Static and Free Vibration Analysis of Composite Straight Beams on the Pasternak Foundation

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Abstract: - The objective of this study is to investigate the static and free vibration analysis of the cross-ply laminated straight beams on a two-parameter foundation, namely Pasternak. The curved element formulation is based on Timoshenko beam theory including the shear influence and the rotary inertia. The degrees of freedom of the two nodded element are three translations, three rotations, two shear forces, one axial force, two bending moment and one torque (12 DOF). A parametric study is performed on the static and the natural frequencies of cross-ply laminated straight beams with various foundation parameters. Support conditions are simply supported, fixed-fixed and fixed-roller.

Key-Words: - composite beam, finite element method, Timoshenko beam theory, elastic foundation

1 Introduction

The increased use of laminated composite beams in many applications due to their attractive properties in strength, stiffness and lightness has resulted in a growing demand for engineers in the design of structures. The free vibration analysis of laminated composite beams with no foundation are studied by several researchers [1-8]

Beams on elastic foundation are presented in [9]. Elastic and viscoelastic foundation models are proposed in [10]. [11] investigated the effects of rotary inertia, shear deformation and foundation constants on the natural frequencies of Timoshenko beam on Pasternak foundation. [12] investigated the free vibration of beam-columns on two-parameter elastic foundations. [13] is presented the static analysis of beams on two-parameter elastic foundation using the exact displacement function. [14] used a finite element procedure for the free vibration analysis of Timoshenko beam-columns fully or partially supported two-parameter elastic foundation. [15] employed the exact and the approximate shape functions to analyze the free vibration of beams on two-parameter elastic foundation. Free vibration analysis of initially stressed beams resting two-parameter elastic foundation is considered using the finite element method in [16]. [17] employed two different differential equation of motion for Timoshenko beam based on elastic foundation parameters. [18] is performed the exact free vibration analysis of Euler beams on Pasternak foundation. The bending, buckling and free vibration problems of Timoshenko and Euler beams on different elastic foundation models are presented by using Green's functions in [19]. [20] presented a mixed method, which composes the state space method and the differential quadrature method, for bending and free vibration of beams on a Pasternak elastic foundation. [21] presented analytic solutions for the static analysis of beams on Pasternak foundation. A symplectic method based on two-dimensional elasticity theory is used for analytic solutions.

The bending and free vibration of functionally graded beams on elastic foundation is considered by using exact two-dimensional elasticity theory in [22]. The stability and free vibration of functionally graded sandwich beams on two-parameter elastic foundation is studied using Chebyshev collocation method in [23]. Free vibration and buckling analysis of double functionally graded Timoshenko beams on elastic foundation is studied in [24]. The exact natural frequencies and buckling loads are obtained using Wittrick-William algorithm.

Nonlinear free flexural and post-buckling analysis of laminated orthotropic cross-ply beams on two-parameter elastic foundation is presented in [25]. The static analysis of thick composite beams on Winkler foundation using higher order shear deformation theory is investigated analytically and experimentally in [26]. The differential quadrature method for nonlinear free vibration analyses of laminated composite beams on elastic foundation which has cubic nonlinearity with shearing layer is employed in [27]. Free vibration analysis of cross-ply laminated beams on elastic foundation is studied using finite element method in [28,29].

In this study, the static and free vibration analyses of cross-ply laminated straight beams based on Timoshenko beam theory, resting on Pasternak foundation are investigated. The constitutive equations of layered orthotropic beams are derived by reducing the constitutive relations of orthotropic materials for three-dimensional body [30]. As a numerical investigation, the static and free vibration analysis of isotropic thin/thick beams and cross-ply laminated beams on Pasternak foundation are performed via the mixed finite element method (MFEM) and the results are compared with the literature [18-20, 28-29] and the static analysis results of cross-ply laminated beams on resting Winkler and Pasternak are given as a contribution for the literature.

2 Formulation

2.1 The constitutive relations

The constitutive equation yields

$$\boldsymbol{\sigma} = \mathbf{E} : \boldsymbol{\varepsilon} \tag{1}$$

$\boldsymbol{\sigma}$ is the stress tensor, $\boldsymbol{\varepsilon}$ is the strain tensor and $\mathbf{E}$ is the function of elastic constants. In order to derive the constitutive equations of a laminated composite beam, firstly the assumptions made on stress, in accordance with beam geometry [3], secondly some reductions made on the constitutive relation of orthotropic materials for the three dimensional body by incorporating the Poisson's ratio [30].

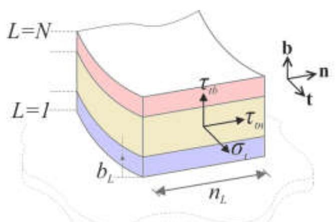

Fig.1 The stresses in the Frenet Coordinate System (N: Total number of layers)

In Frenet coordinate system (see Fig.1), paying attention to $\sigma_n = \sigma_b = \tau_{nb} = 0$, the constitutive relations yield

$$\begin{Bmatrix} \sigma_t \\ \tau_{bt} \\ \tau_{tn} \end{Bmatrix} = [\boldsymbol{\beta}] \begin{Bmatrix} \varepsilon_t \\ \gamma_{bt} \\ \gamma_{tn} \end{Bmatrix} \tag{1}$$

In (1), $[\boldsymbol{\beta}]_{3\times 3}$ matrix is the function of orthotropic material constants. Timoshenko beam theory requires shear correction factors and it is assumed to be $5/6$ for a general rectangular cross-section. By means of the kinematic equations

$$\begin{aligned} u_t^* &= u_t + b\,\Omega_n - n\,\Omega_b \\ u_n^* &= u_n - b\,\Omega_t \\ u_b^* &= u_b + n\,\Omega_t \end{aligned} \tag{2}$$

u_t^*, u_n^*, u_b^* are displacements at the beam continuum and u_t, u_n, u_b are displacements on the beam axis and Ω_t, Ω_n and Ω_b present the rotations of the beam cross-section around the *t, n* and *b* Frenet coordinates, respectively. The strains which are derived from (2)

$$\begin{Bmatrix} \varepsilon_t \\ \gamma_{bt} \\ \gamma_{tn} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_t}{\partial t} \\ \dfrac{\partial u_t}{\partial b} + \dfrac{\partial u_b}{\partial t} \\ \dfrac{\partial u_t}{\partial n} + \dfrac{\partial u_n}{\partial t} \end{Bmatrix} + b \begin{Bmatrix} \dfrac{\partial \Omega_n}{\partial t} \\ 0 \\ -\dfrac{\partial \Omega_t}{\partial t} \end{Bmatrix} + n \begin{Bmatrix} -\dfrac{\partial \Omega_b}{\partial t} \\ \dfrac{\partial \Omega_t}{\partial t} \\ 0 \end{Bmatrix} \tag{3}$$

By obtaining strains for beam geometry due to displacements [31], the forces and moments for a layer can be derived by analytical integration of the stresses in each layer through the thickness of the cross-section, respectively.

$$T_t = \sum_{L=1}^{N} \left(\int_{-0.5n_L}^{0.5n_L} \left(\int_{b_{L-1}}^{b_L} \sigma_t \mathrm{d}b \right) \mathrm{d}n \right) \tag{4}$$

$$T_b = \sum_{L=1}^{N} \left(\int_{-0.5n_L}^{0.5n_L} \left(\int_{b_{L-1}}^{b_L} \tau_{bt} \mathrm{d}b \right) \mathrm{d}n \right) \tag{5}$$

$$T_n = \sum_{L=1}^{N} \left(\int_{-0.5n_L}^{0.5n_L} \left(\int_{b_{L-1}}^{b_L} \tau_{tn} \mathrm{d}b \right) \mathrm{d}n \right) \tag{6}$$

$$\begin{aligned} M_t = &\sum_{L=1}^{N} \left(-\int_{-0.5n_L}^{0.5n_L} \left(\int_{b_{L-1}}^{b_L} b\,\tau_{tn} \mathrm{d}b \right) \mathrm{d}n \right) \\ &+ \sum_{L=1}^{N} \left(\int_{b_{L-1}}^{b_L} \left(\int_{-0.5n_L}^{0.5n_L} n\,\tau_{tb} \mathrm{d}n \right) \mathrm{d}b \right) \end{aligned} \tag{7}$$

$$M_n = \sum_{L=1}^{N} \left(\int_{-0.5n_L}^{0.5n_L} \left(\int_{b_{L-1}}^{b_L} b\sigma_t \mathrm{d}b \right) \mathrm{d}n \right) \tag{8}$$

$$M_b = -\sum_{L=1}^{N} \left(\int_{b_{L-1}}^{b_L} \left(\int_{-0.5n_L}^{0.5n_L} n\sigma_t \mathrm{d}n \right) \mathrm{d}b \right) \tag{9}$$

N is the number of the layer, n_L is the width of the layer, b_L and b_{L-1} are the directed distances to the bottom and the top of the L^{th} layer where b is positive upward. The constitutive equation in a matrix form:

$$\begin{Bmatrix} T_t \\ T_n \\ T_b \\ \hdashline M_t \\ M_n \\ M_b \end{Bmatrix} = \sum_{L=1}^{N} \left[\begin{array}{c:c} \mathbf{E}_m^L & \mathbf{E}_{mf}^L \\ \hdashline \mathbf{E}_{fm}^L & \mathbf{E}_f^L \end{array} \right] \begin{Bmatrix} \dfrac{\partial u_t}{\partial t} \\ \dfrac{\partial u_t}{\partial n} + \dfrac{\partial u_n}{\partial t} \\ \dfrac{\partial u_t}{\partial b} + \dfrac{\partial u_b}{\partial t} \\ \hdashline \dfrac{\partial \Omega_t}{\partial t} \\ \dfrac{\partial \Omega_n}{\partial t} \\ \dfrac{\partial \Omega_b}{\partial t} \end{Bmatrix} \tag{10}$$

or, since $[\mathbf{C}] = [\mathbf{E}]^{-1}$, in accordance with (1) and (3), (10) yields to the form

$$\begin{Bmatrix} \varepsilon_t \\ \gamma_{tn} \\ \gamma_{bt} \\ \hdashline \kappa_t \\ \kappa_n \\ \kappa_b \end{Bmatrix} = \left[\begin{array}{c:c} \mathbf{C}_m & \mathbf{C}_{mf} \\ \hdashline \mathbf{C}_{fm} & \mathbf{C}_f \end{array} \right] \begin{Bmatrix} T_t \\ T_n \\ T_b \\ \hdashline M_t \\ M_n \\ M_b \end{Bmatrix} \tag{11}$$

$\kappa_t, \kappa_n, \kappa_b$ are curvatures.

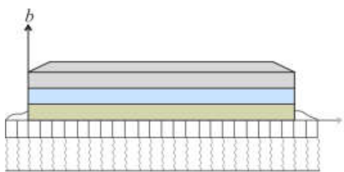

Fig.2 Pasternak model for the two- parameter foundation

2.2 The field equations and functional

The field equations and functional for the isotropic homogenous spatial beam, which are based on Timoshenko beam theory exists in [32-34]. The field equations and the functional are extended to laminated composite beams in [35-36]. Winkler and Pasternak foundation (Fig.2) terms inserted to the field equations of spatial beam as follows:

$$\left.\begin{aligned} -\mathbf{T}_{,s} - \mathbf{q} + k_W \mathbf{u} - k_P \mathbf{u}_{,ss} + \rho A \ddot{\mathbf{u}} = \mathbf{0} \\ -\mathbf{M}_{,s} - \mathbf{t} \times \mathbf{T} - \mathbf{m} + \rho \mathbf{I} \ddot{\mathbf{\Omega}} = \mathbf{0} \end{aligned}\right\} \tag{12}$$

$$\left.\begin{aligned} \mathbf{u}_{,s} + \mathbf{t} \times \mathbf{\Omega} - \mathbf{C}_m \mathbf{T} - \mathbf{C}_{mf} \mathbf{M} = \mathbf{0} \\ \mathbf{\Omega}_{,s} - \mathbf{C}_{fm} \mathbf{T} - \mathbf{C}_f \mathbf{M} = \mathbf{0} \end{aligned}\right\} \tag{13}$$

s is the arc axis of the spatial beam, $\mathbf{u}(u_t, u_n, u_b)$ is the displacement vector, $\mathbf{\Omega}(\Omega_t, \Omega_n, \Omega_b)$ is the cross section rotation vector. k_W and k_P are Winkler and Pasternak foundation constants, respectively. $\ddot{\mathbf{u}}$ and $\ddot{\mathbf{\Omega}}$ are the accelerations of the displacement and rotations, $\mathbf{T}(T_t, T_n, T_b)$ defines the force vector, $\mathbf{M}(M_t, M_n, M_b)$ is the moment vector, ρ is the material density. A is the area of the cross section, $\mathbf{I}$ stores the moments of inertia, $\mathbf{C}_m$, $\mathbf{C}_f$, $\mathbf{C}_{mf}$ and $\mathbf{C}_{fm}$ are compliance matrices where $\mathbf{C}_{mf}$, $\mathbf{C}_{fm}$ are coupling matrices [37]. $\mathbf{q}$ and $\mathbf{m}$ are the distributed external force and moment vectors, respectively. Once the motion is considered as harmonic for the free vibration of the beam, the conditions $\mathbf{q} = \mathbf{m} = \mathbf{0}$ are satisfied. Incorporating Gâteaux differential in terms of (12)-(13) with potential operator concept [38] yields to an original functional for the literature.

$$\begin{aligned} \mathbf{I}(\mathbf{y}) = & -\left[\mathbf{u}, \mathbf{T}_{,s}\right] - \left[\mathbf{M}_{,s}, \mathbf{\Omega}\right] + \left[\mathbf{t} \times \mathbf{\Omega}, \mathbf{T}\right] \\ & -\tfrac{1}{2}\left\{\left[\mathbf{C}_m \mathbf{T}, \mathbf{T}\right] + \left[\mathbf{C}_{mf} \mathbf{M}, \mathbf{T}\right] + \left[\mathbf{C}_{fm} \mathbf{T}, \mathbf{M}\right]\right. \\ & \left. + \left[\mathbf{C}_f \mathbf{M}, \mathbf{M}\right]\right\} - \tfrac{1}{2}\rho A \omega^2 \left[\mathbf{u}, \mathbf{u}\right] - \tfrac{1}{2}\rho \omega^2 \left[\mathbf{I\Omega}, \mathbf{\Omega}\right] \\ & + \tfrac{1}{2} k_W \left[\mathbf{u}, \mathbf{u}\right] + \tfrac{1}{2} k_P \left[\mathbf{u}_{,s}, \mathbf{u}_{,s}\right] + \left[\left(\mathbf{T} - \hat{\mathbf{T}}\right), \mathbf{u}\right]_\sigma \\ & + \left[\left(\mathbf{M} - \hat{\mathbf{M}}\right), \mathbf{\Omega}\right]_\sigma - k_P \left[\hat{\mathbf{u}}_{,s}, \mathbf{u}\right]_\sigma + \left[\hat{\mathbf{u}}, \mathbf{T}\right]_\varepsilon \\ & + \left[\hat{\mathbf{\Omega}}, \mathbf{M}\right]_\varepsilon - k_P \left[\mathbf{u}_{,s}, \left(\mathbf{u} - \hat{\mathbf{u}}\right)\right]_\varepsilon \end{aligned} \tag{14}$$

For a static analysis, the above functional needs to be modified by excluding the terms $\frac{1}{2}\rho A \omega^2 [\mathbf{u}, \mathbf{u}]$, $\frac{1}{2}\rho \omega^2 [\mathbf{I\Omega}, \mathbf{\Omega}]$ and inserting $[\mathbf{q}, \mathbf{u}]$, $[\mathbf{m}, \mathbf{\Omega}]$. In (14), the square brackets indicate the inner product, the terms with hats are known values on the boundary and the subscripts ε and σ represent the geometric and dynamic boundary conditions, respectively.

2.3 Mixed finite element formulation

The linear shape functions are employed in the finite element formulation. The curvatures are satisfied exactly at the nodal points and linearly interpolated through the element [33]. Calculation of the natural free vibration frequencies of a structural system yields to the following standard eigenvalue problem,

$$\left([\mathbf{K}]-\omega^2[\mathbf{M}]\right)\{\mathbf{u}\}=\{\mathbf{0}\} \tag{15}$$

where, $[\mathbf{K}]$ and $[\mathbf{M}]$ are the system and mass matrix of the entire domain, respectively. $\mathbf{u}$ is the eigenvector (mode shape) and ω depicts the natural angular frequency of the system.

3 Numerical Examples

The static and dynamic analysis of isotropic straight beams resting on two-parameter elastic foundation are verified with the literature [18-20] and then the natural frequencies of cross-ply laminated straight beams on Winkler and Pasternak foundations are verified with the literature [28-29]. As far as the knowledge of the authors, the static analysis of cross-ply laminated straight beams on Pasternak foundation presented in this study is as a contribution for the literature.

3.1 Isotropic beams on elastic foundation

Common parameters for the beam problem on elastic foundation are: Two different the length of beam-to-the thickness of beam ratio ($L/h=120$ and $L/h=5$) is considered. Two different boundary conditions are utilized, namely, fixed-fixed (C-C) and simply supported-simply supported (S-S). The Winkler $\bar{K}_W\,(0,10^2,10^4,10^6)$ and Pasternak $\bar{K}_P\,(0,10,25)$ foundation parameters are considered. The non-dimensional parameters, which are used in tables, are defined as:

$$\bar{u}_b=\frac{u_b EI_n}{qL^4}\,,\qquad \bar{\omega}=\omega\sqrt{\frac{\rho AL^4}{EI_n}}$$
$$\bar{K}_W=\frac{k_W L^4}{EI_n},\qquad \bar{K}_P=\frac{k_P L^2}{EI_n} \tag{16}$$

3.1.1 Static analysis

The straight beam subjected to a q uniformly distributed load on two-parameter foundation is solved and compared the literature results [19-20]. The non-dimensional maximum displacements ($\bar{u}_b\times10^{-2}$) are listed in Tables 1-4. The MFEM results are an excellent agreement with the literature.

Table 1 The non-dimensional maximum displacements for the isotropic thin beam (boundary condition: S-S)

foundation parameters		$L/h=120$, $\bar{u}_b\times10^{-2}$			
$\bar{K}_W$	$\bar{K}_P$	MFEM	[19]	[20]	[20] exact
0	0	1.3024	1.3023	1.3023	1.3023
10	0	1.1807	1.1814	1.1806	1.1806
	10	0.6134	0.6141	0.6133	0.6133
	25	0.3557	0.3566	0.3557	0.3557
10^2	0	0.6401	0.6403	0.6400	0.6400
	10	0.4256	0.4261	0.4256	0.4256
	25	0.2829	0.2836	0.2828	0.2828

Table 2 The non-dimensional maximum displacements for the isotropic moderately thick beam (boundary condition: S-S)

foundation parameters		$L/h=5$, $\bar{u}_b\times10^{-2}$		
$\bar{K}_W$	$\bar{K}_P$	MFEM	[20]	[20] exact
0	0	1.4322	1.4203	1.4202
10	0	1.2857	1.2826	1.2773
	10	0.6388	0.6464	0.6402
	25	0.3631	0.3721	0.3657
10^2	0	0.6671	0.6961	0.6685
	10	0.4363	0.4593	0.4388
	25	0.2869	0.3052	0.2894

Table 3 The non-dimensional maximum displacements for the isotropic thin beam (boundary condition: C-C)

foundation parameters		$L/h=120$, $\bar{u}_b\times10^{-2}$		
$\bar{K}_W$	$\bar{K}_P$	MFEM	[19]	[20]
0	0	0.2609	0.2616	0.2606
10	0	0.2557	0.2565	0.2555
	10	0.2053	0.2062	0.2053
	25	0.1588	0.1597	0.1588
10^2	0	0.2169	0.2174	0.2167
	10	0.1793	0.1800	0.1794
	25	0.1427	0.1435	0.1427

Table 4 The non-dimensional maximum displacements for the isotropic moderately thick beam (boundary condition: C-C)

foundation parameters		$L/h=5$ $\bar{u}_b \times 10^{-2}$	
$\bar{K}_W$	$\bar{K}_P$	MFEM	[20]
0	0	0.3906	0.3881
10	0	0.3789	0.3782
	10	0.2854	0.2887
	25	0.2094	0.2148
10^2	0	0.2979	0.3091
	10	0.2366	0.2482
	25	0.1817	0.1930

3.1.2 Dynamic analysis

The free vibration analysis of isotropic straight beams on elastic foundation is carried out and the non-dimensional natural frequencies are tabulated in Tables 5-8. Only for the moderately thick beam, in the case of increasing values of Winkler and Pasternak foundation parameters the results of our study with respect to [20] diverged (Tables 6 and 8). For the rest, the results are all in agreement.

Table 5 The non-dimensional fundamental frequencies ϖ for the isotropic thin beam (boundary condition: S-S)

foundation parameters		$L/h=120$			
$\bar{K}_W$	$\bar{K}_P/\pi^2$	MFEM	[18]	[20]	[20] exact
0	0	3.1414	3.1415	3.1414	3.1414
10^2	0	3.7482	3.7483	3.7482	3.7482
	1.0	4.1436	4.1437	4.1436	4.1436
	2.5	4.5824	4.5824	4.5823	4.5823
10^4	0	10.024	10.024	10.024	10.024
	1.0	10.048	10.048	10.048	10.048
	2.5	10.084	10.084	10.084	10.084
10^6	0	31.623	31.623	31.622	31.622
	1.0	31.624	31.624	31.622	31.622
	2.5	31.625	31.625	31.625	31.625

Table 6 The non-dimensional fundamental frequencies ϖ for the isotropic moderately thick beam (boundary condition: S-S)

foundation parameters		$L/h=5$		
$\bar{K}_W$	$\bar{K}_P/\pi^2$	MFEM	[20]	[20] exact
0	0	3.0453	3.0480	3.0480
10^2	0	3.6798	3.6705	3.6705
	1.0	4.0839	4.0664	4.0664
	2.5	4.5280	4.4991	4.4991
10^4	0	9.9267	7.3408	7.3408
	1.0	9.9502	7.3410	7.3410
	2.5	9.9851	7.3412	7.3412
10^6	0	13.3516	7.3508	7.3508
	1.0	13.3516	7.3508	7.3508
	2.5	13.3516	7.3508	7.3508

Table 7 The first three non-dimensional natural frequencies ϖ for the isotropic thin beam (boundary condition: C-C)

foundation parameters		$L/h=120$		
$\bar{K}_W$	$\bar{K}_P/\pi^2$	MFEM	[18]	[20]
0	0	4.7289	4.7300	4.7314
		7.8488	7.8540	7.8533
		10.985	10.996	10.991
10^2	0	4.9494	4.9500	4.9515
		7.9000	7.9040	7.9044
		11.004	11.014	11.010
	1.0	5.1816	5.1820	5.1834
		8.1208	8.1240	8.1247
		11.183	11.192	11.188
	2.5	5.4767	5.4770	5.4783
		8.4200	8.4230	8.4234
		11.436	11.444	11.440
10^4	0	10.123	10.123	10.123
		10.837	10.839	10.838
		12.518	12.526	12.522
	1.0	10.152	10.152	10.152
		10.925	10.927	10.926
		12.641	12.648	12.644
	2.5	10.194	10.194	10.194
		11.053	11.055	11.054
		12.818	12.825	12.821

Table 8 The first three non-dimensional natural frequencies $\bar{\varpi}$ for the isotropic moderately thick beam (boundary condition: C-C)

foundation parameters		$L/h=5$	
$\bar{K}_W$	$\bar{K}_P/\pi^2$	MFEM	[20]
0	0	4.2420	4.2634
		6.4179	6.4648
		8.2853	7.4013
10^2	0	4.5325	4.5418
		6.5062	6.5472
		8.3260	7.4018
	1.0	4.7925	4.7910
		6.8293	6.8471
		8.6590	7.4091
	2.5	5.1104	5.0974
		7.2389	7.2228
		9.0948	7.4261
10^4	0	10.018	7.4054
		10.261	8.5458
		10.863	10.112
	1.0	10.047	7.4091
		10.347	8.6492
		11.011	10.192
	2.5	10.087	7.4135
		10.470	8.7616
		11.221	10.306

3.2 Laminated composite straight beams on Pasternak foundation

The static and dynamic analysis of symmetrically (0°/90°/90°/0°) and non-symmetrically (0°/90°/0°/90°) layered cross-ply beams with four orthotropic lay-ups on Winkler and Pasternak foundation is investigated. The material properties of orthotropic material are as follows: $E_t = 144.8\,\mathrm{GPa}$, $E_n = 9.65\,\mathrm{GPa}$, $E_b = 9.65\,\mathrm{GPa}$, $G_{tn} = 4.14\,\mathrm{GPa}$, $G_{tb} = 4.14\,\mathrm{GPa}$, $G_{nb} = 3.45\,\mathrm{GPa}$, $\upsilon_{tn} = \upsilon_{tb} = 0.3$, $\upsilon_{nb} = 0.399$. The density of the material $\rho = 1389.23\,\mathrm{kg/m^3}$. The length of beam-to-the thickness of beam ratio $L/h = 15$, where the width of rectangular cross-section is $b = 0.15\,\mathrm{m}$, is considered. Two different boundary conditions C-C (fixed-fixed) and C-S (fixed-simply supported) are employed.

3.2.1 Dynamic analysis

Firstly, the free vibration analysis of symmetrically and non-symmetrically layered cross-ply straight beams with no foundation is considered and compared with literature [1,2,5,28,29] and then the first three non-dimensional natural frequencies are tabulated in Tables 9-10. The definition of non-dimensional natural frequency parameter is

$$\bar{\varpi} = \omega L^2 \sqrt{\frac{\rho}{E_t h^2}} \tag{17}$$

Table 9 The first three non-dimensional natural frequencies of symmetrically and non-symmetrically cross-ply straight beams (no foundation, boundary condition: C-C)

	Ref.	Non-dimensional frequencies		
		$\bar{\varpi}_1$	$\bar{\varpi}_2$	$\bar{\varpi}_3$
0°/90°/90°/0°	MFEM	4.5870	10.281	16.955
	[1]	4.5940	10.291	16.966
	[2]	4.6180	10.796	16.984
	[5]	4.5869	10.281	16.955
	[28,29]	4.6170	10.471	18.160
0°/90°/0°/90°	MFEM	3.7030	8.8150	15.045
	[2]	3.7360	9.1870	15.102
	[28,29]	3.7320	9.1810	15.097

Table 10 The first three non-dimensional natural frequencies of symmetrically cross-ply straight beams (no foundation, boundary condition: C-S)

	Ref.	Non-dimensional frequencies		
		$\bar{\varpi}_1$	$\bar{\varpi}_2$	$\bar{\varpi}_3$
0°/90°/90°/0°	MFEM	3.5180	9.4300	16.370
	[1]	3.5250	9.4420	16.384
	[2]	3.6130	9.5690	16.482
	[5]	3.5183	9.4299	16.370
	[28,29]	3.7060	9.6500	17.384

Next, the free vibration analysis of symmetrically and non-symmetrically layered cross-ply straight beams on elastic foundation is investigated and the results compared with [28,29]. k_P is constant along to the length of beam in our MFEM solution for cross-ply laminated beams on Pasternak foundation. The tabulated results are given in Tables 11-12. The foundation constants for Winkler and Pasternak are $k_W = 100\,\mathrm{kN/m^2}$ and $k_P = 200\,\mathrm{kN}$, respectively. The all results are in a good agreement.

Table 11 The first three non-dimensional natural frequencies of symmetrically and non-symmetrically cross-ply straight beams on elastic foundation (boundary condition: C-C)

	foundation constants		non-dimensional frequencies			
	k_W kN/m^2	k_P kN	Ref.	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$
0°/90°/90°/0°	100	0	MFEM	4.597	10.285	16.958
			[29]	4.627	10.475	18.162
		200	MFEM	4.603	10.296	16.971
			[28]	4.714	10.632	18.346
0°/90°/0°/90°	100	0	MFEM	3.715	8.820	15.048
			[29]	3.636	8.626	15.687
		200	MFEM	3.723	8.833	15.063
			[28]	3.752	8.832	15.918

Table 12 The first three non-dimensional natural frequencies of symmetrically and non-symmetrically cross-ply straight beams on elastic foundation (boundary condition: C-S)

	foundation constants		non-dimensional frequencies			
	k_W kN/m^2	k_P kN	Ref.	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$
0°/90°/90°/0°	100	0	MFEM	3.531	9.435	16.373
			[29]	3.718	9.655	17.387
		200	MFEM	3.539	9.446	16.387
			[28]	3.819	9.811	17.578
0°/90°/0°/90°	100	0	MFEM	2.773	7.817	14.192
			[29]	2.840	7.738	14.610
		200	MFEM	2.783	7.830	14.208
			[28]	2.973	7.938	14.849

3.2.2 Static analysis

The static analysis of symmetrically (0°/90°/90°/0°) and non-symmetrically (0°/90°/0°/90°) cross-ply laminated straight beams under q uniformly distributed load with and without foundation is carried out. The results for no foundation are validated with ANSYS and the results are given in Tables 13-15 for non-dimensional maximum vertical displacement ($\bar{u}_b$) and shear force ($\bar{T}_b$) and bending moment ($\bar{M}_n$) at the fixed end. The results for the beam resting on elastic foundation are given in Tables 16-17. The variation of $\bar{u}_b$ along the beam span is depicted for symmetrically and non-symmetrically cross-ply laminated beams on Pasternak foundation with C-C and CS boundary conditions in Figs.3-4, respectively. The definition of non-dimensional values is

$$\bar{u}_b = \frac{u_b E_t I_n}{qL^4}, \qquad \bar{T}_b = \frac{T_b}{qL}, \qquad \bar{M}_n = \frac{M_n}{qL^2} \tag{18}$$

Table 13 The non-dimensional $\bar{u}_b$ displacements for symmetrically and non-symmetrically cross-ply straight beams without foundation
[diff.% = (MFEM-ANSYS)*100/MFEM]

	$\bar{u}_b \times 10^{-3}$ (non-dimensional displacement)					
	C-C			C-S		
	MFEM	ANSYS	diff.%	MFEM	ANSYS	diff. %
0°/90°/90°/0°	5.07	5.05	0.3	8.57	9.17	-7.0
0°/90°/0°/90°	7.83	8.21	-4.9	14.0	16.2	-15.5

Table 14 The non-dimensional $\bar{T}_b$ displacements for symmetrically and non-symmetrically cross-ply straight beams without foundation
[diff.% = (MFEM-ANSYS)*100/MFEM]

	$\bar{T}_b$ (non-dimensional shear force)					
	C-C			C-S		
	MFEM	ANSYS	diff.%	MFEM	ANSYS	diff.%
0°/90°/90°/0°	0.50	0.50	0.0	0.62	0.64	-3.2
0°/90°/0°/90°	0.50	0.50	0.0	0.62	0.65	-4.8

Table 15 The non-dimensional $\overline{M}_n$ displacements for symmetrically and non-symmetrically cross-ply straight beams without foundation
[diff.% = (MFEM-ANSYS)*100/MFEM]

	$\overline{M}_n$ (non-dimensional bending moment)					
	C-C			C-S		
	MFEM	ANSYS	diff.%	MFEM	ANSYS	diff.%
0°/90°/90°/0°	0.08	0.08	0.0	0.120	0.126	-5.0
0°/90°/0°/90°	0.08	0.08	0.0	0.117	0.125	-6.8

Table 16 The non-dimensional $\overline{u}_b$ displacement and $\overline{T}_b$, $\overline{M}_n$ internal forces for symmetrically and non-symmetrically cross-ply straight beams on elastic foundation (boundary condition: C-C)

	foundation constants		non-dimensional values		
	k_W kN/m^2	k_P kN	$\overline{u}_b$ $\times 10^{-3}$	$\overline{T}_b$	$\overline{M}_n$
0°/90°/90°/0°	100	0	5.047	0.498	0.083
		200	5.033	0.498	0.083
0°/90°/0°/90°	100	0	7.777	0.498	0.083
		200	7.737	0.498	0.082

4 Conclusion

The finite element algorithm is verified for isotropic and composite straight beams resting on Winkler and Pasternak foundation [1,2,5,18-20,28,29]. Next, the static and free vibration analysis of symmetrically and non-symmetrically cross-ply laminated composite straight beams on Pasternak foundation is investigated via mixed FE algorithm. The static analysis of cross-ply laminated beams on resting Pasternak beam results are presented as a contribution for the literature. It is observed that, in composite materials, layer-wise material placement along the thickness direction has great importance on the response the structure.

Table 17 The non-dimensional $\overline{u}_b$ displacement and $\overline{T}_b$, $\overline{M}_n$ internal forces for symmetrically and non-symmetrically cross-ply straight beams on elastic foundation (boundary condition: C-S)

	foundation constants		non-dimensional values		
	k_W kN/m^2	k_P kN	$\overline{u}_b$ $\times 10^{-3}$	$\overline{T}_b$	$\overline{M}_n$
0°/90°/90°/0°	100	0	8.502	0.616	0.119
		200	8.465	0.616	0.118
0°/90°/0°/90°	100	0	13.834	0.612	0.116
		200	13.733	0.612	0.115

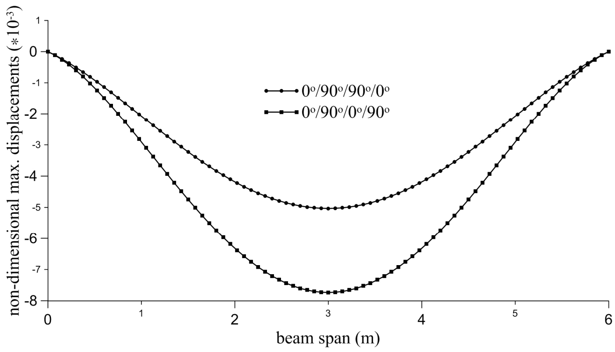

Fig.3 The variation of non-dimensional maximum displacements of laminated composite straight beam with C-C boundary condition

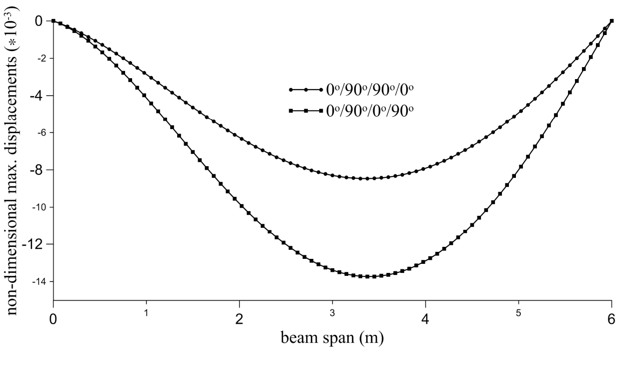

Fig.4 The variation of non-dimensional maximum displacements of laminated composite straight beam with for C-S boundary condition

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