Features and Computational Differences of Critical Points on The Equilibrium Curve

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Abstract: - In this paper, we discuss the features of investigating the type of the critical point on the equilibrium curve (bifurcation point or limit point). There are new ideas about the completion of the initial equilibrium at the limit point or bifurcation point. Examples as the Mises's truss, a shallow arch and a shallow cylindrical panel show the features of the birth and movement of bifurcation points along the curve of equilibrium.

Key-Words: -bifurcation, critical point, limit point, loss of stability, nonlinear system.

1 Introduction

The structure instability is an important and long-standing part of mechanics. The instability phenomenon in most cases means overall failure of a structure that is why it draws a lot of attention.

There is a difference between limit and bifurcation points.The indispensable and sufficient condition of instability will be discussed.

2 Stability of the Equilibrium of Elastic Systems

It is known that the coordinates of equilibrium points of mechanical and elastic systems are the real roots of a system of geometrically nonlinear equations

$$F_j(q_1, q_2, ..., q_n, P, e) = 0,\ j = 1,2,...,n. \quad (1)$$

Here $q_1, q_2, ..., q_n$ - the coordinates of the system in the *n* - dimensional space, *P* and *e* - the parameters of the load and the initial imperfection. Every j – equilibrium equation defines a certain hypersurface. The coordinates of the points of simultaneous intersection of all hypersurfaces are the coordinates of the equilibrium states of the elastic or mechanical system. These intersections can be points, lines or surfaces, forming zero-dimensional, one-dimensional or two-dimensional sets. In the case of a zero-dimensional and nowhere dense equilibrium set, the elastic systems are isolated from each other. If these equilibria are simple (not multiple), then they are isolated and nondegenerate (the Jacobi matrix of system (1) is nonsingular). Continuous sets of n-dimensional equilibria are sets of uninsulated and degenerate (or "indifferent") equilibria.

For conservative systems, equilibrium points are projections of extremum points or stationary points of the total potential energy of the system E ($q_1, q_2, ..., q_{n_n}$). If these points are nondegenerate (the first variation $\delta E = 0$, and the Hessian matrix $\{\partial^2 E / \partial q_i \partial q_k\}$ and the Jacobi matrix $\{\partial F_j / \partial q_k\} j,k = 1,2,...n$ are non-singular), then Sard's theorem implies that these projections form a set of measure zero. In this case all the equilibria of the conservative system are isolated, nondegenerate, and fairly "rare" in the configuration space. According to A.M. Lyapunov and M. Mors [1], near an isolated and non-degenerate equilibrium the total potential energy of an elastic system is strongly equivalent to the quadratic form

$$E(q_1, q_2, ..., q_n) \approx \sum_i \lambda_i q_i^2 + E_0. \quad (2)$$

If all the "stability coefficients" λ_i are strictly positive, then the equilibrium is stable ($E_0 = \min E$, the Lagrange-Dirichlet theorem). If among these coefficients there is at least one negative, then the isolated equilibrium is unstable. A special case appears when all the stability coefficients (except the last) are positive and the last coefficient is zero. This situation corresponds to a critical equilibrium in which the Hessian matrix (or the Jacobi matrix) becomes singular ($\det H_E = \det J = 0$), and the equilibrium becomes a multiple, degenerate, but externally isolated. However, the most important thing is that at a critical point (load $P = P_{cr}$), or arbitrarily close to it (load $P = P_{cr} \pm \delta P$), the previously stable subcritical equilibrium becomes unstable.

The stability margin of the isolated equilibrium of a conservative system at a fixed load is determined by the magnitude of the kinematic perturbation that can overcome the potential barrier – the height of the potential well in the direction of the perturbation. The minimum potential barrier is always given by the height of the so-called criterial saddle - unstable equilibrium, which is located on the crest, separating the two potential wells of neighboring stable equilibria and having the lowest "height".

Let us draw a line of the potential energy level $E = const$ along the inner surface of the potential well at the height of the "criterial saddle". If we project this line in the configuration space, we get the stability boundary of this equilibrium. Any kinematic perturbation that does not lead the system beyond the indicated boundary will cause only damped or undamped oscillations near stable equilibrium.

If, at some load, a stable equilibrium is unique for a given system, then the potential well and the stability region have unlimited scale in all directions, and any kinematic perturbation will lead to the appearance of these oscillations.

In general, the potential well is confined by the "watershed" line separating it from neighboring potential wells. The projection of the "watershed" line into the configuration space generates the boundary of the "region of attraction" around the point of stable equilibrium, including its stability region. If the watershed line is closed, then it contains at least two unstable equilibria: a "criterial saddle" and an equilibrium with a local maximum of the potential energy.

The number of isolated equilibrium states of an elastic system is determined by the number of real roots of system (1). When the coefficients of the system change, this number may increase or decrease. But the birth of new equilibria or their disappearance always occur in pairs (at the limit points of the curves of equilibrium). Geometrically, the moment of birth of two new equilibria corresponds to the appearance of the point of tangency of one of the hypersurfaces with the previously formed line of intersection points of the remaining hypersurfaces. Until the moment of tangency, the nascent pair of equilibria did not exist, since the corresponding pair of roots of system (1) was still complex conjugate. At the moment of contact, the complex parts of these roots disappear, and the identical real parts determine the birth of double and externally isolated equilibrium. With further variation of the coefficients of equations (1), the point of tangency splits into two intersection points of all hypersurfaces, and double and degenerate equilibrium splits into two simple, isolated and non-degenerate equilibria. One of them is always a stable equilibrium, the other is unstable. The described birth sequence of two isolated equilibria corresponds to the lower limit point. The development of a near-critical situation in the opposite direction leads to the fusion of two simple equilibria into one double, followed by its disappearance, which takes place at the upper limit point.

Let’s discuss some features of degenerate "indifferent" equilibria forming connected continual sets. The equilibrium points of the ball on a smooth horizontal plane form a two-dimensional continuum. The set of equilibria of a body floating in a fluid is a three-dimensional continuum. All the equilibria of the physical pendulum, the horizontal axis of which passes through its center of gravity, form a one-dimensional continuum, etc. For an Euler rod, in the case of a double critical load with identical constraints, postcritical curved bent equilibria also form a one-dimensional continuum, because as a result of transverse deflections the center of any cross-section of the rod can be at an arbitrary point of a circle perpendicular to the subcritical axis of the rod. In this sense, the critical equilibrium of this rod is "indifferent". The potential well of this equilibrium is a circular trough in the form of half the surface of a cut torus, which creates only orbit stability of the equilibrium. Therefore, this rod can be "moved along the bottom of a circular trough" without any energy costs. This feature is a consequence of the structural instability of "indifferent" equilibria. This explains the well-known paradox of Nikolai - the lack of equilibrium states of a cantilever "postcritical" rod with a circular cross-section in the case of applying an additional arbitrarily small nonconservative twisting moment at the free end.

An arbitrarily small slope of the plane "destroys" all the equilibriums of the ball in this plane. A small curvature of the plane in a cylindrical surface with a horizontal tangent "at the bottom" narrows the two-dimensional continuum of the indifferent balances of the ball to a one-dimensional one. Curvature of the plane in the surface of positive Gaussian curvature translates all indifferent equilibria into one isolated and non-degenerate (stable on the "bottom" of the well and unstable at the top of the "hill" of this surface). If the weight of the floating body decreases by a small amount in comparison with the weight of the liquid displaced by it, the body will float to the surface, and the continuum of indifferent equilibria instead of three-dimensional will turn out to be two-dimensional. For an Eulerian rod (and any other two-parameter elastic system that loses stability at the point of symmetric stable bifurcation) the critical equilibrium at load $P = P_{cr}$ is not indifferent. In fact, this equilibrium is triple, degenerate but externally isolated, and at the last time is still stable. There is no "new" (or "adjacent") equilibria infinitely close to the original then $P = P_{cr}$. However, for an arbitrarily small increase in the load $P = P_{cr} + dP$, the triple equilibrium breaks up into three simple isolated equilibria (two new, mirror-like and stable equilibria, located very close to the third – the initial equilibrium, which becomes "watershed" and unstable).

Loss of stability at bifurcation point is only if the initial equilibrium is incomplete. This means that there are many "new" energetically orthogonal equilibria, supplementing the original equilibrium to the complete equilibrium. For a centrally compressed Euler rod, the initial equilibrium is one-dimensional and incomplete ($N_z = P \neq 0$, $M_z = M_x = M_y = Q_x = Q_y = 0$). At the load $P = P_{cr} + dP$, this equilibrium becomes unstable. The rod with an arbitrary asymmetric cross section is transformed into a new stable compression-flexural-torsion and complete equilibrium, in which all 6 internal forces $N_z, M_z, M_x, M_y, Q_x, Q_y$ in the cross sections of the rod are not equal to zero. For conservative elastic systems, the geometric symmetry of the system and the incompleteness of its single subcritical equilibrium are sufficient conditions for the realization of the stability loss scenario "in the small" (i.e., stable symmetric bifurcation). Here the necessary condition is the symmetry and incompleteness of subcritical equilibrium.

If the initial equilibrium of a one-parameter elastic system is complete (relatively complete), then the loss of stability can occur only at the limit point (i.e. "in the large"), when the system tends to fall into a distant stable equilibrium. An example is a flat loss of stability of symmetric equilibrium of a semicircular arch with pinched ends, loaded in a lock section by a vertical force. In this case, the critical load at the limit point is approximately 1.5 times smaller than the "pseudo-bifurcation" critical load obtained from the linearized analysis.

3 Features of Bifurcation Points

In analysis of a conservative elastic system that depends on one or several parameters, the theorems of J.M.T. Thompson [2], Appel-Vozlinsky [3, 4], N.F. Morozov [5] are important in determining the type of critical points.

The theorem of J.M.T. Thompson. If the equilibrium of the elastic system strictly monotonically develops with the growth of the load parameter, this equilibrium can lose its stability only at the bifurcation point. In other words, if the load is a generalized coordinate on the equilibrium curve, there are no local extremum and there are no inflection points with a horizontal tangent, then the stable equilibrium of the system can become unstable only at the bifurcation point.

However, it should be added that Thompson's theorem is valid if the growing subcritical equilibrium is incomplete (there is an energetically orthogonal complement, as will be discussed later). Only in this case a bifurcation loss of stability is possible.

Under certain additional conditions, one can predict whether the symmetric bifurcation point is stable or unstable (this also refers to incomplete equilibria, Fig. 1). The asymmetric bifurcation point is always unstable. This statement came out in the work of P. Appel [3], and later in the work of V.I. Vozlinsky [4].

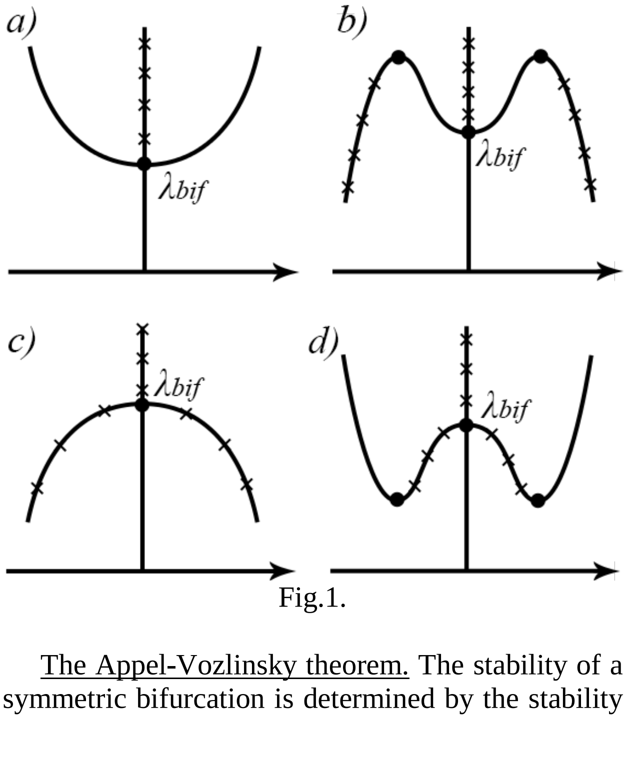

Fig.1.

The Appel-Vozlinsky theorem. The stability of a symmetric bifurcation is determined by the stability of the approaching branch of equilibrium curve. In other words, if the subcritical equilibrium is unique and stable, then at the bifurcation point there is a loss of stability "in the small". Crossing the initial equilibrium, the curve of new equilibria will consist of stable post-bifurcation equilibria (at least, at some initial section of the postcritical loads).

We should note that this theorem gives only sufficient conditions for the stability of a symmetric bifurcation. In the case when the elastic conservative system depends on four parameters (catastrophe "butterfly" [6]), the germ of the catastrophe (the highest term in the expansion in the bifurcation coordinate) having the sixth degree is positive ($+q^6$), then the loss of stability will be at the point of symmetric stable bifurcation in spite of the fact that the subcritical equilibrium is not unique one (Fig.1b).

The Appell-Vozlinsky theorem allows us to formulate the opposite assertion. If a single equilibrium half-branch is unstable (this is always so!), the new equilibria at the bifurcation point are also unstable (at least on some portion of the subcritical loads). But this is also only a sufficient condition. An example is the equilibrium curve for a catastrophe of a dual "butterfly" ($-q^6$, Fig.1d).

Here are a few examples of the use of the above-mentioned statements for the analysis of bifurcation points.

1. The stability problem of the usual elastic Euler rod. Its subcritical equilibrium is single, incomplete and it monotonically develops as the compressive load increases. According to Thompson's theorem, the Euler rod can lose stability only at the bifurcation point. On the basis of the Appel-Vozlinsky theorem, due to the uniqueness of the subcritical equilibrium of the central compression, the loss of stability will be "soft" ("in the small", at the point of symmetric stable bifurcation). Its post-bifurcation (and more complete) compressed-curved equilibrium will be stable and observable, which is confirmed experimentally.

2. A uniformly compressed circular ring is compressed monotonically as the load increases. The loss of stability will be at the bifurcation point. This subcritical equilibrium of the ring is unique. Hence, the bifurcation of the circular form of the ring into the initial elliptic ring will be stable (Appel-Vozlinsky theorem). Then the ellipse passes into a stable oval, and later takes a stable "double" figure-eight type shape. The diagram of near-critical equilibria, obtained by the authors with the help of FEM, is close in character to the diagram for the Euler rod.

3. For the axisymmetric equilibrium of a bent round plate Morozov [5] proved its uniqueness for large deflections. However, this does not imply the absence of other forms of equilibrium (for example, cyclically symmetric) of a strongly curved plate with a load that is less than critical. In this case the axisymmetric equilibrium develops monotonically. Consequently, the bifurcation loss of stability of this partial equilibrium is expected. But because of the possible nonuniqueness of the subcritical axisymmetric equilibrium, the bifurcation point can be either stable or unstable. The results of calculations with the NASTRAN showed that a stable initial post-bifurcation equilibrium (circular wave formation, fig.2) under the action of a distributed load and a concentrated force was observed in the case of contour constraints of a sliding-type [7]. Postcritical equilibrium of plate and graph of vertical displacements of points along a parallel with a radius of 0.85R for the plate under distributed load (a) and under concentrated load (b) are shown at fig.2.

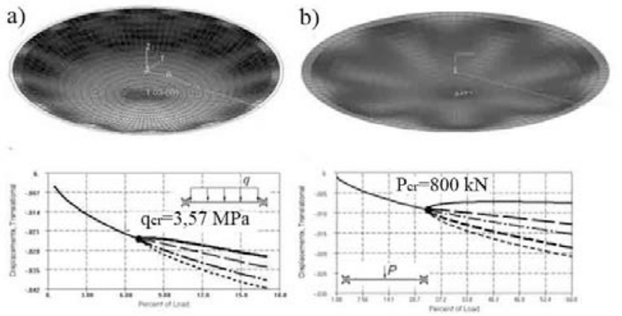

Fig.2.

If the edge is hinged the wave formation under the same loads occurred instantly at much greater deflections (fig.3). This means that with the hinged edge, the bifurcation point was unstable. Large radial forces, shape of model at postcritical equilibrium and graph of vertical displacements are shown on the fig.3.

4. A similar in character pattern of loss of stability of axisymmetric equilibrium develops when the concentrated force is at the apex of the segment of a non-regular shell of revolution with a circular base. Around the force a relatively small circular dent is formed almost instantly (Fig.4a). Its dimensions and deflections grow monotonously along with the increase in the load. One can expect a loss of stability of the axisymmetric equilibrium at the bifurcation point.

Subcritical equilibrium is not single. Therefore, it is not possible to formally use the Appel-Vozlinsky theorem to determine the nature of the bifurcation point. But locally it is unique. The loss of stability of axisymmetric equilibrium and the transition to a cyclically symmetric equilibrium occurs at the point of symmetric stable bifurcation.

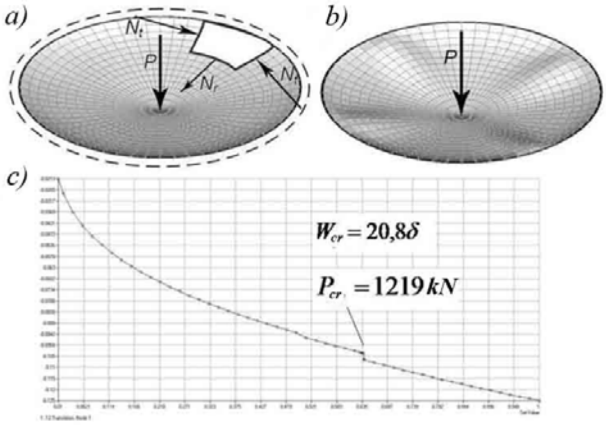

Fig.3.

Three waves are formed (Fig.4b). With a further increase in the load this form changes into four-wave stable shape (Fig.4c) as a result of secondary bifurcation. Further, a shape with 5 circumferential waves is also formed, and so on.

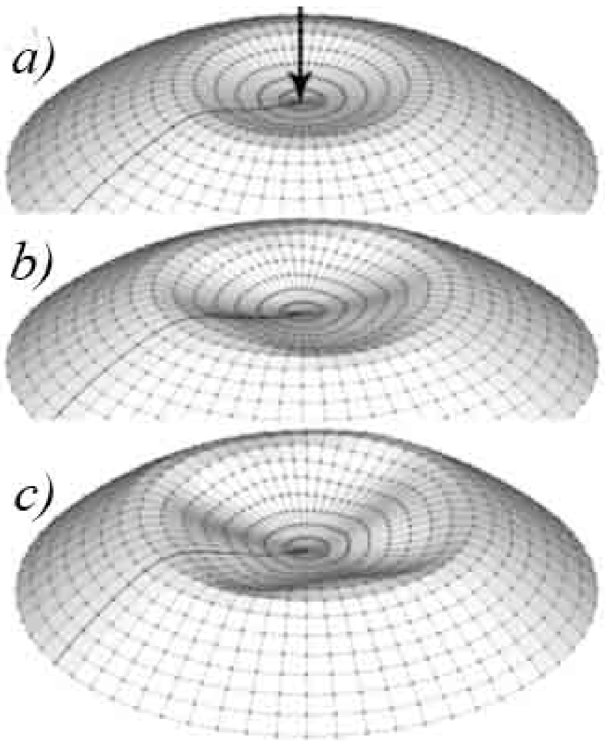

Fig.4.

The described pattern of successive wave formation was obtained numerically for a clamped shell (R = 500 cm, δ = 0.25 cm, R / δ = 1000, a = 353.5 cm – the radius of the base) is shown in Fig.4.

A.V. Pogorelov [8] described an experiment with a copper shell that was close in size (but significantly thinner). He observed three transverse waves formed as a result of the loss of stability of axisymmetric equilibrium.

4 Computational Differences of Bifurcation Points and Limit Points

In this paragraph, we discuss the computational differencesof critical points (bifurcation point or limit point).

Let the system of nonlinear equilibrium equations for a finite-element model -of an elastic system be represented as depending on the node displacements ν, the generalized coordinate q (on which the equilibrium state curve is constructed), the load parameter λ, and the vector of unit nodal loads $\vec{P}$.

$$G(\nu(q), \lambda(q)\vec{P}) = 0 \qquad (3)$$

The derivative of a nonlinear operator (the Jacobi matrix or the tangent rigidity matrix) is expressed as follows:

$$K(\nu, \lambda) = \frac{\partial G(\nu, \lambda)}{\partial \nu} \qquad (4)$$

The derivative with respect to the load parameter λ gives the aforementioned unit load vector $\vec{P}$:

$$\frac{\partial \vec{G}}{\partial \lambda} = \vec{P} \qquad (5)$$

The points of critical equilibrium (λ_{cr}, ν_{cr}) are determined by two main expressions:

$$\det K(\lambda_{cr}, \nu_{cr}) = 0 \qquad (6)$$

$$K(\lambda_{cr}, \nu_{cr}) \cdot \vec{W}_1^0 = 0 \qquad (7)$$

Here $\vec{W}_1^0$ is the first "zero" eigenvector ofloss of stability, corresponding to the "zero" eigenvalue of the stiffness matrix in pre-critical equilibrium.

The main relation determining the type of the critical point isobtained by multiplying zero eigenvector $(W_1^0)^T$ from the left by the derivative of the nonlinear operator with respect to the coordinate q. After some transformations, taking into account relation (7), we obtain the basic formula [9]:

$$(W_1^0)^T \frac{\partial \vec{G}}{\partial q} \rightarrow ... \rightarrow \left((W_1^0)^T \cdot \vec{P}\right) \cdot \frac{\partial \lambda}{\partial q} = 0 \qquad (8)$$

If in this formula, the scalar product is not zero $((\overrightarrow{W_1^0} \cdot \vec{P}) \neq 0)$, and the derivative turns to zero $(\partial\lambda/\partial q = 0)$, then the critical point is the limit point.

If on the contrary, the scalar product turns to zero $((\overrightarrow{W_1^0} \cdot \vec{P}) = 0)$, and the derivative is not

$(\partial\lambda/\partial q \neq 0)$, then the critical point is a bifurcation point.

Another sign of the bifurcation point is the fulfillment of the orthogonality between the pre-critical form of equilibrium and the "zero" eigenvector:

$$\left(W_1^0 \cdot W_{precr}\right) = 0 \qquad (9)$$

In the case of a limit point, the orthogonality condition does not hold:

$$\left(W_1^0 \cdot W_{precr}\right) \neq 0 \qquad (10)$$

Note that formula (6) in its most general form was first given in the works of A. Jepson and A. Spence (1982, 1985) [10, 11].

Regardless of the work of these authors, the relation (8) was formulated in the works of G.A. Manuilov, S.B. Kositsyn and K.A. Zhukov [12]. An energy interpretation of formula (8) was also given there.

If the work of an external load on displacements given by the eigenvector of the tangent stiffness matrix for pre-critical equilibrium is zero, then the critical point is a bifurcation point. Otherwise, this critical point is a limit point.

A formal proof was given in the paper of M. Deml and B. Wunderlich [13]. An overall review of the issues under consideration is given in the book by V. Galishnikova, P. Dunaiski and P.J. Pahl in 2009 [14] and in the work of the authors [9].

P. Vriggers and J. Simo [15] connected the mathematical results of Jepson and Spence to the FEM method. The authors also indicated signs of differences in the types of limit points and bifurcation points under special additional conditions that are expressedby the constants a, b, c, d calculated according to the following formulas:

$$a = \left(\vec{W}_1^0\right)^T \nabla_\nu (K\,\vec{W}^0)\vec{W}_1^0 \qquad (11)$$

$$b = \left(\vec{W}_1^0\right)^T \nabla_\lambda (K\,\vec{W}_1^0) + (\vec{W}_1^0)^T \nabla_\nu (K\,\vec{W}_1^0)\tilde{\nu} \qquad (12)$$

$$c = \left(\vec{W}_1^0\right)^T \nabla_\lambda P + 2\left(\vec{W}_1^0\right)^T \nabla_\lambda (K\tilde{\nu}) + \\ + \left(\vec{W}_1^0\right)^T \nabla_\nu (K\tilde{\nu})\tilde{\nu} \qquad (13)$$

$$d = b^2 - ac \qquad (14)$$

Here $\tilde{\nu} = K^{-1}\vec{P}$, $\nabla_\nu\ \nabla_\lambda$ are the geometric derivatives of the matrix K"along the direction"

$$\nabla_\nu (K\,\vec{W})\Delta\nu = \frac{d}{d\varepsilon}\left(K(\nu + \varepsilon\Delta\nu, \lambda)\vec{W}_1^0\right)\Big|_{\varepsilon=0} \qquad (15)$$

$$\nabla_\lambda (K\,\vec{W})\Delta\lambda = \frac{d}{d\varepsilon}\left(K(\nu, \lambda + \varepsilon\Delta\lambda)\vec{W}_1^0\right)\Big|_{\varepsilon=0} \qquad (16)$$

To distinguish the types of limit points, Jepson and Spence [11] suggested the following conditions.

If the scalar product $\left(\left(\vec{W}_1^0\right)^T \cdot \vec{P}\right) \neq 0$ and coefficient is not zero $a \neq 0$ at the critical point, then this is a simple (quadratic) limit point (turning point) (Fig.5b).

If the scalar product $\left(\left(\vec{W}_1^0\right)^T \cdot \vec{P}\right) \neq 0$, but the coefficient $a = 0$, then the critical point is a double (cubic) limit point (the point of inflection from the horizontal tangent, Fig.5a). It is called a "weakly stable" equilibrium point.

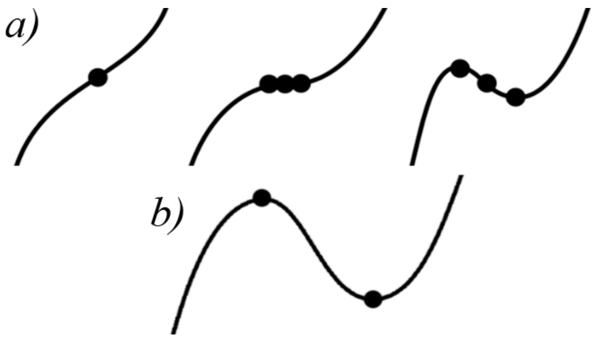

Fig.5.

The phenomenon of loss of stability of shallow arches and shells with an increase in the elevation parameter begins from this point.The emergence and development of limit points can be represented using a cusp catastrophe (Fig.6).

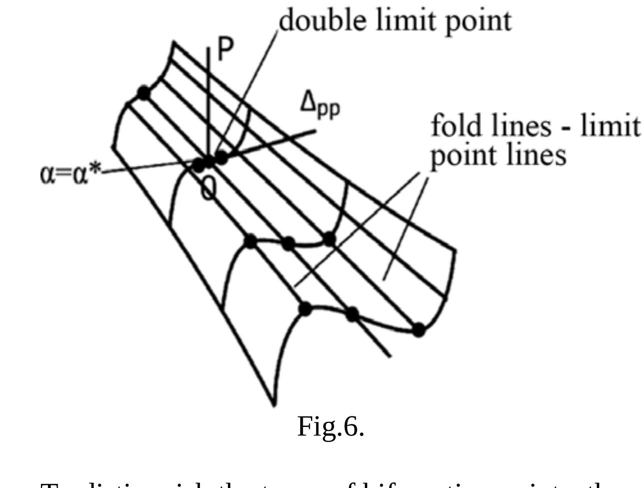

Fig.6.

To distinguish the types of bifurcation points, the following conditions are used.

If: $\left(\left(\vec{W}_1^0\right)^T \cdot \vec{P}\right) = 0$, $a \neq 0$, $d > 0$,

then the bifurcation point is simple and asymmetric (Fig.7c).

If: $\left(\left(\vec{W}_1^0\right)^T \cdot \vec{P}\right) = 0$, $a = 0$, $b \neq 0$,

then the bifurcation point is symmetric (unstable for $b < 0$, Fig.3a, stable if $b > 0$, Fig.7b).

If: $\left(\left(\vec{W}_1^0\right)^T \cdot \vec{P}\right) = 0,\ d < 0,$

then the isolated point of the bifurcation occurs(Fig.7d).

We note that, despite the apparent exclusivity of isolated bifurcation points, they sometimes occur in the most unexpected cases. For example, the bifurcation point of the symmetric equilibrium of a circular, high-rise two-hinged arch (2α>135°), loaded with two identical symmetrically located forces, is isolated.

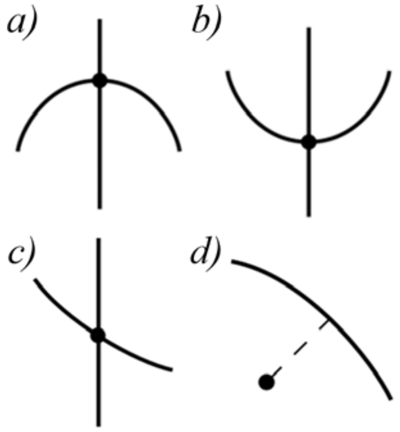

Fig.7.

Consider another specific point of bifurcation, which occurs when the type of the critical point changes (limit point is replaced by the bifurcation point). The described transition from the limit point to the bifurcation point is accomplished through a double critical point of the type "branching at the top of the hill" (variant of the hyperbolic umbilic catastrophe) [6, 16].

This double critical point (Fig.8) is characterized by the following formulas:

$$(P_1 \cdot \overrightarrow{w_1^0}) \neq 0,\ \frac{\partial \lambda}{\partial q_1} = 0,\ \frac{\partial^2 \lambda}{\partial q_1^2} < 0, \qquad (17)$$

$$(P_1 \cdot \overrightarrow{w_2^0}) = 0,\ \frac{\partial \lambda}{\partial q_2} = 0,\ \frac{\partial^2 \lambda}{\partial q_2^2} < 0. \qquad (18)$$

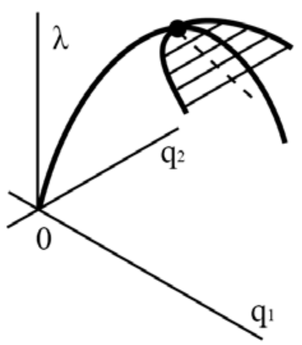

Fig.8.

5 Complement of the initial post-critical equilibrium

As it is known [9,17], the initial post-critical equilibrium is the sum of the pre-critical equilibrium and the zero eigenvector with a scale factor ξ:

$$\vec{W}_{init.p/cr} = \vec{W}_{precr} + \xi \overrightarrow{W_1^0} + ... \qquad (19)$$

At the point of bifurcation, the pre-critical equilibrium W_{precr} is the incomplete equilibrium. Therefore, there is an energetically orthogonal addition for it. Incomplete equilibrium can lose its stability both at the bifurcation point and at the limiting point. If the pre-critical equilibrium is "full" (the energy orthogonal complement is equal to zero), then the instability is possible only at the limit point. These positions follow from the orthogonality relations (8), (9).

After the passage of the bifurcation point, the addition of pre-critical equilibriumis possible in two versions:

1) new material components are added (for example before mentioned Euler rod, for shells momentless equilibrium becomes moment equilibrium);

2) relatively new ("organized" in another way) components that were previously presented in the pre-critical equilibrium (symmetric equilibria + antisymmetric from the composition of the complement give a sum of asymmetrical post-critical equilibrium) are added.

These considerations can be represented in the form of the formulas

$$W_1^0 \in Add^{E\perp}\left\{\vec{W}_{precr}\right\} \ \rightarrow \ \left(\overrightarrow{W_1^0} \cdot \vec{W}_{precr}\right) = 0 \quad (20)$$

At the limit point, the orthogonality condition (9) is not valid $\left(W_1^0 \cdot W_{precr}\right) \neq 0$. However, for this point we can symbolically write down some (conditional) decomposition of the zero eigenvector

$$W_1^0 = c_1 W_{precr} + c_2 W_1^{0\,bif} \qquad (21)$$

If the limit point is close to the unstable bifurcation point (to which it is "subordinate" according to the catastrophe hierarchy [18]), then $c_2 >> c_1$, $c_1 \neq 0$, and in this case the initial postcritical equilibrium is more complete than the pre-critical equilibrium. For a detailed analysis of the example related to the loss of stability at the limit point of a cylindrical shell, see paper [17].

If the limit point is not connected with the bifurcation point or is located far enough from it ($c_2 = 0$ or $c_2 << c_1$), then the zero eigenvector is proportional (or almost proportional) to the pre-critical equilibrium vector $(W_1^0 = c_1 W_{precr})$. Then the initial postcritical equilibrium (after passing through the limit point) repeats the pre-critical equilibrium, but will be unstable

$$\vec{W}_{init.p/cr} = \vec{W}_{precr} + \xi c_1 \vec{W}_{precr} = \left(1 + \xi c_1\right) \vec{W}_{precr} \quad (22)$$

6 The Emergence of Bifurcation Points and Their Movements on the Equilibrium Curve

Suppose that some elastic system has an equilibrium curve with two limit points. Bifurcation points are generated in pairs and can appear at some point of the stable part of the equilibrium curve (before the upper limit point) or after passing through this limit point (on the unstable part of the equilibrium curve). The example of the first variant is the "high" Mises's truss ($\alpha_0 \geq 67°.36$, Fig.9). As long as the angle of the rods is less than 67°.36 the Mises's truss loses stability only at the limit point. It is known that the critical angle of inclination of the rods at the limit point, as well as the value of the critical force, satisfy the relations [19]:

$$\cos^3 \alpha_{cr}^* = \cos\alpha_0 \,, P_{cr}^* = 2cl \sin^3 \alpha_{cr} \,.$$

The value of stiffness c for rods:

$$c = \frac{EA}{l} \,.$$

When the angle 67°,36 is reached, a new pair of critical points defining a symmetric unstable bifurcation is created at a certain point in the ascending stable equilibrium branch. This follows from an analysis of the solutions of equation [19]

$$\cos^3 \alpha^{bif} - \cos\alpha^{bif} = \cos\alpha_0 \quad (23)$$

Which can be represented as a cubic equation:

$$x^3 - x + a = 0 \quad (24)$$

The discriminant of this equation is D = 0 when $\alpha_1^{bif} = \alpha_2^{bif}$ for $\alpha_0 = 67.36°$. This equality indicates the birth of a double real root (Fig.10a). From this moment, at the high Mises's truss, the minimal critical load corresponds to the loss of stability at a symmetric unstable bifurcation point. The corresponding critical bifurcation load is calculatedaccording to:

$$P_{bif} = 2cl\left(1 - \frac{\cos\alpha_0}{\cos\alpha^{bif}}\right)\sin\alpha^{bif} \quad (25)$$

With the increase of the angle $\alpha_0 > 67.36°$, two bifurcation points begin to diverge along the stable branch (Fig.10b). With an increase in the initial angle to $\alpha_0 = 69.29$ °, the second bifurcation point reaches the limit point, becoming a double critical point such as "branching at the top of the hill". Further, this bifurcation point passes through the limit point and reaches the unstable branch of equilibria curve (Fig.10d). Through these points, pass two loops of unstable bifurcations.

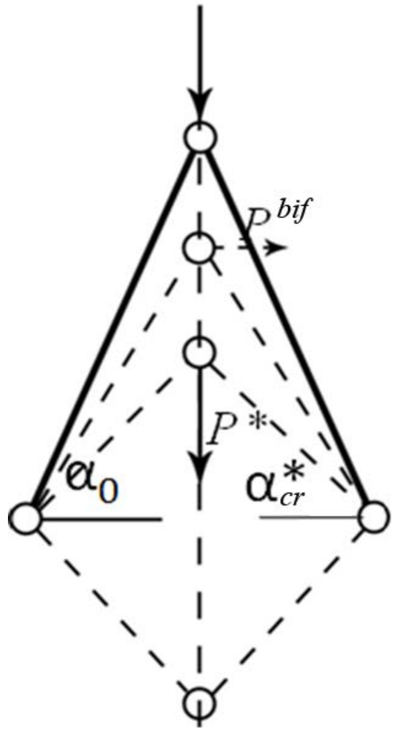

Fig.9.

However, in many cases (arched structures, shell structure) bifurcation points are born on an unstable branch of equilibria. However, their birth does not mean an immediate change of the type of loss of stability, as was in the previous case (the Mises's truss).

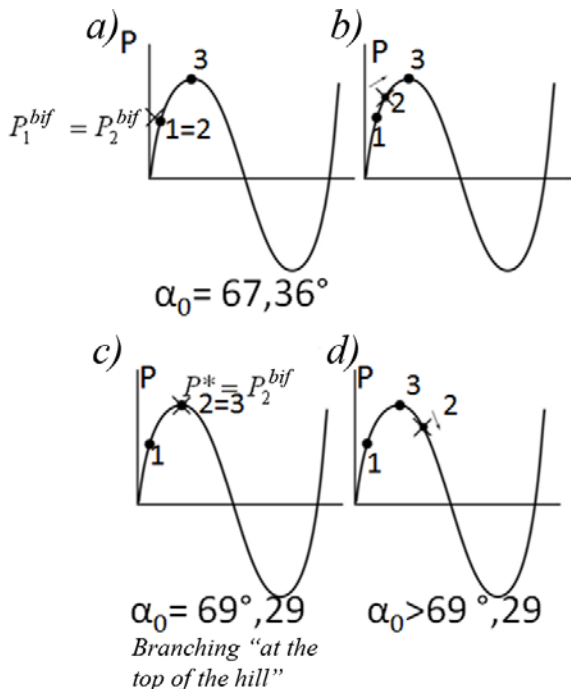

Fig.10.

Let us consider the simplest shallow sinusoidal outline arch:

$$y = f \sin\frac{\pi z}{l}, \tag{26}$$

loaded with a sinusoidal distributed load

$$p = p_0 \sin\frac{\pi z}{l}. \tag{27}$$

A detailed analysis of the development of critical points for such an arch is given in book[19].

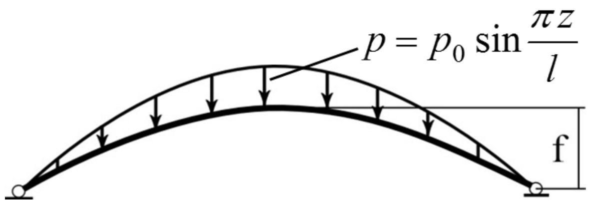

Fig.11.

It is convenient to analyze the birth of bifurcation points using a dimensionless parameter $s = 4I/Af^2$.

While s> 1.0, a very shallow arch is stable. For s = 1.0, a double limit point appears on the equilibrium curve (the inflection point with the horizontal tangent). From this point, all the arcs having the parameter s <1.0 up to the value s*= 2/11 (≈0.182) are unstable at the limit points. However, at s**=0.25, a double bifurcation point occurs on the unstable part of the branch of the equilibrium curve (Fig.12).

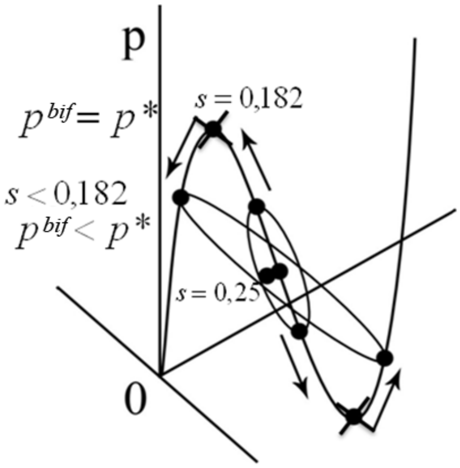

Fig.12.

With further decrease of the parameter s (s<0.25), the bifurcation points diverge from each other. Moving along the unstable part of the equilibrium curve, one of them tends to the upper limit point, the other – to the lower limit point. When the value s becomes s*=2/11 (≈0.182) the bifurcation points reach the limit points. At this moment, two (upper and lower) double critical points ("branching at the top of the hill") are formed.

With an even smaller decrease of the parameter s (s<2/11), the minimal critical load will correspond to the loss of stability of the arch at the point of a symmetric unstable bifurcation. The corresponding loops of unstable post-bifurcation equilibria are shown in Fig.12. The dimensionless critical loads for the arch at the limit point and the bifurcation point are determined by the relations:

$$P_{cr}^{*} = 1 \pm \sqrt{\frac{1(1-s)^3}{27s^2}}; \tag{28}$$

$$P_{cr}^{bif} = 1 \pm 3\sqrt{1-4s}. \tag{29}$$

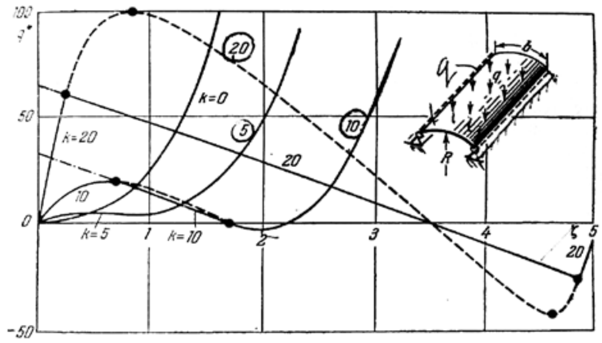

Fig.13.

P is the dimensionless load parameter:

$$P = \frac{P_0 l^4}{\pi^4 EJf}. \tag{30}$$

A similar picture of the behavior of bifurcation points takes place in the case of loss of stability of a shallow elongated cylindrical panel hinged along straight edges and loaded with a uniformly distributed load [20] (Fig.13). While the parameter $k = b^2/R\delta$ is less than 5, the panel does not lose stability of symmetric equilibrium. For k = 5, a double limit point appears, and in the range from k = 5 to k≈9, the panel looses stability at the limit point. At a certain value of k close to k≈8, a double bifurcation point appears on the unstable branch of equilibria (Fig.10). As the parameter k increases, the double bifurcation point splits into two simple bifurcations diverging from each other along the unstable branch, and at k≈10 the upper bifurcation point reaches the upper limit point. From this point, the panel becomes unstable at the bifurcation points (symmetric and unstable). The corresponding lines of unstable bifurcation equilibria are shown in Fig.13. We note that the behavior of bifurcation points of an elongated cylindrical panel actually repeats the behavior of these points for a sinusoidal arch.

In conclusion, we note that in the work of Kerr A.D. and Soifer M.T. [21] is the investigation of the stability of a circularly clamped arch under the action of a radial load (taking into account the geometric nonlinearity).It was established that when parameter k reaches 5.024, a double bifurcation point appears. However, the creation of the bifurcation parameter does not yet mean the beginning of the loss of stability of the arch at the bifurcation point. In fact, this double bifurcation point is born on an unstable branch of equilibrium curve, and in order to go through the limit point it is necessary to increase the parameter k to the value k = 5.18. Only after this, the minimum critical load will correspond to the loss of stability of the arch in a symmetrical bifurcation point. This is confirmed by the first proper forms of loss of stability for such an arch (Fig.14), calculated at the parameters k = 5.17 and k = 5.2.

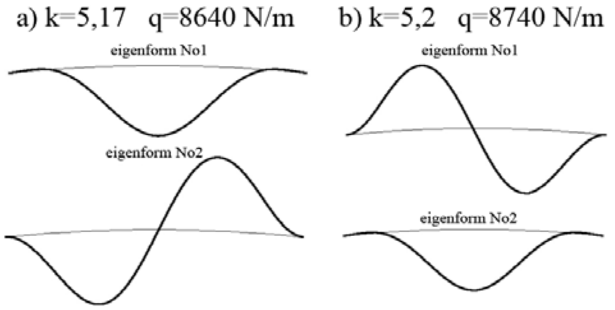

Fig.14.

With the parameter k = 5.17, the first eigenform is symmetric, it corresponds to the loss of stability at the limit point. The second form is skew-symmetric. At a parameter value of 5.2 for pre-critical equilibrium, the first form is already skew-symmetric, indicating a bifurcation loss of stability. We emphasize once again that the moment of birth of bifurcation points does not mean an immediate change in the type of loss of stability. Such a change will occur only if this bifurcation point is born on a stable branch of the equilibrium curve.

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