Spatial and Temporal Instability of Shallow Mixing Layers in the Presence of Non-Constant Resistance Force

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Abstract: - Two methods for linear stability analysis of shallow mixing layers are presented in the paper: temporal stability analysis and spatial stability analysis. The combined effect of small curvature and non-constant friction force in the transverse direction is analyzed. Both spatial and temporal linear stability problems are solved numerically by means of a collocation method based on Chebyshev polynomials. Marginal stability curves are obtained for different values of the parameters of the problem. It is shown that both curvature and variable friction stabilize the flow.

Key-Words: - Linear stability analysis, small curvature, non-uniform friction, shallow mixing layer

1 Introduction

Shallow mixing layers are often observed in nature and civil engineering. Examples include flows at river junctions and flows in compound and composite channels. Different methods are used to analyze flow patterns and structure of shallow mixing layers [1]: experimental investigations, numerical modeling and stability analysis. In the present paper we analyze linear stability of slightly curved shallow mixing layers.

Linear stability of straight shallow mixing layers under the rigid-lid assumption is analyzed in [2], [3]. It is assumed in [2] and [3] that free surface acts as a rigid lid so that water depth is constant. The rigid-lid assumption is relaxed in [4] where it is shown that the rigid-lid assumption (from a linear stability point of view) works well for small Froude numbers. Such a condition is usually satisfied for many shallow flows observed in practice.

The analyses in [2]-[4] have shown that bottom friction stabilizes the flow. The influence of bottom friction on the stability of shallow flows is usually described by the bed-friction number $S = c_f b / h,$ where c_f is the friction coefficient which is related to roughness of the surface and the Reynolds number of the flow by empirical formulas [5], h is water depth and b is the characteristic length scale of the flow (for example, mixing layer width).

The effect of large Froude numbers on the linear stability of shallow mixing layers is analyzed in [6] where it is shown that

In many cases shallow flows are not straight. The effect of small curvature on linear stability characteristics of deep mixing layers in analyzed in [7] where it is shown that curvature has a stabilizing effect on stably curved mixing layer and destabilizing effect on unstable curved mixing layer. Spatial and temporal instability of curved shallow mixing layers is analyzed in [8], [9].

In all the above mentioned studies the friction coefficient is assumed to be constant. There are cases, however, where resistance force varies considerably in the transverse direction. One example of such a situation is a flow in a compound channel where the friction force is usually larger in the floodplain and smaller in the main channel. Similar situation occurs during floods where water occupies vegetated areas along river banks. Series of experimental papers by MIT group led by prof. H. Nepf appeared in the literature [10]-[14] in the last few years. It is shown in [10]-[14] that the variation of the friction force in the transverse direction has an important effect on the mass and momentum exchange.

Recently the effect of non-constant resistance force in the transverse direction on the linear stability characteristics of straight shallow mixing layers is analyzed in [15]-[17]. Spatial stability analysis is used in [15]. Temporal stability analysis of particle-laden shallow flows with non-constant resistance force is performed in [16]. Preliminary results of the spatial stability of shallow mixing layers with non-uniform resistance force are presented in [17].

In the present paper both temporal and spatial stability analysis is performed for the case where the friction force varies in the transverse direction. The flow is assumed to be slightly curved with large radius of curvature. Linear stability problem is solved using both temporal and spatial method. Results of numerical computations are presented.

2 Mathematical Formulation of the Problem

Consider the following form of shallow water equations under the rigid-lid assumption

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \qquad (1)$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{\partial p}{\partial x}+\frac{c_f(y)}{2h}u\sqrt{u^2+v^2}=0, \qquad (2)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}-\frac{1}{R}u^2+\frac{\partial p}{\partial y}+\frac{c_f(y)}{2h}v\sqrt{u^2+v^2}=0, \qquad (3)$$

where p is the pressure, h is water depth, u and v are the depth-averaged velocity components, $c_f(y)$ is the friction coefficient which is assumed to be dependent on the transverse coordinate y, and R is the dimensionless radius of curvature ($R >> 1$).

Equation (1) is equivalent to the continuity equation in a two-dimensional hydrodynamics so that it is natural to introduce the stream function $\psi(x,y,t)$ by the relations

$$u=\frac{\partial \psi}{\partial y},\ v=-\frac{\partial \psi}{\partial x} \qquad (4)$$

Using (1)-(4) we obtain

$$(\Delta\psi)_t+\psi_y(\Delta\psi)_x-\psi_x(\Delta\psi)_y+\frac{2}{R}\psi_y\psi_{xy}+$$
$$+\frac{c_f(y)}{2h}\Delta\psi\sqrt{\psi_x^2+\psi_y^2}+\frac{c_f(y)}{2h\sqrt{\psi_x^2+\psi_y^2}}(\psi_y^2\psi_{yy}+$$
$$+2\psi_x\psi_y\psi_{xy}+\psi_x^2\psi_{xx})+\frac{c_{fy}(y)}{2h}\psi_y\sqrt{\psi_x^2+\psi_y^2}=0, \qquad (5)$$

where $c_{fy}(y)$ is the derivative of the function $c_f(y)$ with respect to y. The friction coefficient $c_f(y)$ is modeled by the formula

$$c_f(y)=c_{f_0}\gamma(y), \qquad (6)$$

where $\gamma(y)$ is a differentiable shape function.

A perturbed solution $\psi(x,y,t)$ is assumed to be of the form

$$\psi(x,y,t)=\psi_0(y)+\varepsilon\psi_1(x,y,t)+... \qquad (7)$$

where $\psi_0(y)$ is the stream function of the base flow $U(y)$, so that $U(y)=\psi_{0y}(y)$. In the present study the function $U(y)$ is given by

$$U(y)=\frac{1}{2}(1+\tanh y). \qquad (8)$$

Substituting (6) and (7) into (5) and linearizing the resulting equation in the neighborhood of the base flow we obtain

$$\psi_{1xxt}+\psi_{1yyt}+\psi_{0y}(\psi_{1xxx}+\psi_{1yyx})-\psi_{0yyy}\psi_{1x}+$$
$$+\frac{c_f(y)}{2h}(\psi_{0y}\psi_{1xx}+2\psi_{0yy}\psi_{1y}+2\psi_{0y}\psi_{1yy}) \qquad (9)$$
$$+\frac{c_{fy}(y)}{h}\psi_{0y}\psi_{1x}+\frac{2}{R}\psi_{0y}\psi_{1xy}=0.$$

The perturbed solution is represented in the form of a normal mode

$$\psi_1(x,y,t)=\varphi(y)e^{i\alpha(x-ct)}, \qquad (10)$$

where α is the wave number and c is the wave speed of the perturbation. . It follows from (9) and (10) that

$$\varphi_{yy}[\alpha(U-c)-iSU\gamma)-iS(\gamma U_y+\gamma_y U)\varphi_y$$
$$+\varphi[\alpha^3(c-U)-\alpha U_{yy}+i\alpha^2 US\gamma/2)=0, \qquad (11)$$

where $S=\frac{c_{f_0}b}{h}$ is the stability parameter and b is a length scale of the problem.

The boundary conditions have the form

$$\varphi(\pm\infty)=0. \qquad (12)$$

Problem (11), (12) is an eigenvalue problem. In general, both parameters α and c in (10) can be complex. However, two ways to solve (11), (12) are usually used in practice: (a) temporal stability analysis and (b) spatial stability analysis. In case (a) it is assumed that the wave number α is real while the eigenvalue $c=c_r+ic_i$ is complex. The sign of the imaginary part of the complex eigenvalue c is used to decide whether flow (8) is linearly stable or unstable: if all eigenvalues satisfy the inequality $c_i < 0$ then base flow (8) is said to be linearly stable. On the other hand, if at least one eigenvalue has a positive imaginary part ($c_i > 0$) then base flow (8) is said to be linearly unstable. Finally, if one eigenvalue satisfies the condition $c_i = 0$ while all other eigenvalues have negative imaginary parts, base flow is said to be marginally stable. The set of all values of the parameters α and S for which flow (8) is marginally stable determines the marginal stability curve.

In case (b) the wave speed is assumed to be real while the wave number α is complex: $\alpha = \alpha_r + i\alpha_i$. Flow (8) is said to be spatially unstable if at least one $\alpha_i < 0$.

Numerical solution of (11), (12) for the case of temporal stability is simpler since the discretized eigenvalue problem is linear in c. In this case standard software packages for the solution of a generalized matrix eigenvalue problem can be used for calculations. On the other hand, a nonlinear eigenvalue problem has to be solved in case of spatial stability analysis. As a result, a computational algorithm should be modified.

In the present paper both approaches are presented. In Section 3 we describe the numerical method used to discretize (11), (12). The results of linear stability calculations for temporal stability are shown in Section 4 while spatial stability is analyzed in Section 5.

3 Numerical Method

Numerical solution of (11), (12) is obtained by a collocation method. It is convenient to map the interval $-\infty < y < +\infty$ onto the interval $-1 \le \xi \le 1$ using the new variable $\xi = \frac{2}{\pi}\arctan y$. In terms of the transformed variable the solution to (11) is sought in the form

$$\varphi(\xi) = \sum_{k=0}^{N-1} a_k (1-\xi^2) T_k(\xi), \tag{13}$$

where $T_k(\xi) = \cos k \arccos \xi$ is the Chebyshev polynomial of the first kind of order k and a_k are unknown coefficients. The factor $(1-\xi^2)$ in (13) guarantees that zero boundary conditions at $\xi = \pm 1$ will be satisfied automatically:

$$\varphi(\pm 1) = 0. \tag{14}$$

The collocation points are chosen in the form

$$\xi_m = \cos \frac{\pi m}{N}, \quad m = 1,2,..., N-1. \tag{15}$$

Using (11), (13) and (15) we obtain the following generalized eigenvalue problem

$$(A + cB)a = 0, \tag{16}$$

where A and B are complex-valued matrices and $a = (a_0 a_1 ... a_{N-1})^T$. Note that both A and B are non-singular (this fact simplifies the solution of the generalized eigenvalue problem (16)).

4 Temporal Stability Analysis

Numerical results are presented in the paper for the case where variability of the friction coefficient in the transverse direction is described by (6), where

$$\gamma(y) = \frac{\beta+1}{2} + \frac{(\beta-1)}{2}\tanh \mu y, \tag{17}$$

with $\beta = \frac{c_{f_1}}{c_{f_0}} \ge 1$ and the parameter μ in (17) represents how sharp is the transition from the region of larger friction to the region of smaller friction.

Here c_{f_1} and c_{f_0} are the friction coefficients in the vegetated area and main channel, respectively. Note that in [16] the friction coefficient varied is such a way that $c_{f0} = 0$ as $y \to -\infty$. Variability of the friction coefficient given by (17) is consistent with the velocity profile (8) since higher velocity is expected in the region where friction force is smaller.

Marginal stability curves for the case $R = \infty, \mu = 1$ and three values of the parameter β, namely, $\beta = 1, 1.5$ and 2 are shown in Fig. 1. Note that $\beta = 1$ corresponds to a constant friction coefficient in the whole region of the flow while values $\beta > 1$ represent the degree of non-uniformity of the friction force in the transverse direction. It is seen from Fig. 1 that the case with non-constant friction is more stable than the case of a uniform friction. In addition, the critical value of S decreases as the parameter β increases.

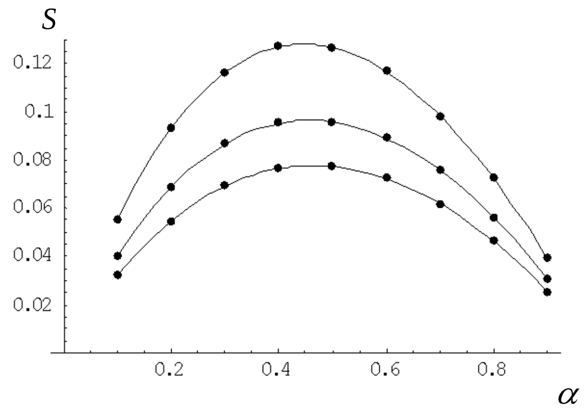

Fig. 1. Marginal stability curves for the case $R=\infty$ and three values of $\beta:\beta=1,\beta=1.5$ and $\beta=2$ (from top to bottom).

Fig. 2 plots the marginal stability curves for the case $1/R=0.03,\mu=1$ and three values of $\beta:1,1.5$ and 2. It is also seen from the graph that non-uniformity of the friction coefficient stabilizes the flow in the presence of a small curvature.

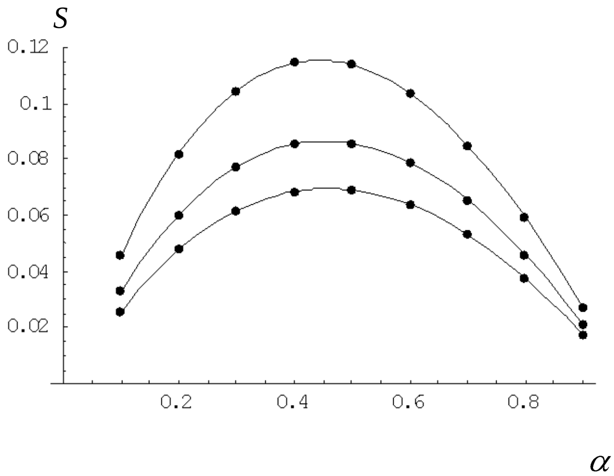

Fig. 2. Marginal stability curves for the case $1/R=0.03$ and three values of $\beta:\beta=1,\beta=1.5$ and $\beta=2$ (from top to bottom).

The effect of small curvature on the stability boundary is shown in Fig. 3 for the case of a non-constant friction. The marginal stability curves in Fig. 3 correspond to one value $\beta=2$. It is seen from the graph that the increase in curvature leads to a more stable flow.

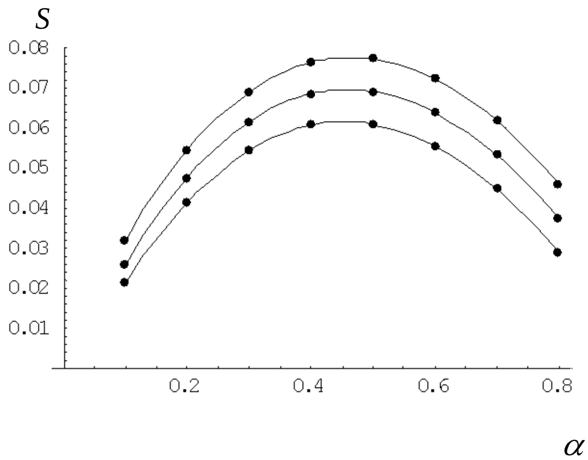

Fig. 3. Marginal stability curves for the case $\beta=2,\mu=1$ and three values of $1/R:0,0.03,0.06$ (from top to bottom).

The combined effect of the variable friction and small curvature on the stability boundary is shown in Fig. 4 where the critical value of the parameter S (defined as $S_{cr}=\max_k S(k)$) is plotted versus β for three different values of $1/R:0,0.03,0.06$ (from top to bottom). Both parameters (small curvature $1/R$ and non-uniformity of the friction coefficient β) have a stabilizing influence on the flow.

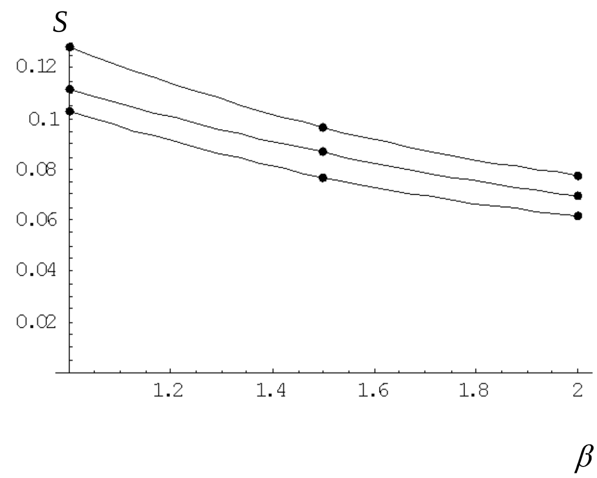

Fig. 4. Critical values S_{cr} versus β for three values

of $1/R: 1, 1.5$ and 2 (from bottom to top).

5 Spatial Stability Analysis

$R = \infty$

In this section we consider numerical results that are obtained using spatial stability approach.

Fig. 5 plots the spatial growth rates for the case $S = 0.15, \mu = 1$ and (no curvature). The case $\beta = 1$ (top curve) corresponds to uniform friction. As can be seen from the graph, non-uniform friction of the form (17) has a stabilizing influence on the flow: the growth rate for the most unstable mode decreases as the parameter β increases. Note that β represents the degree of non-uniformity of the friction force in the transverse direction. Thus, flow with non-uniform friction is more stable than flow with uniform friction.

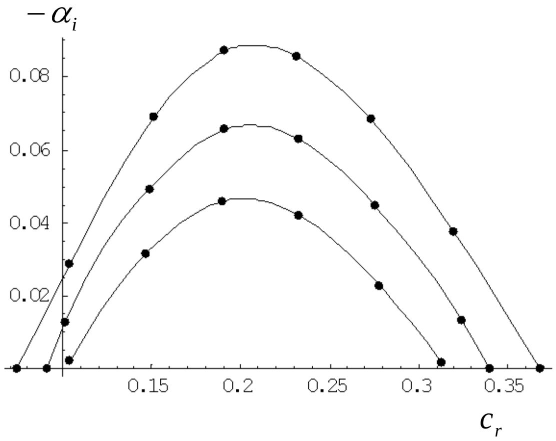

Fig. 5. Spatial growth rates $-\alpha_i$ versus c_r for three values of $\beta: \beta = 1, \beta = 1.5$ and $\beta = 2$ (from top to bottom).

The role of curvature on the stability characteristics of the flow is seen from Fig. 6 where the spatial growth rate for the case $S = 0.15$ and $\beta = 1.5$ is shown for three values of the parameter $1/R$, namely, $1/R = 0$ (straight flow with no curvature), 0.01 and 0.02.

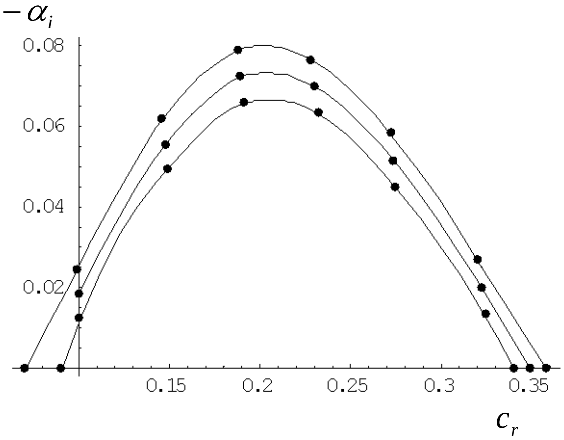

Fig. 6. Spatial growth rates $-\alpha_i$ versus c_r for three values of $1/R: 0, 0.01$ and 0.02 (from bottom to top).

The bottom curve in Fig. 6 corresponds to the case of no curvature and is the most stable among the three cases considered. Thus, increase in curvature has a destabilizing effect on the flow.

The effect of the parameter μ on the spatial growth rates is shown in Fig. 7 for the case $S = 0.15, \beta = 2, 1/R = 0$.

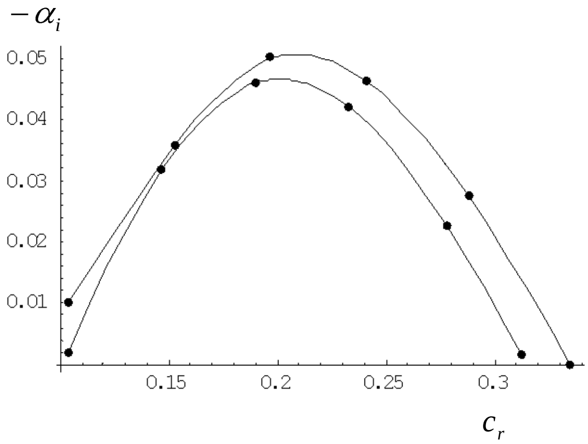

Fig. 7. Spatial growth rates $-\alpha_i$ versus c_r for two values of $\mu: 1$ and 10 (from bottom to top).

It is seen from Fig. 7 that steeper friction gradients result in less stable flow.

6 Conclusion and Direction of Future Work

Spatial and temporal linear stability analysis of shallow mixing layers is performed in the present paper. The effect of several parameters of the stability characteristics of the flow is investigated. In particular, it is shown that non-uniform friction coefficient in the transverse direction of the flow has a stabilizing influence in comparison with the case of a uniform friction. In addition, numerical computations demonstrate that slightly curved mixing layers are more stable than layers without curvature. Finally, it is shown that steepness of the change of the friction coefficient in the transverse direction has a destabilizing influence on the flow.

Linear stability analysis is performed in the present paper under the assumption of a parallel flow. In other words, the base flow profile (8) is assumed to be independent on the longitudinal coordinate. Experimental data (see, for example, [5] and [6]) show that the base flow is slightly changing along the longitudinal coordinate. Asymptotic schemes have been developing in the past in order to take into account slow longitudinal variation of the base flow. The authors are currently implementing the asymptotic scheme in order to derive the amplitude evolution equation for the most unstable mode.

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