

Parameter Estimation of Harmonics in Multiplicative and Additive Noise using SVD-based ESPRIT

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Abstract: The frequency estimation of the harmonics in multiplicative and additive noise is investigated in this paper. To improve the frequency resolution of estimation, this paper proposes a SVD-based ESPRIT algorithm to estimate the frequency of harmonics in multiplicative and additive noise. The proposed SVD-based ESPRIT algorithm not only has high-resolution, but also is easy to be implemented in practice because it does not need peak searching. Simulation results clearly show the effectiveness of the proposed algorithm.

Key-Words: Frequency estimation, multiplicative noise, frequency resolution, ESPRIT

1 Introduction

The problem of the parameter estimation of harmonics in noise is commonly encountered in diverse applications in the sciences and engineering. Recently, most of existing algorithms have been developed for the harmonic retrieval on constant amplitude harmonics observed in additive noise. However, multiplicative noise often occurs in a variety of applications (see [1–7], and references therein). For example, in Doppler-radar processing, according to the knowledge of the frequency from a pulse train reflected from a moving object yields the target's velocity, it is more appropriate to model the harmonic as having random amplitude rather than constant amplitude when the target scintillates. In underwater acoustic applications, the multiplicative noise can describe the effects on acoustic waves due to fluctuations caused by the medium, changing orientation, and interference from scatterers of the target [3].

Some feasible methods are proposed to estimate the parameter of harmonics in the presence of the multiplicative noise. The parametric second-order approaches method was presented in [1], the higher order statistics method was presented in [2], and the cyclic statistics method was proposed in [3], respectively. All these methods [1–3] developed that some statistics of the harmonics in multiplicative and additive noise have peaks at corresponding parameters and zeros at the other, and then estimated the harmonic parameters by peak searching. However, due to affection by the pseudo peaks and the Rayleigh limit, all these methods cannot meet the requests of

the high-resolution for the given observed data. This paper considers the high-resolution frequency estimation method of harmonics in multiplicative and additive noise.

It is well-known that the estimation of signal parameter via rotational invariance techniques (ESPRIT) is a high-resolution algorithm for the parameters estimation of the harmonics by exploiting the underlying rotational invariance of signal subspaces [8–11]. In this paper, we first exploit the underlying harmonic signal model in the multiplicative and additive noise. And then, we develop the algorithm of the singular value decomposition (SVD)-based ESPRIT to estimate the frequencies of the harmonics in the multiplicative and additive noise. The proposed SVD-based ESPRIT algorithm has high-resolution. Moreover, due to does not need peak searching, it is easy to be implemented in practice.

The rest of this paper is organized as follows. In Section 2, the signal model is introduced. In Section 3, the SVD-based ESPRIT algorithm is presented to estimate frequencies of the harmonics in the multiplicative and additive noise. In Section 4, simulation examples are conducted to demonstrate the effectiveness of the proposed algorithm. In Section 5, we conclude the paper.

2 Signal Model

Without loss of generality, a discrete-time P -component harmonics in the multiplicative and addi-

tive noise model is considered in this paper as follows.

$$x(t) = \sum_{k=1}^P s_k(t) e^{j(\omega_k t + \phi_k)} + v(t), \quad t = 1, 2, \dots, T, \quad (1)$$

where ω_k and ϕ_k are the normalized frequency and phase of the k th harmonic, respectively. $s_k(t)$ and $v(t)$ are the multiplicative noise and additive noise, respectively. In this paper, we assume the following conditions to be hold for the model (1): (1) ω_k 's are distinct in $(-\pi/2, 0) \cup (0, \pi/2)$, (2) ϕ_k 's are deterministic constant in $(-\pi, \pi]$, (3) $s_k(t)$'s and $v(t)$ are mutually independent stationary real zero-mean white Gaussian random process.

The parameter estimation of harmonics in multiplicative and additive noise includes mainly two contents: the frequency estimation and the harmonic number estimation. In this paper, the harmonic number is assumed to be known. We focus on the frequency estimation in multiplicative and additive noise. The harmonic number estimation in multiplicative and additive noise can be found in [7].

3 The SVD-based ESPRIT Algorithm

To exploit the underlying deterministic nature of the harmonics in multiplicative and additive noise, we define a cyclic covariance of $x(t)$ as following:

$$C_\tau = \bar{E}\{x^2(t)x^{2*}(t+\tau)\} - \bar{E}\{x^2(t)\}\bar{E}\{x^{2*}(t+\tau)\}, \quad (2)$$

where $\tau = 0, 1, \dots, K$, and $(\cdot)^*$ denotes conjugate. $\bar{E}\{\cdot\}$ represents the cyclic mean [3] which is defined as

$$\bar{E}\{y(t)\} = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T E\{y(t)\}. \quad (3)$$

Substituting (1) into (2), C_τ can be written as

$$C_\tau = \sum_{k=1}^P (\sigma_{s_k}^2)^2 e^{-j2\tau\omega_k} + \beta\delta(\tau), \quad (4)$$

where

$$\begin{aligned} \beta = & 2(\sigma_v^2)^2 + \sum_{k=1}^P 2(\sigma_{s_k}^2)^2 + \sum_{k=1}^{P-1} \sum_{l=k+1}^P 4\sigma_{s_k}^2 \sigma_{s_l}^2 \\ & + \sum_{k=1}^P 4\sigma_{s_k}^2 \sigma_v^2, \end{aligned} \quad (5)$$

$\sigma_{s_k}^2$ and σ_v^2 are the variances of multiplicative noise $s_k(t)$ and additive noise $v(t)$, respectively. The detailed derivation of (4) is given in Appendix.

Using the cyclic covariance C_τ , we construct two $K \times K$ matrices as follows

$$\mathbf{R}_1 = \begin{bmatrix} C_0 & C_1 & \cdots & C_{K-1} \\ C_1^* & C_0 & \cdots & C_{K-2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{K-1}^* & C_{K-2}^* & \cdots & C_0 \end{bmatrix}, \quad (6)$$

$$\mathbf{R}_2 = \begin{bmatrix} C_1 & C_2 & C_3 & \cdots & C_{K-1} & C_K \\ C_0 & C_1 & C_2 & \cdots & C_{K-2} & C_{K-1} \\ C_1^* & C_0 & C_1 & \cdots & C_{K-3} & C_{K-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{K-3}^* & C_{K-4}^* & C_{K-5}^* & \cdots & C_1 & C_2 \\ C_{K-2}^* & C_{K-3}^* & C_{K-4}^* & \cdots & C_0 & C_1 \end{bmatrix}. \quad (7)$$

Substituting C_τ into (6) and (7), \mathbf{R}_1 and \mathbf{R}_2 can be written as

$$\mathbf{R}_1 = \mathbf{A}\mathbf{S}\mathbf{A}^H + \beta\mathbf{I}, \quad (8)$$

$$\mathbf{R}_2 = \mathbf{A}\mathbf{S}\mathbf{\Phi}^H\mathbf{A}^H + \beta\mathbf{Z}, \quad (9)$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{j2\omega_1} & e^{j2\omega_2} & \cdots & e^{j2\omega_P} \\ \vdots & \vdots & \ddots & \vdots \\ e^{j2\omega_1(K-1)} & e^{j2\omega_2(K-1)} & \cdots & e^{j2\omega_P(K-1)} \end{bmatrix}, \quad (10)$$

$$\mathbf{S} = \begin{bmatrix} (\sigma_{s_1}^2)^2 & 0 & 0 & \cdots & 0 \\ 0 & (\sigma_{s_2}^2)^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (\sigma_{s_P}^2)^2 \end{bmatrix}, \quad (11)$$

$$\mathbf{\Phi} = \begin{bmatrix} e^{j\omega_1} & 0 & 0 & \cdots & 0 \\ 0 & e^{j\omega_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{j\omega_P} \end{bmatrix}, \quad (12)$$

and \mathbf{Z} is a $K \times K$ matrix with ones on the first sub-diagonal and zeros elsewhere, i.e.,

$$\mathbf{Z} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \quad (13)$$

Matrices \mathbf{R}_1 and \mathbf{R}_2 have the similar forms with \mathbf{R}_{xx} and \mathbf{R}_{xy} that presented in classical ESPRIT algorithm [8]. Here, we detailedly show that the frequency parameters of the harmonics in multiplicative and additive noise can be determined by the generalized eigenvalues of the matrix pencil $\{\mathbf{R}_1, \mathbf{R}_2\}$.

Theorem 1 Define Γ as the generalized eigenvalue matrix associated with the matrix pencil $\{\mathbf{D}_1, \mathbf{D}_2\}$ where $\mathbf{D}_1 = \mathbf{R}_1 - \lambda_{\min} \mathbf{I}$, $\mathbf{D}_2 = \mathbf{R}_2 - \lambda_{\min} \mathbf{Z}$, and λ_{\min} is the minimum (repeated) eigenvalue of \mathbf{R}_1 . Then, if \mathbf{S} is nonsingular, the $K \times K$ matrix Γ is related to the $P \times P$ matrix Φ by

$$\Gamma = \begin{bmatrix} \Phi & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (14)$$

to within a permutation of the elements of Φ .

Proof: The proof is the same as the classical proof given by Roy in [8] with replacing $\lambda_{\min} = 2(\sigma_v^2)^2 + \sum_{k=1}^P 2(\sigma_{s_k}^2)^2 + \sum_{k=1}^{P-1} \sum_{l=k+1}^P 4\sigma_{s_k}^2 \sigma_{s_l}^2 + \sum_{k=1}^P 4\sigma_{s_k}^2 \sigma_v^2$ with $\lambda_{\min} = \sigma^2$. Therefore, the detailed proof is omitted here.

Theorem 1 could be considered as the least squares estimator of an $K \times K$ operator whose action is restricted to a P -dimensional subspace. Here, we propose a novel SVD-based ESPRIT algorithm to solve this problem. The key idea of the proposed novel algorithm is that the SVD is employed to transform the generalized eigenproblem of an $K \times K$ matrix pencil into the one of a $P \times P$ matrix pencil [11]. First, we calculate the SVD of \mathbf{D}_1 :

$$\mathbf{D}_1 = \mathbf{U}\Sigma\mathbf{V}^H = [\mathbf{U}_1, \mathbf{U}_2] \begin{bmatrix} \Sigma_1 & \mathbf{0} \\ \mathbf{0} & \Sigma_2 \end{bmatrix} \begin{bmatrix} \mathbf{V}_1^H \\ \mathbf{V}_2^H \end{bmatrix} \quad (15)$$

where Σ_1 consists of P principal singular values. Without changing the generalized eigenvalues of the matrix pencil $\{\mathbf{D}_1, \mathbf{D}_2\}$, we multiply $\mathbf{D}_1 - \gamma\mathbf{D}_2$ by \mathbf{U}_1^H from the left and by \mathbf{V}_1 from the right to obtain a $P \times P$ matrix pencil $\Sigma_1 - \gamma\mathbf{U}_1^H\mathbf{D}_2\mathbf{V}_1$. Clearly, the P values of the frequencies $\omega_k (k = 1, 2, \dots, P)$ are now determined by the P generalized eigenvalues of the $P \times P$ matrix pencil $\{\Sigma_1, \mathbf{U}_1^H\mathbf{D}_2\mathbf{V}_1\}$.

It's worth noting that C_τ in (2) is calculated when data length $T \rightarrow \infty$. However, in practice, we have only finite observed data. Therefore, it is important to get the estimation of C_τ from a single record $\{x(t)\}_{t=1}^T$. To estimate C_τ from a single record $\{x(t)\}_{t=1}^T$, we need the following lemma in [12]:

Lemma 2 [12] If $x(t)$ is mixing, and $M_{kx}(\alpha; \tau) \triangleq \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T E\{x(t)x(t+\tau_1) \cdots x(t+\tau_{k-1})\} e^{-j\alpha t}$ exists, then the estimator

$$\begin{aligned} \hat{M}_{kx}(\alpha; \tau) &= \frac{1}{T} \sum_{t=1}^T E\{x(t)x(t+\tau_1) \cdots x(t+\tau_{k-1})\} e^{-j\alpha t} \end{aligned} \quad (16)$$

is asymptotically unbiased and mean square sense (m.s.s.) consistent, i.e.,

$$\lim_{T \rightarrow \infty} \hat{M}_{kx}(\alpha; \tau) \stackrel{m.s.s.}{=} M_{kx}(\alpha; \tau). \quad (17)$$

According to Lemma 2, the natural estimator for C_τ is

$$\begin{aligned} \hat{C}_\tau &= \frac{1}{T-\tau} \sum_{t=1}^{T-\tau} x^2(t)x^{2*}(t+\tau) \\ &\quad - \frac{1}{(T-\tau)^2} \left(\sum_{t=1}^{T-\tau} x^2(t) \right) \left(\sum_{t=1}^{T-\tau} x^{2*}(t+\tau) \right), \end{aligned} \quad (18)$$

and it is asymptotically unbiased and mean square sense consistent.

In summary, the key steps of the proposed SVD-based ESPRIT algorithm for the frequency estimation of the harmonics in multiplicative and additive noise is given as follows.

Step 1: Calculate the sample cyclic covariances \hat{C}_τ ($\tau = 0, 1, \dots, K$) from the observed data $\{x(t)\}_{t=1}^T$ using (18).

Step 2: Construct the matrices \mathbf{R}_1 and \mathbf{R}_2 using $\hat{C}_0, \hat{C}_1, \dots, \hat{C}_K$ according to (6) and (7).

Step 3: Calculate the eigendecomposition of \mathbf{R}_1 and denote the minimum eigenvalue as λ_{\min} .

Step 4: Calculate $\mathbf{D}_1 = \mathbf{R}_1 - \lambda_{\min} \mathbf{I}$ and $\mathbf{D}_2 = \mathbf{R}_2 - \lambda_{\min} \mathbf{Z}$.

Step 5: Calculate the singular value decomposition of \mathbf{D}_1 and denote Σ_1 consists of P principal singular values, \mathbf{U}_1 consists of P principal left-singular vectors, and \mathbf{V}_1 consists of P principal right-singular vectors, respectively.

Step 6: Calculate the generalized eigenvalue decomposition of the matrix pencil $\{\Sigma_1, \mathbf{U}_1^H\mathbf{D}_2\mathbf{V}_1\}$ and denote the P generalized eigenvalues as $\gamma_1, \gamma_2, \dots, \gamma_P$.

Step 7: Estimate the frequencies of the harmonics by

$$\hat{\omega}_k = \frac{\angle \gamma_k}{2}, \quad k = 1, 2, \dots, P, \quad (19)$$

where \angle denotes calculation of phase angle.

4 Simulation Results

In this section, simulations are conducted to demonstrate the performance of the proposed SVD-based ESPRIT algorithm for the frequency estimation of the harmonics in multiplicative and additive noise, in which $P = 3$, $\phi_1 = 0.54$, $\phi_2 = 1.83$, $\phi_3 = -2.12$, with the data length $T = 512$. Multiplicative and

Table 1: Frequency estimation results from 1000 Monte Carlo runs of example 1 (mean \pm *std*)

method	$\omega_1 = -0.72$	$\omega_2 = 0.45$	$\omega_3 = 1.36$
the cyclic statistics algorithm	-0.7324 ± 0.2847	0.4618 ± 0.3316	1.3694 ± 0.2536
the proposed algorithm	-0.7190 ± 0.0144	0.4516 ± 0.0152	1.3604 ± 0.0103

Table 2: Frequency estimation results from 100 Monte Carlo runs of example 2 (mean \pm *std*)

method	$\omega_1 = 0.55$	$\omega_2 = 0.66$	$\omega_3 = 0.67$
the cyclic statistics algorithm	0.5525 ± 0.2241	0.6645 ± 0.8574	no estimation result
the proposed algorithm with $K = 150$	0.5501 ± 0.0008	0.6583 ± 0.0111	0.6715 ± 0.0088
the proposed algorithm with $K = 200$	0.5500 ± 0.0006	0.6599 ± 0.0016	0.6701 ± 0.0008

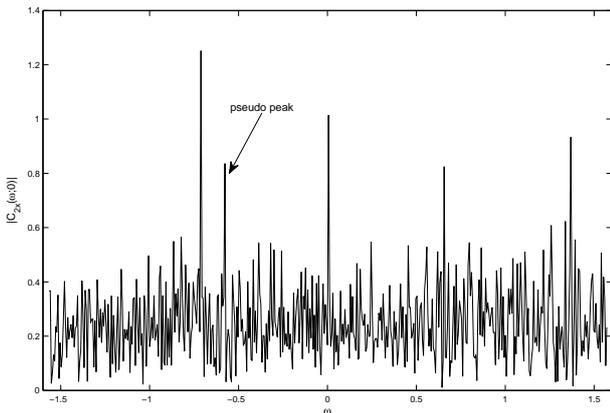


Figure 1: Harmonic estimation using cyclic statistic algorithm of example 1.

additive noise are assumed as real zero-mean white Gaussian with $\sigma_{s_1}^2 = 1.0$, $\sigma_{s_2}^2 = 1.0$, $\sigma_{s_3}^2 = 1.0$, and $\sigma_v^2 = 1.0$. For comparison, we also show the frequency estimation results with the cyclic statistics algorithm developed in [3].

Example 1: The frequencies of harmonics are $\omega_1 = -0.72$, $\omega_2 = 0.45$, and $\omega_3 = 1.36$, respectively. We conduct 1000 Monte Carlo runs to estimate the frequencies of harmonics using the proposed SVD-based ESPRIT algorithm with $K = 40$ and the cyclic statistics algorithm [3]. The mean \pm standard deviation (*std*) of the frequency estimation results is given in Table 1. Due to the occasional appearance of the pseudo peak (as shown in Fig. 1), the cyclic statistics algorithm based on peak-searching cannot always correctly estimate corresponding frequencies of the harmonics. Table 1 shows that the proposed SVD-based ESPRIT algorithm can accurately estimate the frequencies of the harmonics.

Example 2: In this example, we test the frequency resolution of the proposed SVD-based ESPRIT algorithm. The frequencies of the harmonics are very close, $\omega_1 = -1.24$, $\omega_2 = 0.66$, and $\omega_3 = 0.68$. We

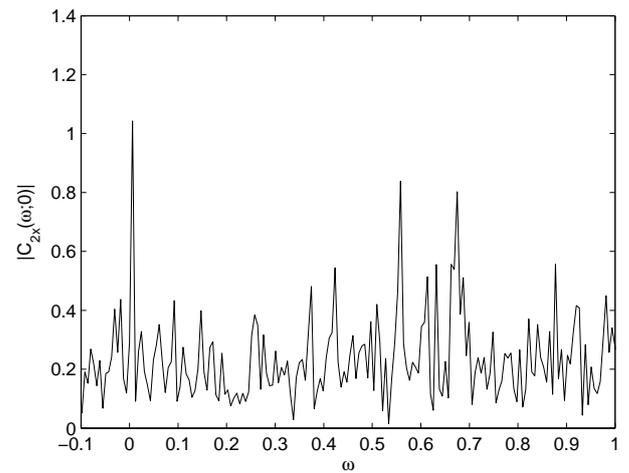


Figure 2: Harmonic estimation using cyclic statistics algorithm of example 2.

firstly estimate the frequencies using the cyclic statistics algorithm. As we can see from Fig. 2, due to the frequencies are very close, the peaks of the cyclic correlation corresponding to the frequencies $\omega_2 = 0.66$ and $\omega_3 = 0.67$ stick together and cannot be separated. Therefore, the cyclic statistics algorithm is invalid in this case. We estimate the frequency using the proposed SVD-based ESPRIT with $K = 150$ and $K = 200$ by 1000 Monte Carlo runs. The mean \pm *std* of the frequency estimation results from 1000 Monte Carlo runs are listed in Table 2. Table 2 shows that the proposed SVD-based ESPRIT algorithm can distinguish effectively the two close frequencies with high resolution.

5 Conclusion

A novel SVD-based ESPRIT algorithm is developed to the frequency estimation of the harmonics in multiplicative and additive noise. The proposed algorithm exploited the underlying signal model of the

harmonics in multiplicative and additive noise. The proposed SVD-based ESPRIT algorithm not only has high-resolution, but also is easy to be implemented in practice because it does not need peak searching. Simulations results showed the proposed SVD-based ESPRIT algorithm has manifest superior performance, compared with the cyclic statistics algorithm.

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Appendix: Derivation of equation (4)

Let $\omega_{P+1} = 0$, $\phi_{P+1} = 0$, and $s_{P+1}(t) = v(t)$, (1) can be written as a simple form as following:

$$x(t) = \sum_{k=1}^{P+1} s_k(t) e^{j(\omega_k t + \phi_k)}. \quad (20)$$

Then, $E\{x^2(t)x^{2*}(t+\tau)\}$ can be calculated by

$$\begin{aligned} & E\{x^2(t)x^{2*}(t+\tau)\} \\ &= \sum_{k=1}^{P+1} ((\sigma_{s_k}^2)^2 + 2(\sigma_{s_k}^2)^2 \delta(\tau)) e^{-j2\omega_k \tau} \\ &+ \sum_{k=1}^P \sum_{l=k+1}^{P+1} 4\sigma_{s_k}^2 \sigma_{s_l}^2 \delta(\tau) e^{-j(\omega_k + \omega_l)\tau} \\ &= \sum_{k=1}^{P+1} (\sigma_{s_k}^2)^2 e^{-j2\omega_k \tau} \\ &+ \left(\sum_{k=1}^{P+1} 2(\sigma_{s_k}^2)^2 + \sum_{k=1}^P \sum_{l=k+1}^{P+1} 4\sigma_{s_k}^2 \sigma_{s_l}^2 \right) \delta(\tau). \end{aligned} \quad (21)$$

In the derivation of (21), we use the assumption (3), all the fourth-order cumulants of Gaussian random process are equal to zero.

Hence, the corresponding cyclic means are

$$\begin{aligned} & \bar{E}\{x^2(t)x^{2*}(t+\tau)\} \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T E\{x^2(t)x^{2*}(t+\tau)\} \\ &= \sum_{k=1}^{P+1} (\sigma_{s_k}^2)^2 e^{-j2\omega_k \tau} \\ &+ \left(\sum_{k=1}^{P+1} 2(\sigma_{s_k}^2)^2 + \sum_{k=1}^P \sum_{l=k+1}^{P+1} 4\sigma_{s_k}^2 \sigma_{s_l}^2 \right) \delta(\tau), \end{aligned} \quad (22)$$

$$\begin{aligned} \bar{E}\{x^2(t)\} &= \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T E\{x^2(t)\} \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T \left(\sum_{k=1}^{P+1} \sigma_{s_k}^2 e^{j2(\omega_k t + \phi_k)} \right) \\ &= \sigma_{s_{P+1}}^2, \end{aligned} \quad (23)$$

$$\begin{aligned} & \bar{E}\{x^{2*}(t+\tau)\} \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T E\{x^{2*}(t+\tau)\} \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T \left(\sum_{k=1}^{P+1} \sigma_{s_k}^2 e^{-j2(\omega_k(t+\tau) + \phi_k)} \right) \\ &= \sigma_{s_{P+1}}^2. \end{aligned} \quad (24)$$

In the derivation of (23) and (24), we use the assumption (3) and

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T e^{j\alpha t} = \delta(\alpha) = \begin{cases} 1, & \alpha = 0, \\ 0, & \text{otherwise.} \end{cases} \quad (25)$$

Therefore, the cyclic covariance is

$$\begin{aligned} C_\tau &= \bar{E}\{x^2(t)x^{2*}(t+\tau)\} - \bar{E}\{x^2(t)\}\bar{E}\{x^{2*}(t+\tau)\} \\ &= \sum_{k=1}^{P+1} (\sigma_{s_k}^2)^2 e^{-j2\omega_k \tau} - (\sigma_{s_{P+1}}^2)^2 \\ &+ \left(\sum_{k=1}^{P+1} 2(\sigma_{s_k}^2)^2 + \sum_{k=1}^P \sum_{l=k+1}^{P+1} 4\sigma_{s_k}^2 \sigma_{s_l}^2 \right) \delta(\tau) \\ &= \sum_{k=1}^P (\sigma_{s_k}^2)^2 e^{-j2\omega_k \tau} + \beta \delta(\tau). \end{aligned} \quad (26)$$

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