Discrete and Three-Parameter Models of Hydraulic Fracture Crack Growth

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Abstract: - An infinite elastic medium with a planar crack is considered. The crack is subjected to the pressure of fluid injected at a point on the crack surface. Description of the crack growth is based on the lubrication equation (balance of the injected fluid and the crack volume), the equation for crack opening caused by fluid pressure on the crack surface, the Poiseuille equation related local fluid flux with crack opening and pressure gradient, and the criterion of crack propagation of linear fracture mechanics. The crack growth is simulated by a discrete process consisting of three basic stages: increasing the crack volume for a constant crack size, jump to a new size defined by the fracture criterion, and filling the new crack configuration by the fluid. First, an isotropic medium with a penny-shaped crack is considered. Dependencies of the crack radius, opening, and pressure distributions on the crack surface on time, fluid viscosity, and fracture toughness of the medium are studied. It is shown that for small fluid viscosity and low injection rates, the pressure distribution can be approximated by a three-parameter model that simplifies substantially the numerical solution. Then, the three-parameter model is applied to the case of heterogeneous media; in this case, the crack shape may be non-circular in the process of hydraulic fracture. Examples of hydraulic fracture crack growth in layered media are presented.

Key-Words: - Fracture mechanics, hydraulic fracture, penny-shape crack, crack in heterogeneous media.

1 Introduction

For importance in gas and petroleum industry, the process of hydraulic fracture has been the object of intense theoretical and experimental studies for about sixty years. The number of publications dedicated to this problem is huge. Publications before 21-st century can be found in the books [1], [2]. More recent publications are mentioned, e.g., in [3], [4], [5]. Mathematically, the problem is reduced to a system of non-linear integro-differential equations in a region with moving boundary. Analytical solutions of this system do not exist even in the simplest cases, and only numerical methods are efficient. By application of conventional numerical methods, the original integro-differential equations are discretized with respect to time and space variables, and then, hydraulic fracture crack geometry should be reconstructed at each discrete time step. The principal unknown of the problem is the pressure distribution on the crack surface. It turns out that construction of this distribution is an ill-posed problem. Application of conventional numerical methods for solution of ill-posed problems can result substantial numerical errors, and only specific methods are efficient [6]. Because the ill-posed problem should be solved at each time step of the crack growth, the errors accumulate, and reliable solution can be lost. In addition to non-linearity and moving boundary, this is another difficulty in numerical solution of the hydraulic fracture problem.

In the present work, growth of a planar crack in an elastic medium is considered. First, the medium is assumed to be isotropic and homogeneous. In this case, the crack is penny-shape, and fluid is injected in the crack center with a positive injection rate. Description of the crack growth is based on the lubrication equation (balance of the injected fluid and the crack volume), the equation for crack opening in elastic media caused by fluid pressure distributed on the crack surface, the Poiseuille equation related local fluid flux with the crack opening, the pressure gradient, and the classical criterion of crack propagation of linear fracture mechanics. Time discretization of these equations is interpreted as an actual process that consists of three stages: growth of the crack volume for a constant crack radius, an instant crack jump to a new radius, and filling the new crack configuration by fluid. For the solution of the ill-posed problem of reconstruction of the pressure distribution at each time step of crack growth, a specific class of approximating functions is used. These positive, monotonically decreasing functions are appropriate for approximation of actual pressure distributions and allow one to solve the ill-posed problem with sufficient accuracy. It is shown that for fluids with small viscosity and low injection rates, the pressure distribution on the crack surface can be approximated by a three-parameter model. This model simplifies numerical solution of the problem. The model can be extended to the case of hydraulic fracture crack growth in heterogeneous media. Examples of application of the model to heterogeneous layered media with varying fracture toughness and elastic moduli are presented.

2 A Crack in Homogeneous Elastic Media Subjected to Fluid Injection

Consider an infinite isotropic homogeneous elastic medium containing an isolated penny shape crack. The crack is subjected to internal pressure caused by the fluid injected at the crack center with given positive injection rate $Q(t)$ (Fig.1).

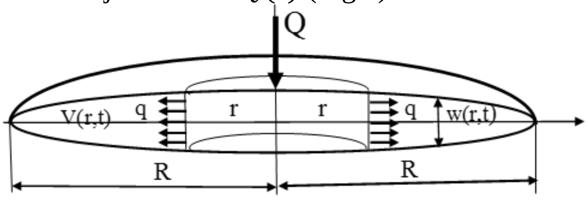

Fig.1

It follows from the symmetry of the problem that the growing crack remains circular with increasing radius $R(t)$. Crack opening $w(r,t)$ and pressure distribution $p(r,t)$ on the crack surface are functions of time t and the distance r from the crack center. We introduce fractional crack volume $v(r,t)$ by the equation

$$v(r,t) = 2\pi \int_r^{R(t)} w(x,t)x dx. \qquad (1)$$

Thus, $v(r,t)$ is the crack volume between the circle of radius r and the crack edge $r = R(t)$. Let $q(r,t)$ be the fluid flux in the radial direction through the crack cross-section with coordinate r. For incompressible fluids and impermeable media, the equation of balance of fractional volume $v(r,t)$ and the injected fluid (lubrication equation) has the form

$$\frac{\partial v}{\partial t} = 2\pi r q(r,t). \qquad (2)$$

The fluid flux $q(r,t)$, crack opening $w(r,t)$, and pressure $p(r,t)$ are related by the Poiseuille law [1]

$$q(r,t) = -\frac{w(r,t)^3}{12\eta}\frac{\partial p(r,t)}{\partial r}. \qquad (3)$$

It is assumed that the fluid is Newtonian with constant viscosity η. From equations (2) and (3) it follows that the lubrication equation can be written in the form

$$\frac{\partial v(\rho,t)}{\partial t} = -2\pi\rho \frac{w(\rho,t)^3}{12\eta}\frac{\partial p(\rho,t)}{\partial \rho}. \qquad (4)$$

Here dimensionless radial coordinate $\rho = r/R(t)$ is introduced. For an isotropic elastic medium and radially symmetric pressure distribution $p(\rho,t)$, crack opening $w(\rho,t)$ and fractional volume $v(\rho,t)$ of a penny-shape crack of radius R are defined by the equations [7], [8]

$$w(\rho,t) = \frac{R(t)}{\pi\mu'}\int_0^1 G(\rho,\varsigma)p(\varsigma,t)d\varsigma, \qquad (5)$$

$$v(\rho,t) = \frac{R(t)^3}{\pi\mu'}\int_0^1 K(\rho,\varsigma)p(\varsigma,t)d\varsigma. \qquad (6)$$

Here $\mu' = \frac{\mu}{4(1-\nu)}$, and μ, ν are shear modulus and Poisson ratio of the medium. The kernel $G(\xi,\varsigma)$ has the form

$$G(\xi,\varsigma) = \begin{cases} \frac{\varsigma}{\rho} F(\sin^{-1}(\kappa), \frac{\varsigma}{\xi}), & \varsigma < \xi \\ F(\sin^{-1}(\kappa^{-1}), \frac{\xi}{\varsigma}), & \varsigma > \xi, \end{cases} \qquad (7)$$

where

$$F(\phi,m) = \int_0^{\phi} \frac{d\theta}{\sqrt{1-(m\sin\theta)^2}}, \quad \kappa = \sqrt{\frac{1-\xi^2}{1-\varsigma^2}}. \qquad (8)$$

The kernel $K(\rho,\varsigma)$ is expressed in terms of $G(\xi,\varsigma)$

$$K(\rho,\varsigma) = 2\pi\int_\rho^1 G(\xi,\varsigma)\xi\, d\xi, \qquad (9)$$

and it is a smooth integrable function of the variables (ρ,ς). The integral operators with kernels $G(\rho,\varsigma)$ and $K(\rho,\varsigma)$ have the following remarkable properties. Actions of these operators on polynomial functions of ρ with even exponents

$$p(\rho) = a_0 + a_1\rho^2 + a_2\rho^4 + \ldots + a_n\rho^{2n} \qquad (10)$$

are polynomials of the same power $2n$ multiplied by $(1-\rho^2)^{\frac{1}{2}}$ (for the G-kernel) and $(1-\rho^2)^{\frac{3}{2}}$ (for the K-kernel). Coefficients of these polynomials are expressed in terms of the coefficients $a_0, a_1, .., a_n$ in equation (10) in explicit analytical forms [8].

Note that calculation of pressure distributions $p(\rho,t)$ from equation (6) with given left hand side $v(\rho,t)$ is in fact solution of Fredgholm integral equation of the first kind with integrable kernel $K(\rho,\varsigma)$. It is a well-known ill-posed problem [6]. For such problems, small deviations (errors) of the left hand sides $v(\rho, t)$ cause large errors in the pressure distribution $p(\rho, t)$.

For calculation of the crack radius in the hydraulic fracture process, the classical criterion of linear fracture mechanics is used. For radial pressure distribution $p(\rho)$, the stress intensity factor K_I for the fracture mode I at the crack edge is [7]

$$K_I(p, R) = \frac{\sqrt{2R}}{\pi} \int_0^1 \frac{p(\rho)\rho}{\sqrt{1-\rho^2}} d\rho, \qquad (11)$$

and the fracture criterion takes the form

$$K_I(p, R) = K_{Ic}, \qquad (12)$$

where K_{Ic} is the so-called fracture toughness. This specific physical parameter defines resistance of the medium to crack propagation.

Lubrication equation (4), equations (5) and (6), and fracture criterion (12) compose a complete system of equations for the growth of a penny-shape crack in a homogeneous isotropic elastic medium by fluid injection. A natural principal unknown of the problem is fluid pressure $p(\rho, t)$ on the crack surface. All other crack parameters (crack radius, crack opening, and fractional volume) are expressed in term of the pressure.

3 Discretization of the Equations of Hydraulic Fracture process

The system of equations of crack growth can only be solved numerically. Conventional numerical methods of solution of partial differential equations are based on (time and space) discretization procedure. The time discretization consists of taking a small finite time step Δt and changing the partial time derivative with the finite difference. As a result, the lubrication equation (4) can be presented in the form

$$v(\rho, t + \Delta t) \approx v(\rho, t) - \frac{2\pi w^3(\rho,t)}{12\eta} \rho \frac{\partial p(\rho,t)}{\partial \rho} \Delta t. \qquad (13)$$

If the right hand side of this equation is known at the moment t, one can calculate the function $v(\rho, t + \Delta t)$ at the moment $t = t + \Delta t$. The difficulty in carrying out this scheme is that the new crack radius at $t = t + \Delta t$ is unknown (the crack has a moving boundary). As the result, the variable ρ on the left hand side of (13) $\rho = r/R(t + \Delta t)$ differs from the similar variable on the right hand side $\rho = r/R(t)$. If the time step Δt is sufficiently small, one can accept that $R(t) \approx R(t + \Delta t)$, calculate fractional volume $v(\rho, t + \Delta t)$ from equation (13), obtain the new pressure distribution from equation (6), and then, find the new crack radius from the fracture criterion (12)

$$K_I\big(p(t + \Delta t), R(t + \Delta t)\big) = K_{Ic}, \qquad (14)$$

and then, go to the next time interval.

Formal discretization of equation (4) allows the following physical interpretation. Let at the moment t the crack radius be $R(t)$, crack volume $V(t) = v(0, t)$, and pressure distribution on the crack surface be $p(\rho, t)$. For such radius and pressure distribution, the stress intensity factor at the crack edge is $K_I(p, R) = K_{Ic}$. Suppose that the process of crack radius growth from $R(t)$ to $R(t + \Delta t)$ consists of three stages (Fig.2). First, during the time interval Δt, the fluid is injected inside the crack but the crack radius does not change. For incompressible fluid, balance of the injected fluid and increment of the crack volume (the lubrication equation (4)) should be satisfied, meanwhile fracture condition (12) is neglected. At the end of this stage, the crack volume increases (dashed line in Fig.2) from $V(t)$ to $V(t + \Delta t)$, and the pressure distribution is $p^+(\rho, t + \Delta t)$. For this pressure, the stress intensity factor at the crack edge is more than K_{Ic}. At this moment, the crack jumps instantly to the new radius $R(t + \Delta t)$ (second stage). Pressure on the crack surface changes, and we assume it is defined by the equation

$$p\left(\frac{r}{R(t+\Delta t)}, t + \Delta t\right) = \alpha p^+\left(\frac{r}{R(t+\Delta t)}, t + \Delta t\right), \qquad (15)$$

where coefficient α ($\alpha < 1$) is to be found from the fracture criterion (14). Because of an instant jump, the fluid inside the crack fills the new region near the crack edge but the total crack volume does not change (third stage). The new crack radius $R(t + \Delta t)$ and the coefficient α in equation (15) are to be found from the equations

$$v(r/R(t + \Delta t), t + \Delta t)|_{r=0} = V(t + \Delta t); \qquad (16)$$

$$K_I[p(r/R(t + \Delta t), t + \Delta t), R(t + \Delta t)] = K_{Ic}. \qquad (17)$$

Left hand sides of these equations are defined in (6) and (11).

At the end of the third stage, the crack volume $V(t + \Delta t)$ is filled with fluid, and the radius $R(t + \Delta t)$ and pressure $p(r, t + \Delta t)$ must satisfy the fracture criterion (17). The total time ΔT of these three stages can be calculated from the equation

$$\Delta T = \frac{V(t+\Delta t) - V(t)}{Q(t)}. \qquad (18)$$

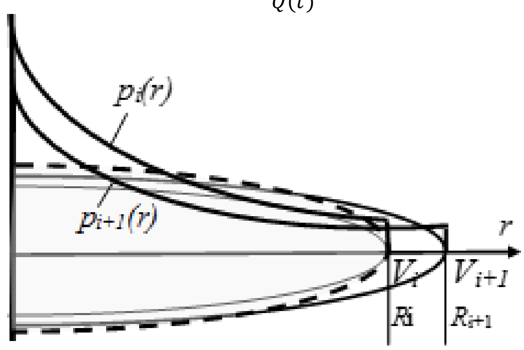

Fig.2

4 Approximation of the pressure distribution and solution of ill-posed problem (6)

For positive injection rate $Q(t)$, the pressure $p(\rho, t)$ is a continuous function of variable ρ that monotonically decreases from the injection point to the crack edge. In addition, at the crack center, the following equation holds

$$\frac{\partial v}{\partial t}|_{r=0} = -2\pi \frac{w^3(0,t)}{12\eta} \lim_{\rho\to 0} \rho \frac{\partial p}{\partial \rho} = Q(t). \quad (19)$$

Because crack opening at the center $w(0, t)$ is finite, the limit in this equation should also be finite. It means that the function $p(\rho, t)$ has logarithmic asymptotics at the crack center. Therefore, the function $p(\rho, t)$ can be approximated by the following series

$$p(r, t) = -p_0(t) ln \frac{r}{R(t)} + \sum_{n=1}^{N} p_n(t)\varphi_n\left(\frac{r}{R(t)}\right), (20)$$

where $p_n(t) \geq 0$ $(n = 0,1,2, \dots, N)$, and $\varphi_n(\rho)$ are monotonically decreasing functions with finite derivatives. For instance, the following ten functions $\varphi_n(\rho)$ can be used for approximation of the pressure distribution

$$\varphi_1 = 1, \varphi_2 = 1 - \rho^{10}, \varphi_3 = 1 - \rho^4,$$
$$\varphi_4 = 1 - \rho^2, \varphi_5 = (1 - \rho^2)^2, \varphi_6 = (1 - \rho^2)^3,$$
$$\varphi_7 = (1 - \rho^2)^4, \varphi_8 = (1 - \rho^2)^8, \varphi_9 = (1 - \rho^2)^{15},$$
$$\varphi_{10} = (1 - \rho^2)^{40}. \quad (21)$$

The graphs of these functions are presented in Fig.3.

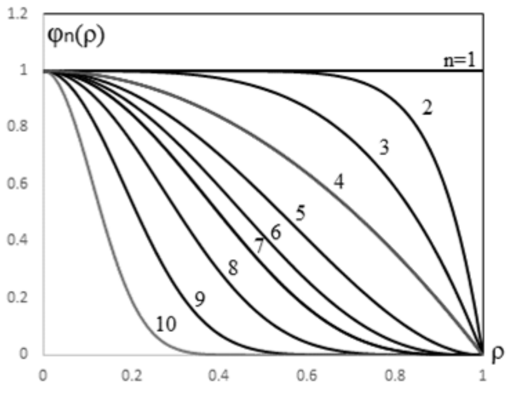

Fig.3

Because all these functions are polynomials similar to (10), action of the operators G and K in equations (5) and (6) on these functions can be presented in explicit analytical forms. The crack opening $w(\rho, t)$ and fractional volume $v(\rho, t)$ take forms

$$w(\rho, t) = \frac{R(t)}{\pi\mu'} \sum_{n=0}^{N} p_n(t) w_n(\rho), \quad (22)$$

$$v(\rho, t) = \frac{R(t)^3}{\pi\mu'} \sum_{n=0}^{N} p_n(t) v_n(\rho), \quad (23)$$

where

$$w_n(\rho) = \int_0^1 G(\rho, \varsigma)\varphi_n(\varsigma) d\varsigma, \quad (24)$$

$$v_n(\rho) = \int_0^1 K(\rho, \varsigma)\varphi_n(\varsigma) d\varsigma, \quad (25)$$

and $\varphi_0(\rho) = -\ln \rho$. For $\varphi_n(\rho)$ in (21), the functions $w_n(\rho)$ and $v_n(\rho)$ can be found in explicit analytical forms [8].

The stress intensity factor $K_I(t)$ at the crack edge is

$$K_I(t) = \frac{\sqrt{2R(t)}}{\pi} \sum_{n=0}^{N} p_n(t) k_n, \quad (26)$$

$$k_n = \int_0^1 \frac{\varphi_n(\rho)}{\sqrt{1-\rho^2}} d\rho. \quad (27)$$

If we denote the right hand side of equation (13) as $rhs(\rho, t)$

$$rhs(\rho, t) = v(\rho, t) - \frac{2\pi w^3(\rho,t)}{12\eta} \rho \frac{\partial p(\rho,t)}{\partial \rho} \Delta t, \quad (28)$$

and take into account equation (6), the lubrication equation (13) can be rewritten in the form

$$\frac{R(t)^3}{\pi\mu'} \int{}_0{}^1 K(\rho, \varsigma) p^+(\varsigma, t + \Delta t) d\varsigma = rhs(\rho, t). \quad (29)$$

The function $p^+(\rho, t + \Delta t)$ is approximated by series similar to (20)

$$p^+(\rho, t + \Delta t) =$$
$$-p_0^+(t + \Delta t) \ln(\rho) + \sum_{n=1}^{N} p_n^+(t + \Delta t)\varphi_n(\rho). \quad (30)$$

Substituting this approximation in (29) and satisfying the resulting equation at M points ρ_k (nodes) homogeneously distributed along the crack radius

$$\rho_k = (k/M) \quad (k = 0,1,2, \dots, M)$$

we obtain the following system of linear algebraic equations for the coefficients $p_n^+(t + \Delta t)$

$$\sum_{n=0}^{N} S_{(k,n)} p_n^+(t + \Delta t) = rhs(\rho_k, t), \quad (31)$$

$$S_{(k,n)} = \frac{4(1-\nu)}{\pi\mu} R(t)^3 \int_0^1 K(\rho_k, \varsigma) \varphi_n(\varsigma) d\varsigma, \quad (32)$$
$$k = 0,1,2, \dots, M.$$

This system can be presented in the following matrix form

$$\mathbf{S} \cdot \mathbf{X} = \mathbf{RHS}, \quad \mathbf{X} = \{p_0{}^+, p_1{}^+, \dots, p_N{}^+\}^T. \quad (33)$$

Here T is the transposition operator. According to the method of solution of ill-posed problems [6], the vector $\mathbf{X}$ can be found from the equation

$$\min_Y \|\mathbf{S} \cdot \mathbf{Y} - \mathbf{RHS}\| = \|\mathbf{S} \cdot \mathbf{X} - \mathbf{RHS}\|, \quad (34)$$

$$\|\boldsymbol{Y}\| = \sum_{n=1}^{N} Y_n^2. \quad (35)$$

The minimum in equation (34) is to be found on vectors $\mathbf{Y}$ with positive components

$$Y_1 \geq 0, Y_2 \geq 0, \dots, Y_{N+1} \geq 0. \quad (36)$$

The matrix $\mathbf{S}$ in equation (32) can be non-square: the numbers of the approximating functions and the nodes on the crack radius can be different. In order to find the minimum in equation (34) with restrictions (36) standard methods of linear programming can be used.

According to the discrete model, at the first stage of the *(i+1)*th step of crack growth, the crack radius remains fixed $R = R(t_i)$, and pressure distribution $p^+(\rho, t_i + \Delta t)$ is to be constructed from equations (30), (34) and (36). The appropriate value of the time interval Δt in equations (28)-(31) should be taken in such a way that the relative error δ of the solution of equation (33)

$$\delta = \frac{\|\boldsymbol{S}\cdot\boldsymbol{X}-\boldsymbol{RHS}\|}{\|\boldsymbol{RHS}\|} \qquad (37)$$

does not exceed a prescribed tolerance (in the calculations, $\delta<0.01$ was taken). Then, the new crack radius $R(t_i+\Delta t)$ and pressure distribution are calculated from equations (16) and (17).

Example of evolution of the pressure distributions on the crack surface in the process of hydro fracture is shown in Fig.4-6 for the fluid with viscosity $\eta = 0.01 Pa\cdot sec$, $0.1 Pa\cdot sec$, $1 Pa\cdot sec$, $\mu = 6.25 GPa$, $\nu = 0.2$ the material fracture toughness $K_{Ic} = 1.6 MPa\sqrt{m}$, $Q = 0.1\frac{m^3}{sec}$.

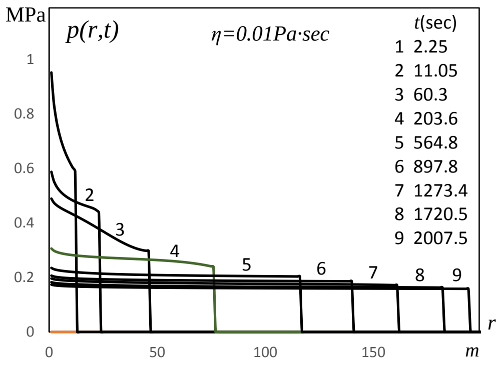

Fig.4

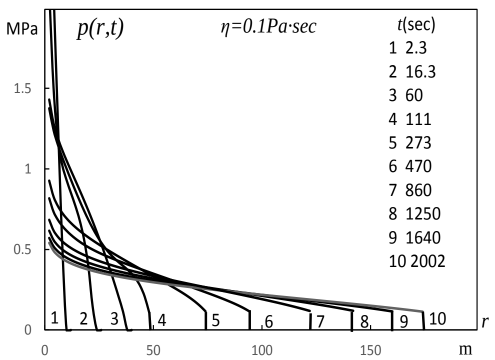

Fig.5

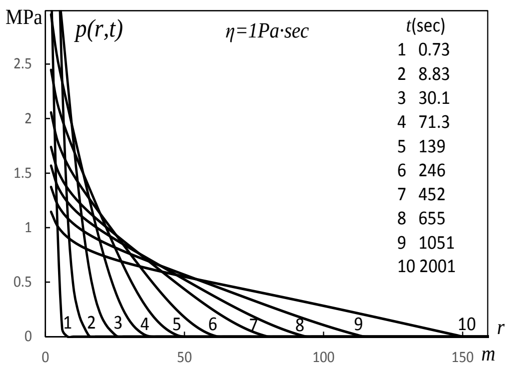

Fig.6

Dependences of the crack radius R on time and fracture toughness K_{Ic} are shown in Fig.7 ($\eta = 0.1 Pa\cdot sec$), and on time and fluid viscosity η in Fig.8 ($K_{Ic} = 1 MPa\sqrt{m}$). Dependence of crack opening *w(r,t)* on time is shown in Fig.9 for $\eta = 0.01 Pa\cdot sec$, and in Fig.10 for $\eta = 1 Pa\cdot sec$, Q=0.1m^3/sec, $\mu = 6.25 GPa$, $\nu = 0.2$, $K_{Ic} = 1 MPa\sqrt{m}$.

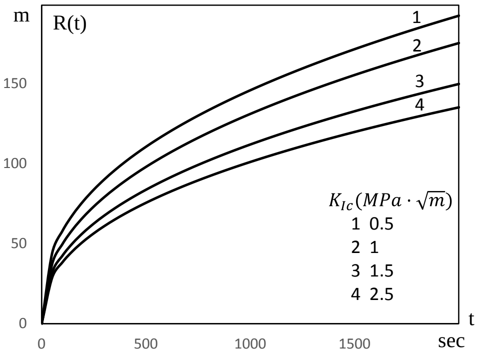

Fig.7

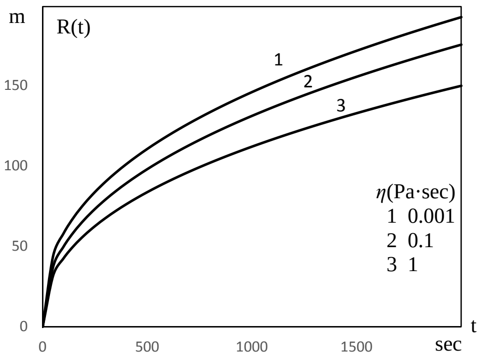

Fig.8

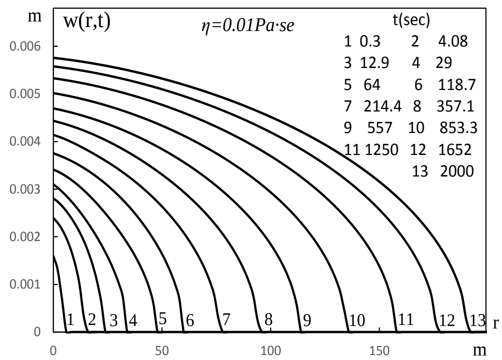

Fig.9

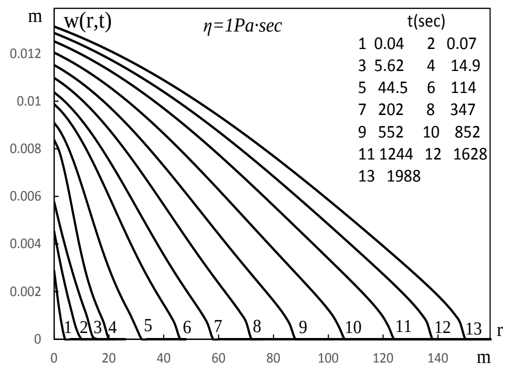

Fig.10

5 Three-parameter model of pressure distribution

Dependences of the coefficients $p_n(t)$ in equation (21) for pressure distribution $p(\rho,t)$ on time are presented in Figs.11-14 for fluid viscosity $\eta = 0.001,\ 0.01,\ 0.1,\ 1 Pa \cdot sec$, ($\mu = 6.25 GPa$, $\nu = 0.2$, $K_{Ic} = 1 MPa\sqrt{m}$, $Q = 0.1 m^3/sec$).

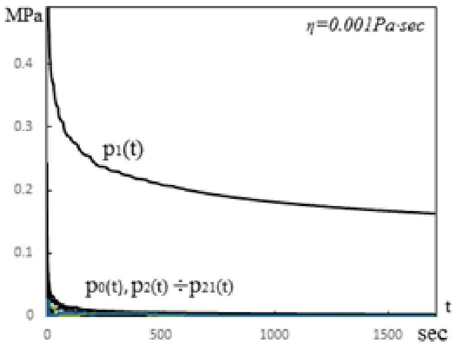

Fig.11

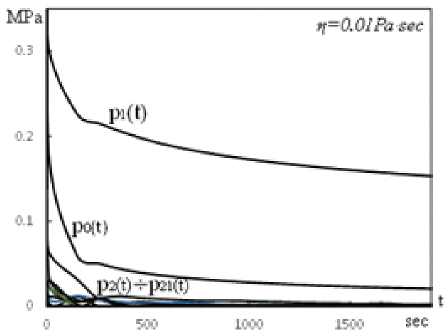

Fig.12

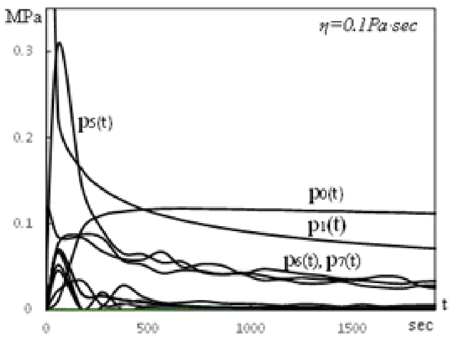

Fig.13

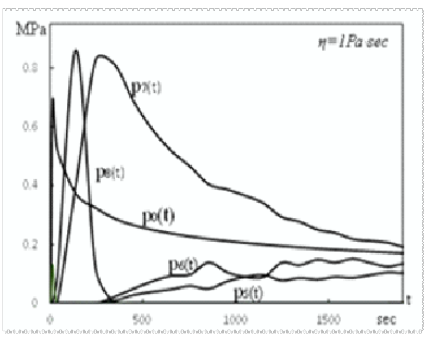

Fig.14

It is seen from these figures that for small viscosity ($\eta < 0.01 Pa \cdot sec$), the coefficients $p_0(t)$ and $p_1(t)$ in series (21) dominate. For larger viscosity, these coefficients become comparable with the others in equation (21). Thus in the region of small viscosity, the pressure distribution can be approximated by the two first terms in series (21)

$$p(r,t) = -p_0(t) ln(\frac{r}{R(t)}) + p_1(t). \qquad (38)$$

This equation contains three unknown functions of time $p_0(t)$, $p_1(t)$, and $R(t)$. For the pressure distribution (38), the total crack volume $V(t) = v(0,t)$, crack opening at the center $w(0,t)$, and the stress intensity factor at the crack edge $K_I(t)$ are

$$V(t) = \frac{2R(t)^3}{9\mu'}[(4 - 3ln2)p_0(t) + 3p_1(t)], \quad (39)$$

$$w(0,t) = \frac{R(t)}{\pi\mu'}[(2 - ln2)p_0(t) + p_1(t)], \quad (40)$$

$$K_I(t) = \frac{\sqrt{2R(t)}}{\pi}[(1 - ln2)p_0(t) + p_1(t)]. \quad (41)$$

The three functions $p_0(t)$, $p_1(t)$, $R(t)$ can be found from the condition of equivalence of crack volume $V(t)$ and the total volume of injected fluid

$$V(t) = \int_0^t Q(\tau)d\tau, \qquad (42)$$

the fracture criterion (14), and equation (19) for injection rate $Q(t)$ at the crack center

$$K_I(t) = K_{Ic}, \qquad (43)$$

$$p_0(t) = \frac{12\eta Q(t)}{2\pi w(0,t)^3}. \qquad (44)$$

The system of equations (39)-(44) can be solved numerically.

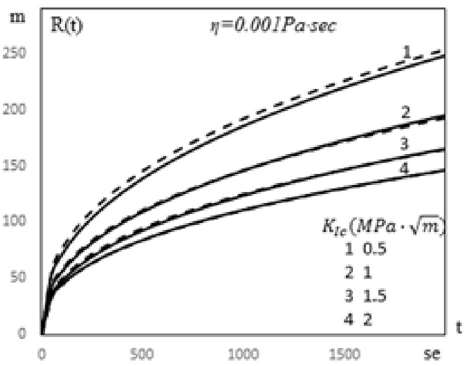

Fig. 15

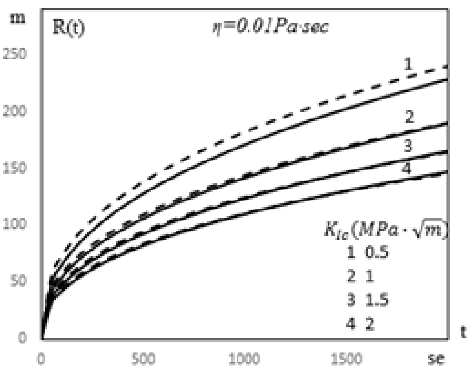

Fig.16

In Figs.15-17, time dependencies of crack radius $R(t)$ for the three-parameter model and the discrete model of crack growth are shown for fluid viscosity $\eta = 0.001$, 0.01, $0.1 Pa \cdot sec$ and various values of fracture toughness K_{Ic}. In these figures, solutions of the system (39)-(44) are dashed lines, and the results of the discrete model are solid lines. (Note that viscosity of see water widely used for hydraulic fracture is about $0.001 Pa \cdot sec$). It is seen from these figures that the three-parameter model corresponds better to the discrete model in the case of higher values of the fracture toughness and lower values of fluid viscosity.

In Fig.17, the influence of shear modulus μ of the medium on the time dependence of the crack radius in the process of crack growth is shown. ($\nu = 0.2$, $K_{Ic} = 1 MPa\sqrt{m}$, $\eta = 0.001 Pa \cdot sec$, fluid injection rate is $Q = 0.1 m^3/sec$). In this figure, solid lines correspond to the discrete model, and dashed lines to the three-parameter model. It is seen that the models give close predictions for all the cases.

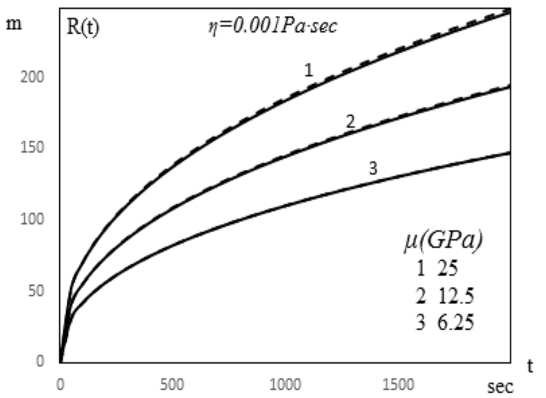

Fig.17

6 Application of the three-parameter model to hydraulic fracture crack grows in heterogeneous media

Consider an isotropic medium with varying in space elastic moduli tensor $C(x)$ and fracture toughness $K_{Ic}(x)$, x is a point of the medium. In this case, the crack shape can be non-circular in the process of hydraulic fracture. For small fluid viscosity and low injection rates, the three-parameter model of pressure distribution can be used. Let the pressure be approximated by the equation similar to (39)

$$p(x,t) = -p_0(t) \ln \frac{r}{R_*(t)} + p_1(t), x \in \Omega. \quad (45)$$

where $r = |x - x^0|$ is the distance from point x on the crack surface Ω to the injection point x^0, $R_*(t)$ is the maximum distance from x^0 to the crack boundary. It follows from the condition at the injection point x^0 that the coefficient $p_0(t)$ in equation (45) is similar to (44)

$$p_0(t) = \frac{12\eta Q(t)}{2\pi w(x^0,t)^3}. \quad (46)$$

If $w(x,t)$ is crack opening, the total crack volume $V(t)$ is calculated as follows

$$V(t) = \int_\Omega w(x,t) d\Omega. \quad (47)$$

For an elastic medium with varying fracture toughness only, crack opening $w(x,t)$ and pressure distribution $p(x,t)$ are related by the equation

$$\int_\Omega T(x - x')w(x',t)d\Omega' = p(x,t),\ x \in \Omega, \quad (48)$$

where the kernel $T(x)$ is a generalized function which regular part has the form [9]

$$T(x) = \frac{\mu'}{\pi |x|^3}. \quad (49)$$

In the case of a medium with varying fracture toughness and elastic moduli, we consider a finite volume W of the heterogeneous medium containing the crack. The sizes of W should be taken such as the stresses induced by the fluid pressure applied to the crack surface practically vanish. outside W.

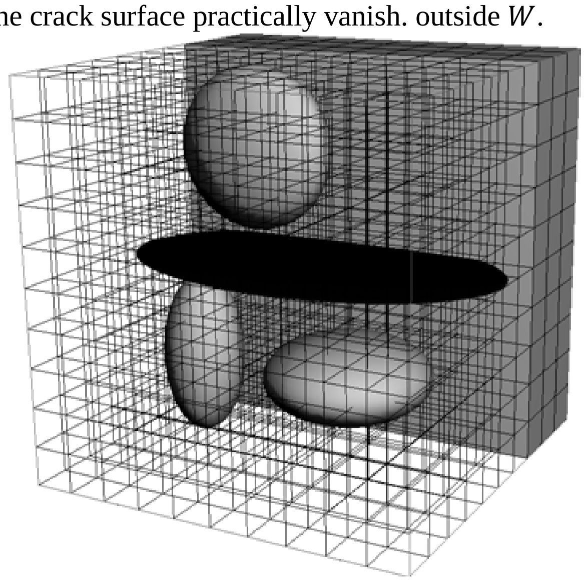

Fig. 18

Let us chose a homogeneous medium with stiffness tensor C_0 and fracture toughness K_{Ic0} as a reference (host) medium. The system of equations for actual stress tensor $\sigma(x)$ in W and crack opening $w(x)$ takes the form (see [10] for details)

$$\sigma(x) = p(x) + \int_W S(x - x')B_1(x')\sigma(x')dx' + \int_\Omega S(x - x')nw(x')d\Omega',\ x \in W \,\&\, x \notin \Omega, \quad (50)$$

$$n(x)\int_\Omega S(x - x')B_1(x')\sigma(x')dx' + \int_\Omega T(x - x')w(x')d\Omega' = p(x),\ x \in \Omega, \quad (51)$$

$$B_1(x) = B(x) - B_0. \quad (52)$$

Here n is the normal vector to crack plane Ω, $B(x) = C(x)^{-1}$ and $B_0 = C_0^{-1}$ are elastic compliance tensors of heterogeneities and host media, and

$$S(x) = -C_0 \nabla\nabla G(x) C_0 - C_0 \delta(x). \qquad (53)$$

Here $G(x)$ is Green function of the host medium C_0, ∇ is gradient operator, $\delta(x)$ is Dirac's delta-function, $T(x) = nS(x)n$. The system (50) and (51) can be solved only numerically. For the numerical solution, the integral equations should be discretized by using an appropriate class of approximating functions. In [10] a class of Gaussian functions concentrated at the nodes of a regular node grid was used for this purpose. The node grid should cover region W and crack surface Ω (Fig.18). The unknown functions (stress tensor $\sigma(x)$ in W and crack opening $w(x)$ on Ω) are approximated by sums of Gaussian functions with coefficients that are the values of the functions at the nodes. The theory of approximation by Gaussian functions was developed in [11]. The discretized system is obtained after substitution of the approximations in equations (50), (51) and satisfaction of the resulting equations at the nodes (collocation method). Advantage of the Gaussian functions is that action of the integral operators in equations (50), (51) on this functions are presented in analytical forms or as combinations of standard integrals that can be tabulated [10]. The discretized problem is a system of linear algebraic equations for the values of stress tensor $\sigma(x^{(s)})$ and crack opening $w(x^{(s)})$ at the nodes. The principal problem in solution of the discretized problem is that for sufficient accuracy, the dimensions of the matrix of the system should be large. However, for regular node grids, this matrix has Toeplitz's structure, and as the result, fast Fourier transform technique can be used for iterative solution of the discretized problem (see details in [9], [10]).

The stress intensity factor (SIF) at the crack edge $K_I(x,t)$ is calculated from the asymptotic value of the crack opening near the crack edge [10], [12]. Let in the polar coordinate system (r,ϕ) with the center at the point of injection x^0 the crack boundary Γ be defined by the equation $r = R(\phi,t)$. Function $R(\phi,t)$ and the coefficients $p_0(t)$, $p_1(t)$ in equation (47) can be found from the following system: integral equations (50), (51), balance of the crack volume $V(t)$ and the volume of injected fluid (47), condition (46) at the point of injection fracture, and the fracture criterion at the crack contour Γ similar to (43)

$$K_I(x) = K_{Ic}(x),\ x \in \Gamma, \qquad (54)$$

Construction of the crack contour and crack opening in the process of hydraulic fracture is based on fast solution of integral equations (50), (51) and calculation of stress intensity factors on the crack edge for a crack of arbitrary shape by the method proposed in [10], [12]. First, we introduce a reference homogeneous medium with constant fracture toughness K_{Ic0} and elastic moduli C_0 and solve the problem for given injection rate $Q(t)$ at discrete time moments $t_1, t_2, ..., t_M$. For the three-parameter model, these solutions determine the coefficients $p_0(t_k)$ and $p_1(t_k)$ in equation (45) and the crack contours $\Gamma(t_k)$ that are circles of radii $R(t_k)$. These values of R, p_0, and p_1 are initial data for iterative construction of the crack contour and pressure distribution at the time moment t_k. In order to satisfy the fracture criterion (54) we define a set of discrete points with polar coordinates $\{r_k, \phi_k\}$, $r_k = R^{(0)}(\phi_k)$, $\phi_k = 2\pi(k/N)$, $k = 1,2,...,N$ on the crack contour $\Gamma(t_1)$. Then, the crack problem is to be solved for the known crack contour $\Gamma_0(t_1)$ and pressure coefficients $p_0(t_1)$ and $p_1(t_1)$. As a result, the values $K_I(\phi)$ at the points $\phi = \phi_k$ are calculated. Then, the distances $R(\phi)$ at $\phi = \phi_k$ ($k = 1,2,...,N$) are changed iteratively according to the following equations (n is the number of the iteration, $n = 0,1,2,...$)

$R^{(n+1)}(\phi_k)=(1-\delta)R^{(n)}(\phi_k)$ if
$\min K_I(\phi_k)>\min K_{Ic}(\phi_k)$ & $\max K_I(\phi_k)>\max K_{Ic}(\phi_k)$;
$R^{(n+1)}(\phi_k)=(1+\delta)R^{(n)}(\phi_k)$ if
$\min K_I(\phi_k)<\min K_{Ic}(\phi_k)$&$\max K_I(\phi_k)<\max K_{Ic}(\phi_k)$;
$R^{(n+1)}(\phi_k)=R^{(n)}(\phi_k)+\delta_K(K_I(\phi_k)-K_{Ic}(\phi_k))$ if
$\min K_I(\phi_k)>\min K_{Ic}(\phi_k)$&$\max K_I(\phi_k)<\max K_{Ic}(\phi_k)$;
$R^{(n+1)}(\phi_k)=R^{(n)}(\phi_k)-\delta_K(K_I(\phi_k)-K_{Ic}(\phi_k))$ if
$\min K_I(\phi_k)<\min K_{Ic}(\phi_k)$&$\max K_I(\phi_k)>\max K_{Ic}(\phi_k)$.

Here the parameters $\delta > 0$ and $\delta_K > 0$ are taken for the fastest convergence of the iteration process, $R^{(0)}(\phi_k) = R^{(0)}(t_1)$. Thus, if K_I at the point ϕ_k of the crack contour is smaller than K_{Ic}, radius $R(\phi_k)$ increases proportionally to the difference $K_I(\phi_k) - K_{Ic}(\phi_k)$. If $K_I(\phi_k) < K_{Ic}(\phi_k)$, radius $R(\phi_k)$ decreases proportionally to the same difference. The process stops at the nth iteration if

$$\Delta = \frac{\sum_k |K_I^{(n)}(\phi_k) - K_{Ic}(\phi_k)|}{\sum_k K_{Ic}(\phi_k)} < \Delta_0. \qquad (55)$$

The tolerance of $\Delta_0 = 0.05$ is taken in the calculations. Here $K_I^{(n)}(\phi_k)$ is SIF at $\phi = \phi_k$ in the nth iteration. Then, keeping $p_1(t_1)$ unchanged, we correct the value of $p_0(t_1)$ in order to satisfy equation (46) at the point of injection. It requires an additional iteration procedure with respect to the coefficient p_0. This iteration process is based on the conditions that p_0 is changed to $0.9p_0$ if $p_0 > 12\eta Q/(2\pi w^3(x^0))$ and to $1.1p_0$ if $p_0 < 12\eta Q/(2\pi w^3(x^0))$. Then, we can calculate crack opening $w_1(r)$ and crack volume V_1 for the calculated pressure coefficients p_0, p_1 and crack boundary $R(\phi)$. The corrected time moment T_1 that corresponds to this crack state is calculated from the equation

$$\int_0^{T_1} Q(\tau)d\tau = \mathrm{V}_1. \tag{56}$$

Then, we proceed to the second moment t_2, etc.

Let a layer $|x_2| < 20$m with fracture toughness $K_{Ic0} = 1$MPa·√m and Young modulus $E_0 = 15$GPa be embedded in the medium with fracture toughness $K_{Ic} = 1.5$MPa·√m and Young modulus $E_1 = 15$GPa (solid lines in Figs.21 and 22) or 30GPa (dashed lines), Poisson ratio of the both media are constant ν=0.2, injection rate is $Q = 0.2$m³/sec, and fluid viscosity is $\eta = 0.01$Pa·sec. The initial crack is penny-shape with radius $R^{(0)} = 1$m, and fluid is injected in the crack center at the point $x_1^0 = x_2^0 = 0$. The crack shapes at various time moments are shown in Fig. 18.

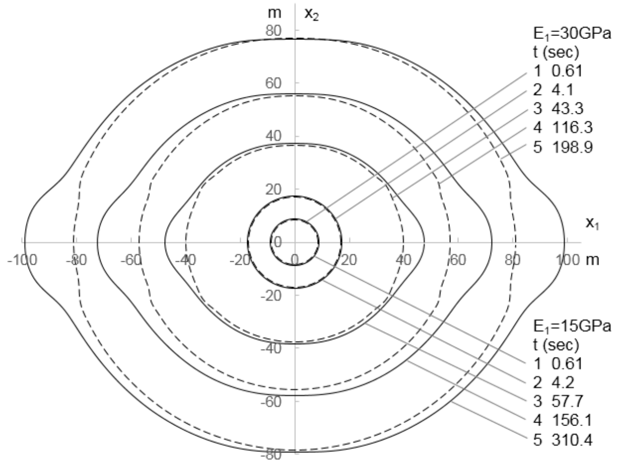

Fig.18

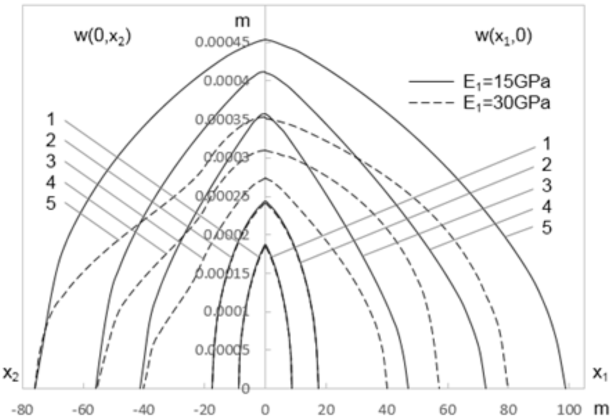

Fig.19

The crack openings along the x_1 and x_2-axes are shown in Fig.19.

In the second example, we consider the medium that consists of two half-spaces with the boundary at $x_2 = 20$m. For $x_2 < 20$m, the Young modulus of the medium is $E_0 = 15$GPa and fracture toughness is $K_{Ic0} = 1$MPa·m$^{1/2}$, for $x_2 > 20$m, $E_1 = 15$GPa (solid lines) or $E_1 = 30$GPa (dashed lines) and $K_{Ic} = 1.5$MPa·m$^{1/2}$. The initial crack is penny-shape with radius $R = 1$m, fluid with viscosity 0.01Pa·sec is injected in the crack center $x_1^0 = x_2^0 = 0$ with rate $Q = 0.2$m^3/sec. The crack shape at various time moments is shown in Fig. 20; the crack opening along the x_2-axis is in Fig.21. It can be noted that for the medium with constant Young modulus $E = 15$GPa and varying fracture toughness, the crack shape is substantially different from the penny-shape when the crack intersects the boundary between the two media. If Young moduli of two media are 15GPa and 30GPa, the crack shape is closer to penny-shape in the process of growing.

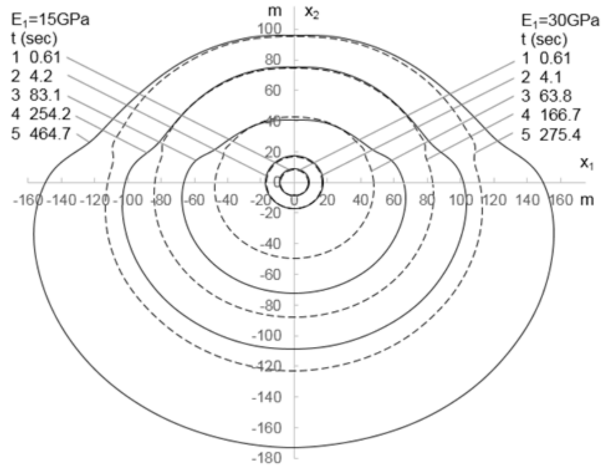

Fig.20

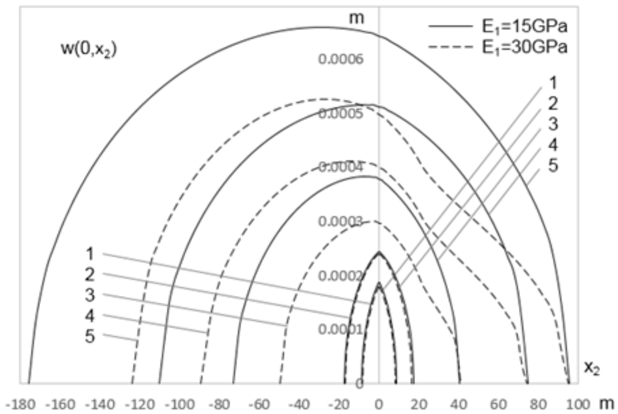

Fig.21

8 Conclusion

Efficient numerical methods for solution of the problem of hydraulic fracture crack growth in homogeneous and heterogeneous isotropic elastic media are proposed.

In the case of homogeneous media, when the crack is penny shape, the numerical algorithm is based on a specific class of approximating functions (21) used for approximation of the pressure distribution on the crack surface. These functions allow excluding numerical integration and differentiation that are sources of numerical errors and solving efficiently the ill-posed problem of reconstruction of pressure distribution in each time step of the crack growth. The proposed discrete model of crack growth can be considered as a physical interpretation of the formal procedure of discretization of the lubrication equation (4). This model results a specific numerical algorithm different from existing in the literature.

For fluids with small viscosity ($\eta < 0.01$Pa·sec) and low injection rates, the pressure distribution on the crack surface can be approximated by the three-parameter model (45). In this case, the problem can be simplified: one can neglect the detailed lubrication equation and find the crack size and pressure distribution from equivalence of the volumes of the crack and the injected fluid, fracture criterion (54), and condition (46) at the point of injection. The three-parameter model can be extended to the case of heterogeneous media. In this case, solution of integral equations for crack opening (50), (51) becomes a complex problem that required a specific numerical method. This method based on Gaussian approximating functions and fast Fourier transform algorithm was developed in [9], [10]. The method allows predicting evolution of the crack boundary in the process of hydraulic fracture crack growth in heterogeneous media.

Note that external stresses (lithographic pressure) that usually act in actual rocks and fluid filtration in the medium were neglected here for simplicity. Accounting these factors in the framework of the proposed numerical algorithm is straightforward.

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