, \mathrm{e}^{-j\frac{N-1-k}{N}2\pi t}$$

$$= \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} y(t)\, \mathrm{e}^{-j\frac{-(1+k)}{N}2\pi t} = Y^*(\omega_{1+k}), \tag{30}$$

where $Y^*(\cdot)$ is the conjugate of $Y(\cdot)$. Consequently, a redundant information has been used in the definition (25) of Σ_Y and only the first $N/2$ samples $Y(\omega_k)$, $k = 0, \ldots, \mathrm{floor}\left(\frac{N-1}{2}\right)$, could be considered. However, from simulation experiences the usage of the whole data set $Y(\omega_k)$, $k = 0, \ldots, N-1$, leads to better results, especially under the conditions described in Remark 3.

4 Analysis in the Frisch Scheme context

Starting from knowledge of the noisy matrix Σ_Y, the determination of the system parameter vector Θ and of the noise variances σ_e^*, σ_w^* can be seen as a Frisch Scheme problem [15, 16].

Consider the set of non–negative definite diagonal matrices of type

$$\tilde{\Sigma} = \mathrm{diag}\left[\,\sigma_s,\, \sigma_w\, I_n,\, 0_n\,\right] \tag{31}$$

such that

$$\Sigma_Y - \tilde{\Sigma} \geq 0 \qquad \det\left(\Sigma_Y - \tilde{\Sigma}\right) = 0. \tag{32}$$

The following statements can be proved. The proofs can be found in [15] and [25].

Theorem 1. The set of all matrices $\tilde{\Sigma}$ satisfying conditions (32) defines the points $P = (\sigma_s, \sigma_w)$ of a convex curve $\mathcal{S}(\Sigma_Y)$ belonging to the first quadrant of the noise space $\mathcal{R}^2$ whose concavity faces the origin. At every point $P = (\sigma_s, \sigma_w)$ can be associated the noise matrix $\tilde{\Sigma}(P)$ and the coefficient vector $\Theta(P)$ satisfying the relation

$$\left(\Sigma_Y - \tilde{\Sigma}(P)\right)\,\Theta(P) = 0.\ \Diamond \tag{33}$$

Theorem 2. Because of the relations (27)–(29), the point $P^* = (\sigma_s^*, \sigma_w^*)$, associated with the true variances of $e(t)$ and $w(t)$, belongs to $\mathcal{S}(\Sigma_Y)$ and the corresponding coefficient vector $\Theta(P^*)$ is characterized (after a normalization of its first entry to 1) by the true system parameter vector, i.e. $\Theta(P^*) = \Theta$. $\Diamond$

In Figure 1 an example of $\mathcal{S}(\Sigma_Y)$ is reported. Note that the points (σ_s, σ_w) of the curve with $\sigma_s \leq \sigma_w$ (dotted line) are non admissible because they do not satisfy the condition $\sigma_e = \sigma_s - \sigma_w > 0$. The set of admissible solutions (continuous line) is thus delimited by the straight lines $\sigma_w = \sigma_s$ and $\sigma_w = 0$.

Theorem 3. Partition the matrix Σ_Y as follows

$$\Sigma_Y = \begin{bmatrix} \sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \tag{34}$$

where Σ_{22} is the square matrix of dimension $2n$. The intersection of $\mathcal{S}(\Sigma_Y)$ with the σ_s axis is the point $P_B = (\sigma_s^{max}, 0)$ given by the least squares solution

$$\sigma_s^{max} = \frac{\det\left(\Sigma_Y\right)}{\det\left(\Sigma_{22}\right)}. \tag{35}$$

Partition the matrix Σ_Y instead as follows

$$\Sigma_Y = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \tag{36}$$

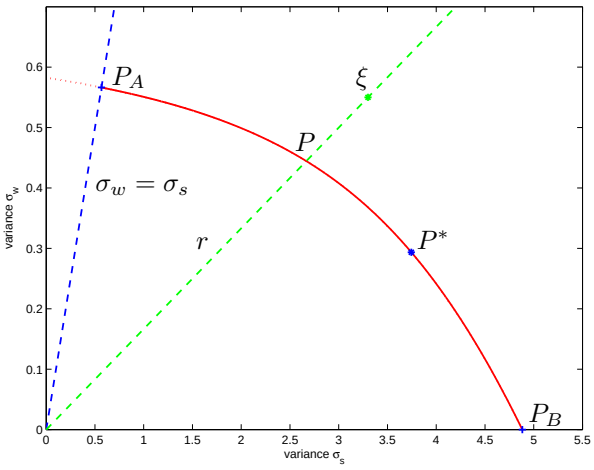

Figure 1: Typical shape of $\mathcal{S}(\Sigma_Y)$.

where Σ_{22} is the square matrix of dimension n. The intersection of $\mathcal{S}(\Sigma_Y)$ with the straight line $\sigma_w = \sigma_s$ is the point $P_A = (\sigma_w^{max}, \sigma_w^{max})$, given by the solution

$$\sigma_w^{max} = \min \text{ eig } \left(\Sigma_{11} - \Sigma_{12}\,\Sigma_{22}^{-1}\,\Sigma_{21}\right). \qquad (37)$$

Since the point P_A corresponds to $\sigma_e = 0$, it is not a solution of Problem 1. $\Diamond$

The next theorem describes a parametrization of the curve $\mathcal{S}(\Sigma_Y)$ that plays an important role in the practical implementation of the identification algorithm [16].

Theorem 4. Let $\xi = (\xi_1, \xi_2)$ be a generic point of the first quadrant of $\mathcal{R}^2$ and r the straight line from the origin through ξ (see Fig. 1). Its intersection with $\mathcal{S}(\Sigma_Y)$ is the point $P = (\sigma_s, \sigma_w)$ given by

$$\sigma_s = \frac{\xi_1}{\lambda_M} \qquad \sigma_w = \frac{\xi_2}{\lambda_M}, \qquad (38)$$

where

$$\lambda_M = \max \text{ eig } \left(\Sigma_Y^{-1}\tilde{\Sigma}_\xi\right) \qquad (39)$$

$$\tilde{\Sigma}_\xi = \begin{bmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2\, I_n & 0 \\ 0 & 0 & 0_n \end{bmatrix}. \Diamond \qquad (40)$$

Remark 3. The described procedure allows to construct the curve $\mathcal{S}(\Sigma_Y)$ in the noise space (σ_s, σ_w) also when only a subset of the whole frequency range is used, on condition that the number of the selected frequencies is large enough. This subset must be chosen by the user on the basis of *a priori* knowledge of the frequency properties of the AR model. In practice, taking into account the observations of Remark 2, two distinct sets of frequencies are jointly considered, the set $W_1 = [\omega_i,\, \omega_f]$, with $i \geq 0$ and $f \leq \text{floor}\left(\frac{N-1}{2}\right)$ and the set $W_2 = [\omega_{N-1-f},\, \omega_{N-1-i}]$, for a total number of the $2L$ frequencies, with $L = f - i + 1$. By considering a new matrix Φ_Y with $2L$ rows, expressions (25)–(27) must be modified as follows

$$\Sigma_Y = \frac{1}{2L}\left(\Phi_Y^H\,\Phi_Y\right) = \bar{\Sigma}_X + \tilde{\Sigma}^*, \qquad (41)$$

where

$$\tilde{\Sigma}^* = \frac{N}{2L}\begin{bmatrix} \sigma_s^* & 0 & 0 \\ 0 & \sigma_w^*\, I_n & 0 \\ 0 & 0 & 0_n \end{bmatrix}. \qquad (42)$$

5 A criterion based on HOYW–type equations

As asserted in Theorem 2, the determination of the point P^* on $\mathcal{S}(\Sigma_Y)$ leads to the solution of Problem 1. For this purpose a search criterion must be introduced. Unfortunately, the theoretic properties of $\mathcal{S}(\Sigma_Y)$ described so far do not allow to distinguish the point P^* from the other points of the curve.

In this section we will describe a possible search criterion. This criterion is analogue to that reported in [13] with reference to time domain identification of AR plus noise models.

Select the integer $q \geq 2n$. Analogously to (11), consider the row vector

$$Z_{q+n+1}(\omega_k) = [1\ \mathrm{e}^{-j\omega_k}\ \ldots \mathrm{e}^{-j(n+q)\omega_k}] \qquad (43)$$

and extract from it the q–dimensional row vector

$$Z_q^h(\omega_k) = [\mathrm{e}^{-j(n+1)\omega_k}\ \ldots \mathrm{e}^{-j(n+q)\omega_k}]. \qquad (44)$$

Then, construct the following $N \times q$ matrix

$$\Pi^h = \begin{bmatrix} Z_q^h(\omega_0) \\ \vdots \\ Z_q^h(\omega_{N-1}) \end{bmatrix}. \qquad (45)$$

and compute the $N \times q$ matrix

$$\Phi_X^h = V_X^{diag}\,\Pi^h. \qquad (46)$$

Define now the $q \times p$ matrix

$$\Sigma_X^h = \frac{1}{N}\left((\Phi_X^h)^H\,\Phi_X\right), \qquad (47)$$

Because of assumption A1, when $N \to \infty$, it follows that

$$\Sigma_X^h\,\Theta = 0. \qquad (48)$$

In an analogous way, we can compute the $N \times q$ matrix

$$\Phi_Y^h = V_Y^{diag}\, \Pi^h \tag{49}$$

and define the $q \times p$ matrix

$$\Sigma_Y^h = \frac{1}{N}\big((\Phi_Y^h)^H\, \Phi_Y\big). \tag{50}$$

Because of Assumptions A3, when $N \to \infty$, it results

$$\Sigma_Y^h = \Sigma_X^h. \tag{51}$$

It is thus possible to write

$$\Sigma_Y^h\, \Theta = 0. \tag{52}$$

Equation (52) constitutes a set of q equations, analogue to the time domain high order Yule–Walker equations, that does not involve the output noise variance σ_w^*.

Thanks to this property the search for P^* along $\mathcal{S}(\Sigma_Y)$ can be performed by introducing the following cost function

$$J(P) = \|\Sigma_Y^h\, \Theta(P)\|_2^2 = \Theta^T(P)(\Sigma_Y^h)^H\, \Sigma_Y^h\, \Theta(P). \tag{53}$$

When $q \geq 2n$ the cost function $J(P)$ in (53) exhibits the following properties

i) $J(P) \geq 0$

ii) $J(P) = 0 \Leftrightarrow P = P^*$.

In fact, if $q \geq 2n$ the conditions stated in [26] hold and the equation (52) admits a unique solution, corresponding to the true system parameter Θ.

On the basis of the previous considerations, it is possible to develop an algorithm for the identification of AR plus noise models. A detailed description of the procedure can be found in [25]. In the following this algorithm is denoted as Alg1-FD.

6 A subspace approach

The approach proposed in this section is analogue to that described in [11] and exploits the set of equations (52) together with the equations (29). It will be shown that the AR plus noise identification problem can be mapped into a quadratic eigenvalue problem that, in turn, can be solved as a generalized eigenvalue problem. The system parameters are thus estimated in one shot, without any search procedure.

Let us partition matrix Σ_Y, defined in (25), as follows

$$\Sigma_Y = \begin{bmatrix} \sigma_{11} & \Sigma_{12} & \Sigma_{13} \\ \Sigma_{21} & \Sigma_{22} & \Sigma_{23} \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} \end{bmatrix}, \tag{54}$$

where σ_{11} is a scalar and Σ_{22}, Σ_{33} are square matrices of dimension n.

By analyzing the structure of matrix (27), it can be observed that, when $N \to \infty$, the last $2n$ equations in (29) can be written as

$$\begin{bmatrix} \Sigma_{21} & \Sigma_{22} - \sigma_w^*\, I_n & \Sigma_{23} \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} \end{bmatrix} \Theta = 0. \tag{55}$$

Equation (55) contains $2n + 2$ unknowns, i.e. σ_w^* and the entries of Θ. By choosing $q \geq 2$, the equations (52) can be combined with the equations (55) in order obtain the following nonlinear system of $2n + q$ equations

$$\begin{bmatrix} \Sigma_{21} & \Sigma_{22} - \sigma_w^*\, I_n & \Sigma_{23} \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} \\ & \Sigma_Y^h & \end{bmatrix} \Theta = 0. \tag{56}$$

This set of equations can be rewritten as

$$(S - \sigma_w^*\, J)\, \Theta = 0, \tag{57}$$

where

$$S = \begin{bmatrix} \Sigma_{21} & \Sigma_{22} & \Sigma_{23} \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} \\ & \Sigma_Y^h & \end{bmatrix} \tag{58}$$

$$J = \begin{bmatrix} 0_{n\times 1} & I_n & 0_n \\ 0_{n\times 1} & 0_n & 0_n \\ & 0_{q\times(2n+1)} & \end{bmatrix}. \tag{59}$$

Multiplying both sides of (57) by $(S - \sigma_w^*\, J)^T$ leads to the equation

$$\big(A_2\, \sigma_w^{2*} + A_1\, \sigma_w^* + A_0\big)\ \Theta = 0, \tag{60}$$

where

$$A_0 = S^T S \tag{61}$$

$$A_1 = -\big(S^T J + J^T S\big) \tag{62}$$

$$A_2 = J^T J. \tag{63}$$

The coefficients of Θ can thus be estimated by solving the following quadratic eigenvalue problem (QEP)

$$\big(A_2\, \lambda^2 + A_1\, \lambda + A_0\big)\ v = 0. \tag{64}$$

The set of $4n + 2$ eigenvalues solving (64) are real or appear in complex conjugate pairs and can also be infinite [27]. If the system is identifiable and the number of data $N \to \infty$, the only real eigenvalue (with multiplicity two) that solves (64) is $\lambda = \sigma_w^*$ [11]. It is thus possible to conclude that the solution of the identification problem is the eigenvector associated with the only real eigenvalue that solves (64).

The QEP (64) can be solved in several ways [27]. The easiest way to solve it consists in rewriting equation (64) as

$$A_2\, v'\, \lambda + A_1\, v\, \lambda + A_0\, v = 0, \tag{65}$$

where $v' = \lambda\, v$. Thus, the following $(4n+2)$–dimensional linear generalized eigenvalue problem (GEP) can be derived [27]

$$(P - \lambda\, Q)\ \eta = 0, \tag{66}$$

where

$$P = \begin{bmatrix} A_0 & 0 \\ 0 & I_{2n+1} \end{bmatrix} \tag{67}$$

$$Q = \begin{bmatrix} -A_1 & -A_2 \\ I_{2n+1} & 0 \end{bmatrix} \tag{68}$$

$$\eta = \begin{bmatrix} v^T & v'^T \end{bmatrix}^T. \tag{69}$$

The only real eigenvalue solving (66) is σ_w^* and the first $2n+1$ entries of the corresponding eigenvector η^* are, after a normalization of the first entry to 1, the entries of Θ, i.e.

$$\eta_0 = \frac{\eta^*}{\eta^*(1)} = \begin{bmatrix} \Theta^T & \sigma_w^* \Theta^T \end{bmatrix}^T. \tag{70}$$

Since only a finite number N of data is available, all the eigenvalues solving (66) will exhibit, in general, a small imaginary part. A criterion leading to good results consists in choosing the eigenvalue having the smallest modulus [11].

On the basis of the previous considerations, it is possible to develop a second algorithm for the identification of AR plus noise models. A detailed description of the procedure can be found in [25]. In the following this algorithm is denoted as Alg2-FD.

7 Numerical examples

In this section, the effectiveness of the proposed identification algorithms is tested by means of numerical simulations.

Example 1. The proposed algorithms have been tested on sequences generated by the following AR model of order $n = 4$, already considered in [13]

$$\begin{aligned} x(t) =& 2.4\, x(t-1) - 3.03\, x(t-2) + 1.986\, x(t-3) \\ & - 0.6586\, x(t-4) + e(t), \end{aligned} \tag{71}$$

where $e(t)$ is a white noise with variance $\sigma_e^* = 1$.

A Monte Carlo simulation of 100 runs has been performed by using, in every run, $N = 1000$ samples of the noisy output $y(t)$. The variance of the observation noise is $\sigma_w^* = 4$, corresponding to a Signal to Noise Ratio (SNR) of about 10 dB, where the SNR is defined as

$$\text{SNR} = 20\, \log_{10}\, \sqrt{\frac{E[x^2(t)]}{E[w^2(t)]}} = 10\, \log_{10}\, \frac{r_x(0)}{\sigma_w^*}\ \text{dB}. \tag{72}$$

Table 1 reports the empirical means of the system parameter estimates and of the noise variance estimates, together with the corresponding standard deviations, obtained with Alg1-FD, and with Alg2-FD. The results are also compared with those obtained by the corresponding time domain algorithms described in [13] and [11] respectively, and denoted with Alg1-TD and Alg2-TD.

The table shows the results obtained when q is equal to the minimum number of equations required for the algorithms' implementation. It can be observed that Alg1-FD and Alg1-TD yield comparable results. Moreover, it can be noted that Alg2-FD does not yield satisfactory results when q is fixed to the minimum admissible value (third line in the Table). However, it is sufficient to select $q = n$ for obtaining parameter estimates that are comparable with those obtained by Alg2-TD and by the other methods.

In Table 1, the last column reports the mean value (in ms) of the time requested to carry out a single run of the Monte Carlo simulation. This value strongly depends on the specific features of the PC used for the simulations and, moreover, it may slightly change in different Monte Carlo sessions. However, it provides the correct order of magnitude of the computational efficiency of the algorithms and allows to make a comparison of their performances.

It can be observed that the estimation accuracy of the frequency domain algorithms Alg1-FD, Alg2-FD is comparable to that of the corresponding time domain algorithms Alg1-TD, Alg2-TD. However, as far as the computational efficiency is concerned, it must be observed that Alg2 is always much faster than Alg1, both in time and frequency domain. Moreover, the time domain algorithms are, approximatively, 4–5 times faster than the corresponding frequency domain implementations. Of course, in this respect, particular attention must be given to the coding. For example, it is worth noting that matrices Π_Y (23) and Φ_Y^h (49) must not be computed with the reported expressions, since they are highly time consuming.

It can be conjectured that the major computational burden for the frequency domain approaches is mainly due to the following two facts: the preliminary DFT operations on the data and the computations developed in the complex domain. This aspect is highly compensated by the fact that in the frequency do-

Table 1: True and estimated values of parameters and variances for Alg1-FD, Alg2-FD and Alg1-TD, Alg2-TD. SNR ≈ 10 dB and $N = 1000$.

	α_1	α_2	α_3	α_4	σ_e^*	σ_w^*	Time (ms)
true	-2.4	3.03	-1.986	0.6586	1	4	
Alg1 − FD ($q = 8$)	-2.3794 ± 0.1152	2.9969 ± 0.1693	-1.9569 ± 0.1278	0.6563 ± 0.0281	1.0921 ± 0.3272	4.0135 ± 0.1561	31.5
Alg1 − TD ($q = 4$)	-2.3864 ± 0.0325	3.0049 ± 0.1071	-1.9613 ± 0.1369	0.6553 ± 0.0960	0.9912 ± 0.0546	3.9646 ± 0.2184	8.0
Alg2 − FD ($q = 2$)	-2.3158 ± 0.5049	2.9086 ± 0.8811	-1.8920 ± 0.7709	0.6461 ± 0.2497	1.5656 ± 2.9957	3.9568 ± 0.2557	4.5
Alg2 − FD ($q = 4$)	-2.3913 ± 0.0367	3.0267 ± 0.0776	-1.9877 ± 0.0754	0.6714 ± 0.0366	1.0049 ± 0.4341	3.9769 ± 0.2297	5.5
Alg2 − TD ($q = 4$)	-2.3924 ± 0.0376	3.0234 ± 0.0732	-1.9824 ± 0.0688	0.6672 ± 0.0317	1.0256 ± 0.3870	3.9758 ± 0.2130	1.2

Table 2: True and estimated values of the AR parameters for Alg1-FD, Alg2-FD and Alg1-TD, Alg2-TD. SNR $\approx$ 10 dB and $N = 1000$.

	α_1	α_2	α_3	α_4	Time (ms)
True values	-2.7607	3.8106	-2.6535	0.9238	
Alg1 − FD ($q = 8$) $F_1 = [0\ \ 0.5]$	-2.0911 ± 0.5938	2.4295 ± 1.3386	-1.3965 ± 1.3103	0.5162 ± 0.5371	35.0
Alg1 − TD ($q = 4$)	-2.3307 ± 1.5771	2.7119 ± 3.9835	-1.5336 ± 4.0573	0.4498 ± 1.7307	28.2
Alg2 − FD ($q = 4$) $F_1 = [0\ \ 0.5]$	-2.1589 ± 1.2050	2.8041 ± 2.3264	-1.8611 ± 2.1528	0.7729 ± 0.8505	5.6
Alg2 − TD ($q = 4$)	-2.7820 ± 0.5854	3.9006 ± 1.0575	-2.7603 ± 0.9029	0.9813 ± 0.2983	1.2
Alg1 − FD ($q = 8$) $F_1 = [0.05\ \ 0.2]$	-2.7223 ± 0.0188	3.7624 ± 0.0315	-2.6323 ± 0.0247	0.9372 ± 0.0053	29.0
Alg2 − FD ($q = 4$) $F_1 = [0.05\ \ 0.2]$	-2.7434 ± 0.0802	3.7865 ± 0.2156	-2.6415 ± 0.2341	0.9305 ± 0.1232	2.6

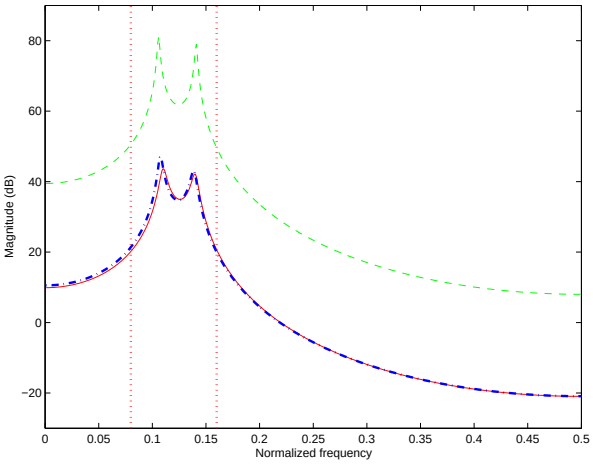

Figure 2: True TF: red, solid; Alg1-FD estimate: green, dashed; Alg2-FD estimate: blue, dash–dotted.

main the filtering operations can be implemented in a straightforward way, with great benefits for the identification, as it will be shown in the next example.

Example 2. In order to verify the selective properties described in Remark 3, the following AR model of order $n = 4$, also proposed in [9], has been considered

$$x(t) = 2.7607\,x(t-1) - 3.8106\,x(t-2) + 2.6535\,x(t-3) - 0.9238\,x(t-4) + e(t), \quad (73)$$

where $e(t)$ is a white noise with variance $\sigma_e^* = 1$.

The model (73) exhibits two pairs of complex poles, with $p_{1,2} = 0.98\,\mathrm{e}^{\pm j\,0.69}$ and $p_{3,4} = 0.98\,\mathrm{e}^{\pm j\,0.88}$. It describes a narrowband AR system with close and sharp spectral peaks, at the frequencies $f_1 = 0.69/(2\pi) = 0.11$ and $f_2 = 0.88/(2\pi) = 0.14$ (see Fig. 2).

This model is particularly difficult to identify under low SNR conditions and poor estimates of the system parameters are obtained if the SNR is lower than 10 dB.

As a proof of this assertion, a first Monte Carlo simulation of 100 independent runs have been carried out, by considering noisy sequences of $N = 1000$ samples. The variance of the observation noise has been fixed to $\sigma_w^* = 90$, corresponding to a SNR of about 10 dB.

The first four lines of Table 2 report the empirical means of the system parameter estimates together with the corresponding standard deviations, obtained with the considered methods. For the sake of simplicity, the estimates of σ_e^* and σ_w^* are not reported. It can be observed that Alg1-TD, Alg1-FD and Alg2-FD have unsatisfactory performances, with bad estimates of the parameters, while Alg2-TD exhibits a greater robustness against noise and still yields a satisfactory result.

In the previous simulations all the available N data, in the whole frequency range $[0\ \ 0.5]$, have been used for the identification. However, it can be observed that the frequency domain methods Alg1-FD

Table 3: True and estimated values of the AR parameters and variance σ_e for Alg1-FD and Alg2-FD. SNR ≈ -5 dB and $N = 1000$.

	α_1	α_2	α_3	α_4	σ_e^*	Time (ms)
True values	-2.7607	3.8106	-2.6535	0.9238	1	
Alg1 − FD ($q = 8$) $F_1 = [0.05\ \ 0.2]$	-2.5223 ± 0.0360	3.3818 ± 0.0469	-2.3656 ± 0.0303	0.8805 ± 0.0110	$--$	35.0
Alg2 − FD ($q = 4$) $F_1 = [0.05\ \ 0.2]$	-2.5137 ± 0.2086	3.3189 ± 0.4673	-2.2576 ± 0.4652	0.8126 ± 0.2287	$--$	2.6
Alg1 − FD ($q = 8$) $F_1 = [0.08\ \ 0.16]$	-2.8217 ± 0.0178	3.9402 ± 0.0254	-2.7838 ± 0.0171	0.9734 ± 0.0039	$--$	32.5
Alg2 − FD ($q = 4$) $F_1 = [0.08\ \ 0.16]$	-2.8046 ± 0.0875	3.8866 ± 0.2017	-2.7181 ± 0.1996	0.9400 ± 0.0851	1.0465 ± 0.6050	2.0

and Alg2-FD yield good parameter estimates when the AR model is identified by using only the data belonging to specific frequency windows defined by the user, $F_1 = [f_i,\ f_f]$ and $F_2 = [1 - f_f,\ 1 - f_i]$, with $f_i = \omega_i/(2\pi) \geq 0$ and $f_f = \omega_f/(2\pi) \leq 0.5$.

The choice of F_1 (and F_2) must be linked to the spectral properties of the AR system to be identified. In particular, for the considered example, the window F_1 will contain the two frequencies f_1 and f_2 that characterize the four poles of model (73).

In order to verify this property, the AR system has been identified by using only the $2L = 300$ frequencies in the windows $F_1 = [f_i,\ f_f]$ and $F_2 = [1 - f_f,\ 1 - f_i]$, where $f_i = 0.05$ and $f_f = 0.2$. The last two lines of Table 2 report the identification results obtained with Alg1-FD and Alg2-FD. It can be noted that in this case both methods give good estimates of the AR parameters. Observe that the computational efficiency of the algorithms is now improved, since only $2L < N$ data are used for the identification.

The choice of the width of the window F_1 is linked to the value of σ_w^*. When more amount of noise $w(t)$ is present on the data then more *a priori* information about the spectral properties of the AR system is required. In particular, for the considered example, when σ_w^* increases a more accurate information about the exact positions of f_1 and f_2 is required.

As a proof of this assertion, two further Monte Carlo simulations of 100 independent runs have been carried out, with $N = 1000$. In this case, however, the variance of the observation noise has been fixed to $\sigma_w^* = 2700$, corresponding to a SNR of about -5 dB. Of course, under these very low SNR conditions also Alg2-TD completely fails the estimates; the results are not reported.

In the first simulation the AR system has been identified by using the same window $F_1 = [0.05\, 0.2]$, as before. In the second simulation a more narrow window $F_1 = [0.08,\ 0.16]$ has been used, with a total number of $2L = 160$ frequencies.

The results of these simulations are reported in Table 3. It can be observed that in the first simulation both methods yield worse estimates for the AR parameters, while in the second simulation they give good estimates, again. As a concluding remark, note that Alg2-FD has to be preferred to Alg1-FD. Not only it is faster, but it gives also a correct estimate of σ_e^*, while Alg1-FD completely fails this estimate, with a consequent error for the static gain of $1/\left|A(\mathrm{e}^{-j\omega_k})\right|$, see Fig. 2. The estimates of σ_w^* are not reported, since they are completely wrong.

With reference to the last simulation, Figure 2 reports the true value of $1/\left|A(\mathrm{e}^{-j\omega_k})\right|_{dB}$, together with the means of the transfer function estimates, obtained with Alg1-FD and Alg2-FD. The advantageous effects of filtering are evident for both methods, in fact they succeed in the identification of the two peaks at the frequencies f_1 and f_2.

8 Conclusions

In this paper a novel frequency domain approach has been proposed for the identification of AR models affected by additive white noise. In particular, two different frequency domain algorithms have been proposed and their estimation properties have been tested and compared by means of Monte Carlo simulations. The numerical results have confirmed the good performances of the new methodology and have shown its effectiveness in the identification of narrowband AR systems with close and sharp spectral peaks.

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