

# A novel algorithm for 2D-IIR filters synthesis via 2D-FIR filters model reduction

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**Abstract:** A digital 2D-FIR filter with linear phase, circularly symmetric with respect to the origin of the frequency plane, is designed using the two-dimensional windowing method. An economical filter with high information efficiency is obtained by applying the balanced realization method to this full-order filter. The result is a linear phase IIR filter whose frequency response is very close to that of the initial filter.

**Keywords:** Digital two-dimensional filter, windowing, state-space, balanced realization, separable denominator, model reduction.

## 1. Introduction

Digital 2D-filters have many applications in the analysis and filtering of radiographic and photographic image data. These data are used in meteorology, seismology, medicine, geophysics, crystallography, etc. Applications include processing X-ray data, magnetic data, gravity records, etc. Visual identification of objects by industrial robotic systems and target following in radar systems process image data too, and require a very high information processing efficiency, usually in real-time. Similarly to the 1D digital filters, the 2D-filters can be divided into two classes: recursive filters (IIR filters) and nonrecursive filters (FIR filters).

The synthesis of a 2D-FIR filter essentially consists in finding the impulse response  $h(m,n)$ , or the transfer-function  $H(z,w)$ , which satisfies the specifications [1,2,3,4]. There are four standard approaches to design 2D-FIR filters: the windowing method, frequency sampling method,

frequency transformation method, and the optimal design method. The first two methods are extensions of the 1D case by a suitable modification of the synthesis procedure.

The windowing method is considered in this paper, due to its simplicity. The 2D window used in the filter design is typically obtained starting with a 1D window, using the classical Huang's method [1] to get a circularly symmetric two-dimensional window; other two-dimensional windows, like the Gaussian window, can also be used in the synthesis of a 2D-FIR filter. Mathematical state-space representation of a 2D-FIR filter has a very important advantage, by enabling to perform equivalence transformations, model reduction, stability test, and other operations; for instance, a canonical state-space form of a 2D-FIR filter which belongs to the class of separable filters (i.e., with separable denominator in the transfer-function) can be obtained [5].

In the second part of this paper, a model order reduction technique, known as

approximation by balanced realization [6, 7], is applied to a full-order 2D-FIR filter designed by the windowing method, to obtain a reduced order separable 2D-IIR filter. As it will be shown, the resulting filter is very close to the initial filter, and the approximation errors of the frequency response are bounded by the sum of the neglected Hankel singular values of the initial filter. Consequently, the frequency response of the resulting filter differs only slightly from that of the initial filter, and the phase remains linear in the bandwidth region. In addition, the obtained 2D-filter is more economical, i.e., has lower complexity.

Therefore, by combining the windowing method and balanced realization method, excellent results are obtained, as it will be shown by simulations performed in MATLAB® 7.1.

## 2. The windowing method

Let us suppose, for all considered types of filters, that the frequency responses can be described by a circularly symmetric function, i.e.,  $H(\omega_1, \omega_2)$  is a function of  $(\omega_1^2 + \omega_2^2)^{1/2}$ . Assume that the desired frequency response  $H_d(\omega_1, \omega_2)$  is known. By inverse Fourier transform of  $H_d(\omega_1, \omega_2)$ , it is possible to determine the desired impulse response of the filter,  $h_d(m, n)$ . Generally,  $h_d(m, n)$  is an infinite sequence. But using the windowing method, a 2D-FIR filter is obtained by multiplying  $h_d(m, n)$  with a window  $w(m, n)$ ,

$$h(m, n) = h_d(m, n)w(m, n). \quad (1)$$

If both functions  $h_d(m, n)$  and  $w(m, n)$  are symmetric with respect to the origin then their product  $h(m, n)$  is also symmetric, so that the resulting filter will have a null phase.

From (1) and the properties of the Fourier transform [8],

$$\begin{aligned} H(\omega_1, \omega_2) &= H_d(\omega_1, \omega_2) \circledast W(\omega_1, \omega_2) \\ &= \frac{1}{(2\pi)^2} \int_{\theta_1=-\pi}^{\pi} \int_{\theta_2=-\pi}^{\pi} H_d(\theta_1, \theta_2) W((\omega_1 \\ &\quad - \theta_1, \omega_2 - \theta_2)) d\theta_1 d\theta_2, \end{aligned}$$

where  $\circledast$  denotes the 2D convolution product. The effect of the window in the frequency domain is to smooth  $H_d(\omega_1, \omega_2)$ .

If the wideness of the principal lobe of  $W(\omega_1, \omega_2)$  is small, the wideness of the transition band of  $H(\omega_1, \omega_2)$  is small too. If the secondary lobes have small amplitudes, then the oscillations in the bandwidth region and the cut-off band have also small amplitudes.

A 2D window used in the filter synthesis is typically obtained starting from a 1D window; the classical Huang's method [1] enables to obtain a 2D window  $w(m, n)$  as follows

$$w(m, n) = w_c(t_1, t_2)|_{t_1=m, t_2=n}, \quad (2.a)$$

$$w_c(t_1, t_2) = w_a(t)|_{t=\sqrt{t_1^2+t_2^2}}. \quad (2.b)$$

The function  $w_a(t)$  in (2.b) is an analog (continuous time) 1D window, which is turned, in this method, to an analog 2D window  $w_c(t_1, t_2)$ . Note that  $W_c(\Omega_1, \Omega_2)$ , the Fourier transform of  $w_c(t_1, t_2)$ , is a circularly symmetric function. However, there is no transformed version of  $W_a(\Omega)$ , the Fourier transform of  $w_a(t)$ . Specifically,  $W_c(\Omega_1, \Omega_2)$  is connected to  $w_a(t)$  by [8]

$$\begin{aligned} W_c(\Omega_1, \Omega_2) &= G(\rho)|_{\rho=\sqrt{\Omega_1^2+\Omega_2^2}} = \\ 2\pi \int_{t=0}^{\infty} t w_a(t) J_0(t\rho) dt |_{\rho=\sqrt{\Omega_1^2+\Omega_2^2}}, \quad (3) \end{aligned}$$

where  $J_0(\cdot)$  is the Bessel function of the first type and zero-order. The function  $G(\rho)$  in (3) is the Hankel transform of  $w_a(t)$ . The analog 2D window  $w_c(t_1, t_2)$  is sampled for obtaining a 2D window  $w(m, n)$ . The resulting sequence  $w(m, n)$  is a circularly symmetric window. From (2.a) and (2.b),

$$\begin{aligned} W(\omega_1, \omega_2) &= \\ \sum_{r_1=-\infty}^{\infty} \sum_{r_2=-\infty}^{\infty} W_c(\Omega_1, \Omega_2) |_{\Omega_1=\omega_1-2\pi r_1, \Omega_2=\omega_2-2\pi r_2}. \quad (4) \end{aligned}$$

There is an aliasing effect in (4). The circular symmetry of  $w(n_1, n_2)$  does not guarantee the circular symmetry of its Fourier transform  $W(\omega_1, \omega_2)$ . Indeed, the function  $W(\omega_1, \omega_2)$  can deviate considerably from the circular symmetry, for  $(\omega_1, \omega_2)$  far from the origin. However, close to the origin, the aliasing effect is reduced and  $W(\omega_1, \omega_2)$  tends to be close to a circular symmetry. Additional

details about two-dimensional windows can be found in [9, 10].

### 3. State-space modeling of 2D-FIR filters

Assuming that

$$H(z, w) = \sum_{i=0}^m \sum_{j=0}^n h(i, j) z^{-i} w^{-j},$$

is the transfer-function of a discrete 2D-FIR filter of order  $(m, n)$ , where  $z^{-1}$  and  $w^{-1}$  are unit backward operators,  $H(z, w)$  can be written in the form

$$H(z, w) = \frac{\sum_{i=0}^m \sum_{j=0}^n h(i, j) z^{m-i} w^{n-j}}{z^m w^n}.$$

This shows that such a 2D-FIR filter belongs to the class of filters with separable denominator, i.e., the denominator polynomial with two independent variables of the transfer-function of a filter in this class can be written as a product of two polynomials, each depending on a single variable only. The transfer-function of these filters is expressed as follows:

$$H(z, w) = \frac{N(z, w)}{D_1(z)D_2(w)}. \tag{5}$$

Any causal 2D system having a transfer-function with a separable denominator can be modeled in the local state-space Roesser's characterisation in the form [5, 11]

$$\begin{bmatrix} x^h(i+1, j) \\ x^v(i, j+1) \end{bmatrix} = \begin{bmatrix} A^{(1)} & 0 \\ A^{(3)} & A^{(4)} \end{bmatrix} \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} + \begin{bmatrix} B^{(1)} \\ B^{(2)} \end{bmatrix} u(i, j), \tag{6.a}$$

$$y(i, j) = [C^{(1)} \quad C^{(2)}] \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} + Du(i, j), \tag{6.b}$$

$i, j \geq 0$ .

It is assumed here that  $A^{(2)} = 0$ ; the case  $A^{(3)} = 0$  is similar.

Above,  $x^h \in \mathbb{R}^m$  and  $x^v \in \mathbb{R}^n$  represent the horizontal and vertical components of the state, respectively,  $u \in \mathbb{R}^p$  is the input vector,  $y \in \mathbb{R}^q$  is the output vector,  $(i, j)$  are nonnegative integer numbers, and  $A^{(1)}, A^{(3)}, A^{(4)}, B^{(1)}, B^{(2)}, C^{(1)}, C^{(2)}$  and  $D$  are constant real matrices with suitable dimensions, given in a canonical form, as follows [5]

$$A^{(1)} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \alpha_1 \\ 1 & 0 & \cdots & 0 & \alpha_2 \\ 0 & 1 & \cdots & 0 & \alpha_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \alpha_m \end{bmatrix},$$

$$A^{(4)} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \beta_1 & \beta_2 & \beta_3 & \cdots & \beta_n \end{bmatrix},$$

$$A^{(3)} = \begin{bmatrix} h_{11} & h_{21} & \cdots & h_{m1} \\ h_{12} & h_{22} & \cdots & h_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ h_{1n} & h_{2n} & \cdots & h_{mn} \end{bmatrix},$$

$$B^{(1)} = [1 \quad 0 \quad \cdots \quad 0]^T, B^{(2)} = [h_{01} \quad h_{02} \quad \cdots \quad h_{0n}]^T, \tag{7}$$

$$C^{(1)} = [h_{10} \quad h_{20} \quad \cdots \quad h_{m0}],$$

$$C^{(2)} = [1 \quad 0 \quad \cdots \quad 0],$$

where  $(.)^T$  denotes the transpose of the matrix  $(.)$ .

The  $\alpha_i$  and  $\beta_i$  parameters are real numbers, which are zero in the 2D-FIR filters case, and the  $h_{ij}$  coefficients are the two-dimensional Markov parameters given by [5]

$$h_{i0} = C^{(1)} A^{(1) i-1} B^{(1)},$$

$$h_{0j} = C^{(2)} A^{(4) j-1} B^{(2)},$$

$$h_{ij} = C^{(2)} A^{(4) j-1} A^{(3)} A^{(1) i-1} B^{(1)}, \quad (i, j) > (0, 0).$$

These parameters are also the coefficients of the two-dimensional impulse response of the 2D-FIR filter, and the corresponding transfer-function is

$$H(z, w) = C(zI_m \oplus wI_n - A)^{-1} B + D,$$

where

$$A = \begin{bmatrix} A^{(1)} & 0 \\ A^{(3)} & A^{(4)} \end{bmatrix}, B = \begin{bmatrix} B^{(1)} \\ B^{(2)} \end{bmatrix},$$

$$C = [C^{(1)} \quad C^{(2)}].$$

### 4. Application of the balanced realization method

In the sequel, a digital separable 2D-IIR filter with a linear phase response is obtained by applying the balanced realization method to a high-order 2D-FIR filter, designed by the windowing method. Several properties associated to the proposed approach are also presented and proven. Moreover, an example is included to illustrate the performance of the method.

The balanced realization method was intensively applied in the past for 1D, as well as 2D dynamical systems [12, 13]. The method delivers economical systems with a reduced information complexity [7, 14]. This paper shows that the method is very useful in approximating digital 2D-FIR filters.

#### 4.1. Method description

The generalization of Gramians for 2D systems is proposed in [6]. The following definition is introduced for a given realization of type (6).

*Definition 4.1:* For a 2D realization  $\{A, B, C, D\}$ , the controllability and observability Gramians  $\mathcal{P}$  and  $\mathcal{Q}$  are given by

$$\begin{aligned} \mathcal{P} &= \frac{1}{(2\pi j)^2} \oint_{|z|=1} \oint_{|w|=1} F(z, w) F^*(z, w) \cdot \frac{dz dw}{z w}, \\ \mathcal{Q} &= \frac{1}{(2\pi j)^2} \oint_{|z|=1} \oint_{|w|=1} G(z, w) G^*(z, w) \cdot \frac{dz dw}{z w}, \end{aligned} \quad (8)$$

where  $F(z, w) = [(zI_m \oplus wI_n) - A]^{-1}B$  and  $G(z, w) = C[(zI_m \oplus wI_n) - A]^{-1}$ .

If  $\{A, B, C, D\}$  is locally controllable and observable [11], the submatrices  $\mathcal{P}^{(1)}$ ,  $\mathcal{P}^{(4)}$ ,  $\mathcal{Q}^{(1)}$  and  $\mathcal{Q}^{(4)}$  of  $\mathcal{P}$  and  $\mathcal{Q}$  are positive definite (PD) matrices [6].

*Definition 4.2:* The model is balanced in the horizontal and vertical directions since the horizontal and vertical controllability and observability Gramians are diagonal and equal, i.e.:

$$\begin{aligned} \mathcal{P}^{(1)} &= \mathcal{Q}^{(1)} = \Sigma^{(1)} = \text{diag}\{\sigma_1^{(1)}, \dots, \sigma_m^{(1)}\}, \\ \mathcal{P}^{(4)} &= \mathcal{Q}^{(4)} = \Sigma^{(4)} = \text{diag}\{\sigma_1^{(4)}, \dots, \sigma_n^{(4)}\}, \end{aligned} \quad (9)$$

where  $\sigma_i^{(1)}$ ,  $i = 1, \dots, m$ , and  $\sigma_j^{(4)}$ ,  $j = 1, \dots, n$  are the *Hankel singular values* of the horizontal and vertical subsystem, respectively.

The aim is to obtain a realization of reduced order  $(\hat{m}, \hat{n})$ , by suitably truncating the original realization. This is done by estimating the order of the reduced model, which satisfies the following conditions of the *Hankel-norm error* [15]:

$$\begin{aligned} \sum_{i=1}^{\hat{m}} (\sigma_i^{(1)})^4 / \sum_{i=\hat{m}+1}^m (\sigma_i^{(1)})^4 &\geq \text{TOL}_1, \quad \text{and} \\ \frac{\sum_{j=1}^{\hat{n}} (\sigma_j^{(4)})^4}{\sum_{j=\hat{n}+1}^n (\sigma_j^{(4)})^4} &\geq \text{TOL}_2. \end{aligned} \quad (10)$$

When  $\hat{m} \neq 0, m$  and  $\hat{n} \neq 0, n$ ,  $\text{TOL}_1$  and  $\text{TOL}_2$  can be specified graphically by plotting the ratio of each left part in (10) as a function of  $\hat{m}$  and  $\hat{n}$ , respectively. Generally,  $\text{TOL}_1$  and  $\text{TOL}_2$  should be much larger than unity.

It is shown in [6] that the 2D Gramians above can be expressed as follows

$$\begin{aligned} \mathcal{P} &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} M_{ij} M_{ij}^T \\ \mathcal{Q} &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \begin{bmatrix} [A_{ij}^T (C^T C) A_{ij}]^{(1)} & [A_{ij}^T (C^T C) A_{i+1, j-1}]^{(2)} \\ [A_{ij}^T (C^T C) A_{i-1, j+1}]^{(3)} & [A_{ij}^T (C^T C) A_{ij}]^{(4)} \end{bmatrix}, \end{aligned}$$

where

$$M_{ij} = \begin{cases} A_{i-1, j} \begin{bmatrix} B^{(1)} \\ 0 \end{bmatrix} + A_{i, j-1} \begin{bmatrix} 0 \\ B^{(2)} \end{bmatrix}, & (i, j) > (0, 0), \\ 0, & (i, j) = (0, 0), \end{cases}$$

and  $A_{ij}$  denoting the transition matrix of the 2D system

$$A_{ij} = A_{10} A_{i-1, j} + A_{01} A_{i, j-1}, \quad (i, j) > (0, 0),$$

and

$$A_{10} = \begin{bmatrix} A^{(1)} & A^{(2)} \\ 0 & 0 \end{bmatrix}, \quad A_{01} = \begin{bmatrix} 0 & 0 \\ A^{(3)} & A^{(4)} \end{bmatrix}.$$

*Theorem 4.1:* The realization  $\{A, B, C, D\}$  and its dual  $\{A^T, C^T, B^T, D^T\}$  satisfy  $\mathcal{P} = \mathcal{Q}_d$  and  $\mathcal{Q} = \mathcal{P}_d$ , where the subscript  $d$  denotes *dual*.

The dual realization is also denoted by  $\{A_d, B_d, C_d, D_d\}$ .

In terms of 2D realization models, using the backward operators  $z^{-1}$  and  $w^{-1}$ , the duality between the systems  $\{A, B, C, D\}$  and  $\{A^T, C^T, B^T, D^T\}$  is analogue to the 1D case. Using the above result, it is possible to redefine the controllability and observability Gramians as follows.

**Definition 4.3:** For a 2D system  $\{A, B, C, D\}$ , the controllability and observability Gramians are defined by

$$\begin{aligned} \mathcal{P} &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} M_{ij} M_{ij}^T, \\ \mathcal{Q} &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} M_{dij} M_{dij}^T, \end{aligned} \quad (11)$$

where

$$M_{dij} = \begin{cases} A_{d_{i-1,j}} \begin{bmatrix} C^{(1)T} \\ 0 \end{bmatrix} + A_{d_{i,j-1}} \begin{bmatrix} 0 \\ C^{(2)T} \end{bmatrix}, & (i,j) > (0,0), \\ 0, & (i,j) = (0,0), \end{cases}$$

with  $M_{dij}$  and  $A_{dij}$  related to the quantities of the dual system, corresponding to  $M_{ij}$  and  $A_{ij}$ , respectively, of the given system, and

$$A_{dij} = \begin{bmatrix} A_{ij}^{(1)T} & A_{i-1,j+1}^{(3)T} \\ A_{i+1,j-1}^{(2)T} & A_{ij}^{(4)T} \end{bmatrix}.$$

Alternatively, the Gramians can be defined as follows

$$\begin{aligned} \mathcal{P} &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \begin{bmatrix} [A_{dij}^T (BB^T) A_{dij}]^{(1)} & [A_{dij}^T (BB^T) A_{d_{i+1,j-1}}]^{(2)} \\ [A_{dij}^T (BB^T) A_{d_{i-1,j+1}}]^{(3)} & [A_{dij}^T (BB^T) A_{dij}]^{(4)} \end{bmatrix}, \\ \mathcal{Q} &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \begin{bmatrix} [A_{ij}^T (C^T C) A_{ij}]^{(1)} & [A_{ij}^T (C^T C) A_{i+1,j-1}]^{(2)} \\ [A_{ij}^T (C^T C) A_{i-1,j+1}]^{(3)} & [A_{ij}^T (C^T C) A_{ij}]^{(4)} \end{bmatrix}. \end{aligned}$$

**Theorem 4.2:**

$$\begin{aligned} (A\mathcal{P}A^T - \mathcal{P})^{(1)} &= -B^{(1)}B^{(1)T}, \\ (A\mathcal{P}A^T - \mathcal{P})^{(4)} &= -B^{(2)}B^{(2)T}, \\ (A\mathcal{Q}A^T - \mathcal{Q})^{(1)} &= -C^{(1)T}C^{(1)}, \\ (A\mathcal{Q}A^T - \mathcal{Q})^{(4)} &= -C^{(2)T}C^{(2)}. \end{aligned}$$

*Proof:* The proof directly follows by replacing  $\mathcal{P}$  and  $\mathcal{Q}$  by their values in (11).

**Theorem 4.3:** For a stable, discrete, separable, locally controllable and observable 2D system,

$\{A, B, C, D\}$ , the off-diagonal submatrices,  $\mathcal{P}^{(2)}, \mathcal{P}^{(3)}, \mathcal{Q}^{(2)}$  and  $\mathcal{Q}^{(3)}$  of the Gramians are zero.

*Proof:* The proof follows by replacing  $A^{(2)} = 0$  in (8).

**Theorem 4.4:** The Gramians  $\mathcal{P}$  and  $\mathcal{Q}$  of a stable, separable, locally controllable and observable 2D realization,  $\{A, B, C, D\}$ , satisfy the following relations:

$$\begin{aligned} A^{(1)}\mathcal{P}^{(1)}A^{(1)T} - \mathcal{P}^{(1)} &= -B^{(1)}B^{(1)T}, \\ A^{(1)}\mathcal{Q}^{(1)}A^{(1)T} - \mathcal{Q}^{(1)} &= -[C^{(1)T}C^{(1)} + A^{(3)T}\mathcal{Q}^{(4)}A^{(3)}] \\ &\equiv -C^{(1)T}C^{(1)} \\ A^{(4)}\mathcal{P}^{(4)}A^{(4)T} - \mathcal{P}^{(4)} &= -[B^{(2)}B^{(2)T} + A^{(3)}\mathcal{P}^{(1)}A^{(3)T}] \\ &\equiv -B^{(2)}B^{(2)T} \\ A^{(4)T}\mathcal{Q}^{(4)}A^{(4)} - \mathcal{Q}^{(4)} &= -C^{(2)T}C^{(2)}, \end{aligned} \quad (12)$$

with

$$C^{(1)} = \begin{bmatrix} C^{(1)} \\ R^{(4)}A^{(3)} \end{bmatrix} \in \mathbb{R}^{(q+n) \times m},$$

$$B^{(2)} = [B^{(2)} \quad A^{(3)}S^{(1)}] \in \mathbb{R}^{n \times (p+m)},$$

where  $R^{(4)T}R^{(4)} = \mathcal{Q}^{(4)}$ , and  $S^{(1)}S^{(1)T} = \mathcal{P}^{(1)}$ . In addition, each of the 1D systems  $\{A^{(1)}, B^{(1)}, C^{(1)}\}$  and  $\{A^{(4)}, B^{(2)}, C^{(2)}\}$  is asymptotically stable, controllable and observable. Therefore, if the 2D system is balanced, then these systems are balanced too.

*Proof:* By replacing  $A^{(2)} = 0$  in Theorem 4.2, and taking into account the results in Theorem 4.3, we obtain

$$\begin{aligned} A\mathcal{P}A^T - \mathcal{P} &= \begin{bmatrix} A^{(1)} & 0 \\ A^{(3)} & A^{(4)} \end{bmatrix} \begin{bmatrix} \mathcal{P}^{(1)} & 0 \\ 0 & \mathcal{P}^{(4)} \end{bmatrix} \begin{bmatrix} A^{(1)T} & A^{(3)T} \\ 0 & A^{(4)T} \end{bmatrix} \\ &\quad - \begin{bmatrix} \mathcal{P}^{(1)} & 0 \\ 0 & \mathcal{P}^{(4)} \end{bmatrix} \\ &= \begin{bmatrix} A^{(1)}\mathcal{P}^{(1)}A^{(1)T} - \mathcal{P}^{(1)} & A^{(1)}\mathcal{P}^{(1)}A^{(3)T} \\ A^{(3)}\mathcal{P}^{(1)}A^{(1)T} & A^{(3)}\mathcal{P}^{(1)}A^{(3)T} + A^{(4)}\mathcal{P}^{(4)}A^{(4)T} - \mathcal{P}^{(4)} \end{bmatrix}. \end{aligned}$$

Then

$$\begin{aligned} (A\mathcal{P}A^T - \mathcal{P})^{(1)} &= A^{(1)}\mathcal{P}^{(1)}A^{(1)T} - \mathcal{P}^{(1)} \\ &= -B^{(1)}B^{(1)T}. \end{aligned}$$

This is the first relation. Since

$$\begin{aligned} (A\mathcal{P}A^T - \mathcal{P})^{(4)} &= -B^{(2)}B^{(2)T} \\ &= A^{(3)}\mathcal{P}^{(1)}A^{(3)T} + A^{(4)}\mathcal{P}^{(4)}A^{(4)T} - \mathcal{P}^{(4)}, \end{aligned}$$

then

$$\begin{aligned} A^{(4)}\mathcal{P}^{(4)}A^{(4)T} - \mathcal{P}^{(4)} &= -B^{(2)}B^{(2)T}A^{(3)}\mathcal{P}^{(1)}A^{(3)T} \\ &= -\begin{bmatrix} B^{(2)} & A^{(3)}S^{(1)} \end{bmatrix} \begin{bmatrix} B^{(2)} & A^{(3)}S^{(1)} \end{bmatrix}^T \\ &= -\mathbf{B}^{(2)}\mathbf{B}^{(2)T}. \end{aligned}$$

Since  $\mathbf{B}^{(2)} = [B^{(2)} \ A^{(3)}S^{(1)}]$ , the third relation is proven. The proof follows the same procedure for the second and fourth relation.

The asymptotic stability of 1D systems is clear. The local controllability and observability requirement implies the controllability of  $\{A^{(1)}, B^{(1)}\}$  and observability of  $\{A^{(4)}, C^{(2)}\}$ . Indeed, if  $\{A^{(4)}, C^{(2)}\}$  is not observable, there is a nonzero vector  $v \in \mathbb{C}^n$  and a scalar  $\lambda \in \mathbb{C}$  so that

$$\begin{aligned} A^{(4)}v &= \lambda v, \quad v^*A^{(4)T} = \lambda^*v^*; \\ C^{(2)}v &= 0, \quad v^*C^{(2)T} = 0. \end{aligned}$$

Pre- and post-multiplying the expression  $A^{(4)T}Q^{(4)}A^{(4)} - Q^{(4)} = -C^{(2)T}C^{(2)}$  by  $v^*$  and  $v$ , respectively, we get  $(|\lambda|^2 - 1)v^*Q^{(4)}v = 0$ . The local controllability and observability imply also that  $Q^{(4)}$  is PD, and hence,  $|\lambda| = 1$ , which is not possible since  $A^{(4)}$  is asymptotically stable. Then,  $\{A^{(4)}, C^{(2)}\}$  is observable. Similarly, it can be proven that  $\{A^{(1)}, B^{(1)}\}$  is controllable. Since the 2D separable system is balanced, (9) is satisfied. The statements above show that the two 1D systems are balanced.

### 4.2. Reduction procedure

**Inputs:** Impulse response  $h(m, n)$  of the 2D full-order system (filter).

**Step 1:** Write  $H(z, w)$ , the transfer-function of  $h(m, n)$ , in the separable form (5).

**Step 2:** Transform  $h(m, n)$  to a state-space realization,  $\{A, B, C, D\}$ , in the canonical form in (7).

**Step 3:** Compute the submatrices  $\mathcal{P}^{(1)}$ ,  $\mathcal{P}^{(4)}$  and  $Q^{(1)}$ ,  $Q^{(4)}$  of the controllability and observability Gramians, respectively, by

solving the pairs of Lyapunov equations in (12).

**Step 4:** Determine nonsingular transformation matrices  $T^{(1)}$  and  $T^{(4)}$  by balancing the horizontal and vertical 1D subsystems represented by the pairs  $(\mathcal{P}^{(1)}, Q^{(1)})$  and  $(\mathcal{P}^{(4)}, Q^{(4)})$  [15], respectively.

**Step 5:** Set  $T = \begin{bmatrix} T^{(1)} & 0 \\ 0 & T^{(4)} \end{bmatrix}$ .

**Step 6:** Find the balanced realization  $\{\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}\} = \{T^{-1}AT, T^{-1}B, CT, D\}$ .

**Step 7:** Estimate the reduced order  $(\hat{m}, \hat{n})$ , of the reduced model using the *Hankel-norm error* criterion [16] in (10).

**Step 8:** Extract the reduced order realization  $\{\hat{A}, \hat{B}, \hat{C}, D\}$  of order  $(\hat{m}, \hat{n})$ , using the following partition of  $\{\tilde{A}, \tilde{B}, \tilde{C}\}$

$$\begin{aligned} \tilde{A} &= \begin{bmatrix} \tilde{A}_1 & 0 \\ \tilde{A}_3 & \tilde{A}_4 \end{bmatrix} = \begin{bmatrix} \hat{A}_1 & * & 0 & 0 \\ * & * & 0 & 0 \\ \hat{A}_3 & * & \hat{A}_4 & * \\ * & * & * & * \end{bmatrix}, \\ \tilde{B} &= \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}, \quad \tilde{C} = [\tilde{C}_1 \quad \tilde{C}_2], \quad \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} = \begin{bmatrix} * \\ * \\ \tilde{B}_2 \\ * \end{bmatrix}, \\ \tilde{C} &= [\tilde{C}_1 \quad \tilde{C}_2] = [\hat{C}_1 \quad * \quad \hat{C}_2 \quad *], \\ \hat{A} &= \begin{bmatrix} \hat{A}_1 & 0 \\ \hat{A}_3 & \hat{A}_4 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix}, \quad \hat{C} = [\hat{C}_1 \quad \hat{C}_2]. \end{aligned}$$

**Outputs:** The transfer-function of the reduced order filter starting from the state-space realization of the procedure in [17], where  $\hat{H}(z, w)$  is given by

$$\hat{H}(z, w) = \frac{[w^{\hat{n}} \ w^{\hat{n}-1} \ \dots \ 1] \hat{N}_t [z^{\hat{m}} \ z^{\hat{m}-1} \ \dots \ 1]^T}{[w^{\hat{n}} \ w^{\hat{n}-1} \ \dots \ 1] \hat{D}_t [z^{\hat{m}} \ z^{\hat{m}-1} \ \dots \ 1]^T}$$

### 4.3. Properties of the reduced model

The reduced model resulting from the balanced realization procedure has some interesting properties. These properties are presented in this subsection.

- a- For a balanced, stable, separable, locally controllable and observable 2D

- realization,  $\{A, B, C, D\}$ , the horizontal and vertical subsystems are also stable.
- b- Given a balanced, stable, separable, locally controllable and observable 2D realization,  $\{A, B, C, D\}$ , the proposed reduction procedure preserves the stability of the resulting reduced order model.
  - c- Given a balanced, stable, separable, locally controllable and observable 2D realization,  $\{A, B, C, D\}$ , if  $\sigma_{\hat{m}}^{(1)} > \sigma_{\hat{m}+1}^{(1)}$  and  $\sigma_{\hat{n}}^{(4)} > \sigma_{\hat{n}+1}^{(4)}$ , then the reduced order realization is locally controllable and observable.
  - d- Given a balanced, stable, separable, locally controllable and observable 2D realization,  $\{A, B, C, D\}$ , if  $\sigma_{\hat{m}}^{(1)} > \sigma_{\hat{m}+1}^{(1)}$  and  $\sigma_{\hat{n}}^{(4)} > \sigma_{\hat{n}+1}^{(4)}$ , then the realization of reduced order  $(\hat{m}, \hat{n})$  is minimal [18].
  - e- The frequency error bound is given below, using the results in [19]. Assume that  $H(z, w)$  and  $\hat{H}(z, w)$  denote the 2D transfer-function matrices of the original and reduced system, respectively. For convenience, assume that the reduction is applied to the vertical subsystem only. Hence a filter of degree  $m\hat{h} - \hat{n}v$  is extracted from the original filter of degree  $mh - nv$ . Then, we have

$$\Delta H(z, w) \equiv H(z, w) - \hat{H}(z, w).$$

Hence, we obtain

$$\|\Delta H(z, w)\|_{\infty} \leq 2 \sum_{i=\hat{n}+1}^n \sigma_i^{(4)} \left[ 1 + 2 \sum_{j=1}^m \sigma_j \langle A^{(1)}, B^{(1)}, (\Sigma^{(1)})^{-1/2} \rangle \right],$$

where  $\sigma_j \langle \cdot, \cdot, \cdot \rangle, j=1, \dots, m$ , correspond to the Hankel singular values of the asymptotically stable, controllable and observable system  $\{., ., .\}$ , and  $\Sigma^{(1)}$  is the diagonal matrix defined in (9).

- f- The proposed method preserves the phase of the 2D-filter after reduction.
- Theorem 4.5* : If  $\hat{\varphi}(\omega_1, \omega_2)$  and  $\varphi(\omega_1, \omega_2)$  are the phases of the reduced filter and of the initial filter, respectively, and  $\Omega_p$  denotes the bandwidth region, then

$$\max_{\Omega_p} |\hat{\varphi}(\omega_1, \omega_2) - \varphi(\omega_1, \omega_2)| \approx \delta_m \leq \frac{2\hat{\epsilon}}{1-\epsilon_p},$$

where  $\hat{\epsilon}$  denotes the estimation of the maximum error in the bandwidth

region between the initial and reduced filter,  $\epsilon_p$  is the tolerance in the bandwidth of the initial filter, and

$$\delta_m = \max_{\Omega_p} \left| 1 - \frac{H(\omega_1, \omega_2) |\hat{H}(\omega_1, \omega_2)|}{\hat{H}(\omega_1, \omega_2) |H(\omega_1, \omega_2)|} \right|.$$

*Proof*: We can write

$$H(\omega_1, \omega_2) = M(\omega_1, \omega_2) e^{j\varphi(\omega_1, \omega_2)},$$

$$\hat{H}(\omega_1, \omega_2) = \hat{M}(\omega_1, \omega_2) e^{j\hat{\varphi}(\omega_1, \omega_2)}.$$

The approximation error in the phase can be written as

$$\begin{aligned} |\hat{\varphi}(\omega_1, \omega_2) - \varphi(\omega_1, \omega_2)| &= \left| \ln \left( \frac{H(\omega_1, \omega_2) \hat{M}(\omega_1, \omega_2)}{\hat{H}(\omega_1, \omega_2) M(\omega_1, \omega_2)} \right) \right| \\ &= |\ln(1 + \delta)|, \end{aligned}$$

where  $\delta$  is estimated as follows

$$\begin{aligned} |\delta| &= \left| 1 - \frac{H\hat{M}}{\hat{H}M} \right| = \left| \frac{\hat{H}M - \hat{H}\hat{M} + \hat{H}\hat{M} - H\hat{M}}{\hat{H}M} \right| \\ &\leq \frac{|\hat{H}(M - \hat{M})| + |\hat{M}(\hat{H} - H)|}{\hat{M}M} \\ &= \frac{\hat{M} \left( |\hat{H}| - |H| \right) + \hat{M} |\hat{H} - H|}{\hat{M}M} \\ &\leq \frac{2|\hat{H} - H|}{M}. \end{aligned}$$

Since the balanced realization method is used, the approximation error  $\hat{\epsilon} \ll 1$  becomes very small if the two terms  $\sum_{i=\hat{m}+1}^m \sigma_i^{(1)}$  and  $\sum_{j=\hat{n}+1}^n \sigma_j^{(4)}$  are also very small; in this case, the phase of the initial filter is preserved after reduction.

#### 4.4. Illustrative example

An example is given in order to show the efficiency of the proposed reduction method. We start with a low-pass 2D-FIR filter designed by the windowing method, and having a full-order  $(m, n) = (20, 20)$ ; this filter is separable, which allows us to represent it in the state-space as a locally controllable and observable canonical form. By applying the developed balanced realization method to the obtained model, this method allows to extract a reduced order model  $(\hat{m}, \hat{n}) = (12, 12)$ , based on the *Hankel-norm error* criterion. We

show the frequency response of the initial filter and of its reduced order approximant, as well as the error between them. Four measures are presented to demonstrate the efficiency of the reduction procedure; the first measure is the Euclidian norm of the error ( $\epsilon_2$ ), and the second is the error criterion in the  $l_\infty$  norm ( $\epsilon_\infty$ ). These measures are given in [12] as follows

$$\epsilon_\infty \triangleq \frac{\max_{(i,j)} |H_{ij} - \hat{H}_{ij}|}{\max_{(i,j)} |H_{ij}|}, \quad \text{and}$$

$$\epsilon_2 \triangleq \left[ \sum_i \sum_j (H_{ij} - \hat{H}_{ij})^2 \right]^{1/2} / \left[ \sum_i \sum_j (H_{ij})^2 \right]^{1/2}.$$

The other measures are the maximal error in the bandwidth region  $R_p$  and the maximal error in the cut-off region  $R_s$ , given as

$$Err(R_p) = \max_{(i,j) \in R_p} |H_{ij} - \hat{H}_{ij}|,$$

and

$$Err(R_s) = \max_{(i,j) \in R_s} |H_{ij} - \hat{H}_{ij}|.$$

The initial model is always stable and with linear phase. We show that the proposed reduction procedure preserves the stability and phase linearity of the reduced filter.

Since the reduced filter is separable, the stability test is very simple: a separable system is stable if  $\rho(\hat{A}_1) < 1$ ,  $\rho(\hat{A}_4) < 1$ , where  $\rho(\cdot)$  denotes the spectral radius of  $(\cdot)$ .

The linearity test in the 2D case follows an indirect procedure: the transfer-function  $\hat{H}(z, w)$  is evaluated in fixed points on the unit circle  $z_i = e^{j2\pi i/N}$  or  $w_i = e^{j2\pi i/M}$  for one of the two variables  $(z, w)$ , and the test becomes one for the one-dimensional systems  $\hat{H}_j(z)$  or  $\hat{H}_i(w)$ .

Figure 1 clearly shows that the frequency behavior of the initial filter and its reduced order approximant are very close, and the error is very small for all frequencies in the bandwidth. Figure 2 shows the graphs of the phase response of the reduced order (12, 12) filter, by evaluating the two-dimensional transfer-function  $\hat{H}(z, w)$  in three fixed points on the unit circle  $w_i = e^{j2\pi i/M}$ , where  $M = 64$  and  $i = \{10, 30, 64\}$ .

We see that the phase is always linear in the bandwidth (0-0.6), and this implies that the reduction procedure preserves the linearity of the phase of the initial filter.

The results shown in Table 1 illustrate the performance of the proposed reduction procedure. It can well be seen that the error is very small even if the chosen order of the reduced model is small compared to the order of the initial model; in addition, the stability is always guaranteed.

## 5. Conclusion

The basic idea is the synthesis of new digital filter using model order reduction. Modeling approach of 2D-FIR filters in state-space, based on the Givone-Roesser's model is used. They belong to the class of separable filters. Then, the model reduction based on balanced realizations is achieved based on two important parameters the Gramians.

For a balanced 2D system, it is possible to apply the criterion of the *Hankel-norm error* for retaining the dominant states and removing the states with low energy; the resulting system represents a simplified version of the initial system.

The following conclusions are drawn from the simulation results:

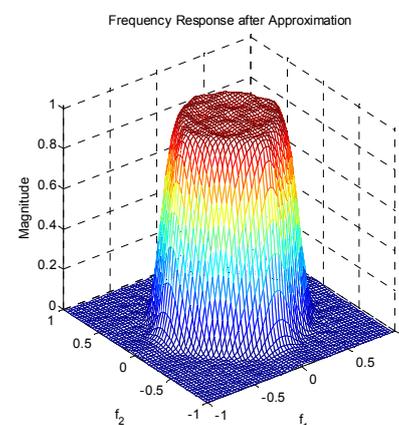
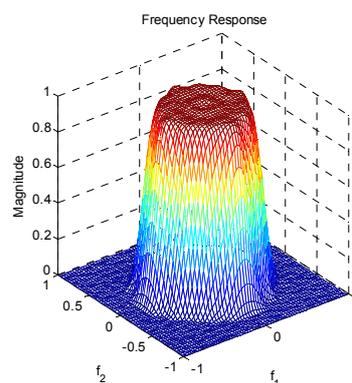
- The reduced order filter is very close to the full-order filter; this is verified by the error gap and other measures. The proposed reduction approach, based on the SVD criterion, presents the advantage of removing the low energy modes, thus offering a good approximation. and preserves stability of the reduced order filter. The initial filter phase is preserved after applying the proposed reduction technique.
- The only inconvenient of the balancing technique is, maybe, the high information complexity needed for computing the controllability and observability Gramians. The derivation of a canonical form before reduction is very important for minimizing this complexity; it is shown that the representation of the 2D-FIR filters in such a form is possible.

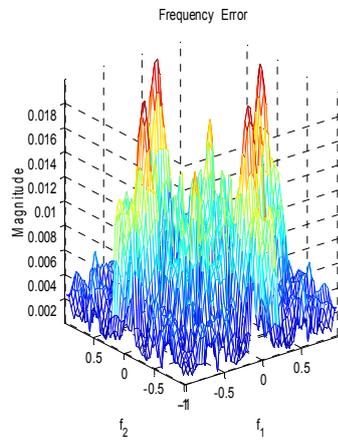
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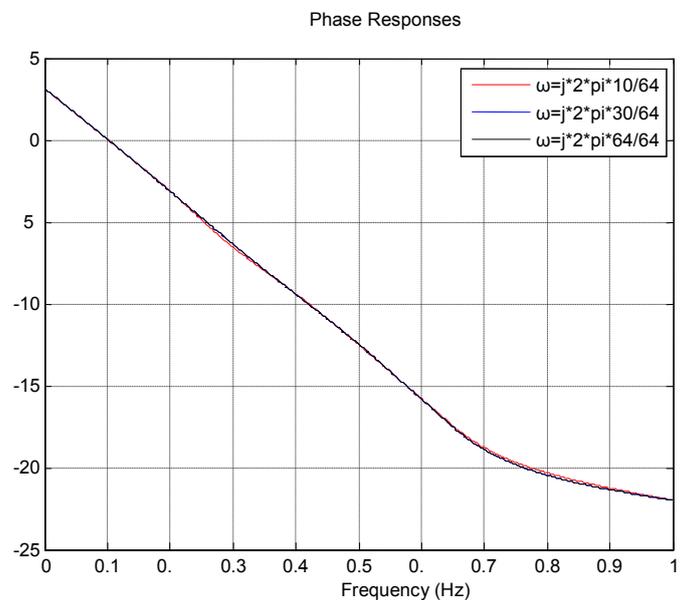




**Fig. 1** – Frequency responses of the original (20, 20) filter, the reduced order (12, 12) filter, and the error between them.

**Table 1** – Performance parameters for different reduced order models.

$(TOL_1, TOL_2)$	$(10^3, 10^3)$	$(5 \times 10^3, 5 \times 10^3)$	$(10^4, 10^4)$
$(\hat{m}, \hat{n})$	(10,10)	(11,11)	(12,12)
$Err(R_p)$	0.06151 3	0.0452 55	0.01346 1
$Err(R_s)$	0.07863 6	0.0415 86	0.01999 7
$\epsilon_s(\%)$	7.84	4.50	1.99
$\epsilon_2(\%)$	4.89	2.86	1.27
$\rho(\hat{A}_1)$	0.391	0.8370	0.8154
$\rho(\hat{A}_4)$	0.8391	0.8370	0.8154



**Fig. 2** – Phase response of the 1D filter after evaluation of the reduced order (12, 12) 2D filter in three fixed points on the unit circle  $\omega_1 = e^{j2\pi \cdot 10/64}$ ,  $\omega_2 = e^{j2\pi \cdot 30/64}$  and  $\omega_3 = e^{j2\pi \cdot 64/64}$ .