

Conditionally Specified Bivariate Kummer-Gamma Distribution

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Abstract: - The Kummer-gamma distribution is an extension of gamma distribution and for certain values of parameters slides to a bimodal distribution. In this article, we introduce a bivariate distribution with Kummer-gamma conditionals and call it the *conditionally specified bivariate Kummer-gamma distribution/bivariate Kummer-gamma conditionals distribution*. Various representations are derived for its product moments, marginal densities, marginal moments, conditional densities, and conditional moments. We also discuss several important properties including, entropies, distributions of sum, and quotient. Most of these representations involve special functions such as the Gauss and the confluent hypergeometric functions. The bivariate Kummer-gamma conditionals distribution studied in this article may serve as an alternative to many existing bivariate models with support on $(0, \infty) \times (0, \infty)$.

Key-Words: - Bivariate distribution; confluent hypergeometric function; gamma distribution; gamma function; Gauss hypergeometric function; Kummer-gamma distribution.

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1 Introduction

There are several bivariate distributions proposed in the statistical literature, see Arnold, Castillo and Sarabia [1], Balakrishnan and Lai [2], Kotz, Balakrishnan and Johnson [3], Hutchinson and Lai [4], and Mardia [5] for good reviews. For more recent work, the reader is referred to Bran-Cardona, Orozco-Castañeda and Nagar [6], Bakouch et al. [7], Bondesson [8], Chen, Tzeng and Lin [9], Franco, Vivo and Kundu [10], Ghosh [11], Gupta and Nadarajah [12], Gupta and Nagar [13, 14], Gupta, Orozco-Castañeda and Nagar [15], Mathai [16], Nadarajah and Kotz [17, 18], Nagar, Arashi and Nadarajah [19], Nagar, Nadarajah and Okorie [20], Orozco-Castañeda, Nagar and Gupta [21], and Semenikhine, Furman and Su [22]. These distributions have attracted useful applications in several areas; for example, in the modeling of the proportions of substances in a mixture, brand shares, i.e., the proportions of brands of some consumer product that are bought by customers, proportions of the electorate voting for the candidate in a two-candidate election and the dependence between two soil strength parameters, and hydrology. They've also been widely used as a prior in Bayesian statistics.

Bivariate distributions have also been applied in areas such as physics, economics, engineering, risk analysis, and medicine. For interesting real life applications the reader is referred to Alsayed and Manzi [23], Danaher and Smith [24], Li et al. [25], Makanda and Shaw [26], Takeuchi [27], and references therein.

Arnold and Strauss [28] considered the most general class of bivariate distributions such that both sets of conditional densities are exponential (also see Nadarajah and Choi [29]). They called their bivariate distribution

the *bivariate exponential conditionals (BEC) distribution*. The conditionally specified bivariate gamma distribution is given in Kotz, Balakrishnan and Johnson [30] (also see Arnold, Castillo and Sarabia [1], Nadarajah [31]). For a review on the construction of bivariate distributions by using the conditional approach, the reader is referred to Arnold, Castillo and Sarabia [1], and Balakrishnan and Lai [2, Chapter 6].

In this article, we introduce a bivariate distribution of positive random variables X and Y such that conditional densities of $X | y$ and $Y | x$ are Kummer-gamma. The bivariate distribution defined in this article is closely connected to gamma and Kummer-gamma distributions, and, therefore we first define these distributions.

The random variable X is said to have a gamma distribution with parameters (α, θ) , denoted as $X \sim Ga(\alpha, \theta)$, if its probability density function (pdf) is given by

$$\frac{x^{\alpha-1} \exp(-x/\theta)}{\theta^\alpha \Gamma(\alpha)}, x > 0,$$

where $\alpha > 0$ and $\theta > 0$.

The random variable X is said to have a Kummer-gamma distribution with parameters (α, γ, ξ) , denoted as $X \sim KG(\alpha, \gamma, \xi)$, if its pdf is given by

$$\frac{x^{\alpha-1} \exp(-\xi x) (1+x)^{-\gamma}}{\Gamma(\alpha) \psi(\alpha, \alpha-\gamma+1; \xi)}, x > 0,$$

where $\alpha > 0$, $\xi > 0$, $-\infty < \gamma < \infty$, and ψ is the confluent hypergeometric functions of the second kind (also known as Kummer's function of the second kind, Tricomi function, or Gordon function), see Luke [32]. The Kummer-gamma distribution is an extension of gamma distribution and for $\alpha < 1$ (and certain values of γ) yields

bimodal distribution. For properties and results on gamma and Kummer-gamma distributions the reader is referred to Gupta and Nagar [33], Gupta, Cardeño and Nagar [34], Johnson, Kotz and Balakrishnan [35], and Koudou [36].

Consider the bivariate distribution of positive random variables X and Y defined by the pdf

$$f(x, y; \alpha, \beta, \nu, \sigma, \phi) = K(\alpha, \beta, \nu, \sigma, \phi)x^{\alpha-1}y^{\beta-1}(x+y)^\nu \times \exp\left[-\left(\frac{x}{\sigma} + \frac{y}{\phi}\right)\right], \quad x > 0, y > 0, \quad (1)$$

where $\alpha > 0, \beta > 0, \sigma > 0, \phi > 0$ and $\nu \geq 0$. The normalizing constant $K(\alpha, \beta, \nu, \sigma, \phi)$, by using Lemma A.1, is given by

$$\begin{aligned} & [K(\alpha, \beta, \nu, \sigma, \phi)]^{-1} \\ &= \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha + \beta + \nu)}{\phi^{-(\alpha+\beta+\nu)}\Gamma(\alpha + \beta)} \\ & \times {}_2F_1\left(\alpha, \alpha + \beta + \nu; \alpha + \beta; 1 - \frac{\phi}{\sigma}\right), \quad \frac{\phi}{\sigma} \leq 1 \\ &= \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha + \beta + \nu)}{\sigma^{-(\alpha+\beta+\nu)}\Gamma(\alpha + \beta)} \\ & \times {}_2F_1\left(\beta, \alpha + \beta + \nu; \alpha + \beta; 1 - \frac{\sigma}{\phi}\right), \quad \frac{\sigma}{\phi} < 1. \end{aligned}$$

From (1), by integrating suitably, the marginal densities of X and Y can be derived as (see Theorem 2.2 and Theorem 2.3),

$$K(\alpha, \beta, \nu, \sigma, \phi)\Gamma(\beta)x^{\alpha+\beta+\nu-1} \exp\left(-\frac{x}{\sigma}\right) \times \psi\left(\beta, \beta + \nu + 1; \frac{x}{\phi}\right), \quad x > 0 \quad (2)$$

and

$$K(\alpha, \beta, \nu, \sigma, \phi)\Gamma(\alpha)y^{\alpha+\beta+\nu-1} \exp\left(-\frac{y}{\phi}\right) \times \psi\left(\alpha, \alpha + \nu + 1; \frac{y}{\sigma}\right), \quad y > 0, \quad (3)$$

respectively. Observe that the densities of X and Y are not gamma and have an additional factor containing the confluent hypergeometric function and can be treated close allies of the gamma distribution. By using (1) and (2), the conditional density of $X|y$ is obtained as

$$\frac{x^{\alpha-1}y^{-(\alpha+\nu)}(y+x)^\nu \exp(-x/\sigma)}{\Gamma(\alpha)\psi(\alpha, \alpha + \nu + 1; y/\sigma)}, \quad x > 0. \quad (4)$$

Likewise, the use of (1) and (3) yields the conditional density of $Y|x$ as

$$\frac{y^{\beta-1}x^{-(\beta+\nu)}(x+y)^\nu \exp(-y/\phi)}{\Gamma(\beta)\psi(\beta, \beta + \nu + 1; x/\phi)}, \quad y > 0. \quad (5)$$

From (4) and (5), it is clear that conditional distributions of $X|y$ and $Y|x$ are Kummer-gamma. Thus, we call the bivariate distribution defined by the density (1) the *conditionally specified bivariate Kummer-gamma (CSBKG) distribution*. This distribution can also be referred to as the *bivariate Kummer-gamma conditionals distribution*. Further, we will write $(X, Y) \sim \text{CSBKG}(\alpha, \beta, \nu, \sigma, \phi)$ if the joint density of X and Y is given by (1). The CSBKG model may serve as an alternative to many existing bivariate distributions with support on $(0, \infty) \times (0, \infty)$ and can have possible applications in areas such as cross-over trials, life testing, hydrology, reliability theory, renewal processes, and stochastic routing problems (see Nadarajah and Kotz [18]).

In this article, in Section 2, we study several properties such as marginal and conditional distributions, joint moments, correlation, and mixture representation of the bivariate distribution defined by the density (1). We also derive distributions of $X + Y$ and $X/(X + Y)$, where $(X, Y) \sim \text{CSBKG}(\alpha, \beta, \nu, \sigma, \phi)$. In Section 4, entropies such as Renyi and Shannon are derived for the conditionally specified bivariate Kummer-gamma distribution defined in this article. Finally, in the Appendix, several known results used in this article are given.

2 Properties

In this section we study several properties of the conditionally specified bivariate Kummer-gamma distribution defined in Section 1.

First, we briefly discuss the shape of (1) for $\sigma = \phi$. The first derivatives of $\ln f(x, y; \alpha, \beta, \nu, \sigma, \sigma)$ with respect to x and y are

$$\begin{aligned} f_x(x, y) &= \frac{\partial \ln f(x, y; \alpha, \beta, \nu, \sigma, \sigma)}{\partial x} \\ &= \frac{\alpha - 1}{x} + \frac{\nu}{x + y} - \frac{1}{\sigma} \end{aligned} \quad (6)$$

and

$$\begin{aligned} f_y(x, y) &= \frac{\partial \ln f(x, y; \alpha, \beta, \nu, \sigma, \sigma)}{\partial y} \\ &= \frac{\beta - 1}{y} + \frac{\nu}{x + y} - \frac{1}{\sigma} \end{aligned} \quad (7)$$

respectively. Setting (6) and (7) to zero, one can compute stationary point of (1) as (a, b) , $a = (\alpha - 1)(\alpha + \beta + \nu - 2)\sigma/(\alpha + \beta - 2)$, $b = (\beta - 1)(\alpha + \beta + \nu - 2)\sigma/(\alpha + \beta - 2)$. Computing second order derivatives of $\ln f(x, y; \alpha, \beta, \nu, \sigma, \sigma)$, from (6) and (7), we get

$$\begin{aligned} f_{xx}(x, y) &= \frac{\partial^2 \ln f(x, y; \alpha, \beta, \nu, \sigma, \sigma)}{\partial x^2} \\ &= -\frac{\alpha - 1}{x^2} - \frac{\nu}{(x + y)^2}, \end{aligned} \quad (8)$$

$$f_{xy}(x, y) = \frac{\partial^2 \ln f(x, y; \alpha, \beta, \nu, \sigma, \phi)}{\partial x \partial y} = -\frac{\nu}{(x + y)^2}, \quad (9)$$

and

$$f_{yy}(x, y) = \frac{\partial^2 \ln f(x, y; \alpha, \beta, \nu, \sigma, \phi)}{\partial y^2} = -\frac{\beta - 1}{y^2} - \frac{\nu}{(x + y)^2}. \quad (10)$$

Further, from (8), (9) and (10), we get

$$f_{xx}(a, b) = -\frac{(\alpha + \beta - 2)^2 + (\alpha - 1)\nu}{(\alpha - 1)(\alpha + \beta + \nu - 2)^2 \sigma^2},$$

$$f_{xy}(a, b) = -\frac{\nu}{(\alpha + \beta + \nu - 2)^2 \sigma^2},$$

$$f_{yy}(a, b) = -\frac{(\alpha + \beta - 2)^2 + (\beta - 1)\nu}{(\beta - 1)(\alpha + \beta + \nu - 2)^2 \sigma^2}$$

and

$$\frac{f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2}{(\alpha + \beta - 2)^3} = \frac{(\alpha - 1)(\beta - 1)(\alpha + \beta + \nu - 2)^3 \sigma^4}{(\alpha - 1)(\beta - 1)(\alpha + \beta + \nu - 2)^3 \sigma^4}.$$

Now, observe that

- If $\alpha > 1, \beta > 1$ and $\alpha + \beta + \nu > 2$, then $f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2 > 0, f_{xx}(a, b) < 0$ and $f_{yy}(a, b) < 0$ and therefore (a, b) is a maximum point.
- If $\alpha > 1, \beta > 1$ and $0 < \alpha + \beta + \nu < 2$, then $f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2 < 0$, and therefore (a, b) is a saddle point.
- If $\alpha > 1, \beta < 1, \alpha + \beta < 2$ and $0 < \alpha + \beta + \nu < 2$, then $f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2 < 0$, and therefore (a, b) is a saddle point.
- If $\alpha > 1, \beta < 1, \alpha + \beta < 2$ and $\alpha + \beta + \nu > 2$, then $f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2 > 0$, and therefore (a, b) is a relative maximum.
- If $\alpha < 1, \beta > 1, \alpha + \beta < 2$ and $\alpha + \beta + \nu > 2$, then $f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2 > 0$, and therefore (a, b) is a relative maximum.

Figure 1 illustrates the shape of the pdf (1) for selected values of $\alpha, \beta, \nu, \sigma$ and ϕ . Here one can appreciate the wide range of shapes that result from the bivariate density defined by (1).

For a non-negative integer ν , we can write (1) as a linear combination of the product of gamma densities; that is

$$\sum_{j=0}^{\nu} C_j \binom{\nu}{j} f_X(x; \alpha + j, \sigma) f_Y(y; \beta + \nu - j, \phi), \quad (11)$$

where $x > 0$ and $y > 0$ with

$$C_j = K(\alpha, \beta, \nu, \sigma, \phi) \sigma^{\alpha+j} \phi^{\beta+\nu-j} \Gamma(\alpha + j) \Gamma(\beta + \nu - j).$$

Further, for $\nu = 0$, the random variables X and Y are independent each having gamma distribution. Thus, the distribution defined by the density (1) is a bivariate generalization of the gamma distribution.

A distribution is said to be negatively likelihood ratio dependent if the density $f(x, y)$ satisfies

$$f(x_1, y_1)f(x_2, y_2) \leq f(x_1, y_2)f(x_2, y_1)$$

(Lehmann [37], Tong [38]). In the case of conditionally specified bivariate Kummer-gamma distribution the above inequality reduces to

$$(x_1 + y_1)(x_2 + y_2) < (x_1 + y_2)(x_2 + y_1), \quad \nu > 0$$

which clearly holds. Hence, for $\nu > 0$, the bivariate distribution defined by the density (1) is negatively likelihood ratio dependent.

Theorem 2.1. Let $(X, Y) \sim \text{CSBKG}(\alpha, \beta, \nu, \sigma, \phi)$, and define $W = X/(X + Y)$ and $S = X + Y$. Then, the density of S is given by

$$K(\alpha, \beta, \nu, \sigma, \phi) \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} s^{\alpha+\beta+\nu-1} \exp\left(-\frac{s}{\phi}\right) \times {}_1F_1\left(\alpha; \alpha + \beta; \left(\frac{1}{\phi} - \frac{1}{\sigma}\right)s\right), \quad s > 0$$

and the density of W is given by

$$K(\alpha, \beta, \nu, \sigma, \phi) \Gamma(\alpha + \beta + \nu) \times \frac{w^{\alpha-1}(1-w)^{\beta-1}}{[w/\sigma + (1-w)/\phi]^{\alpha+\beta+\nu}}, \quad 0 < w < 1,$$

where ${}_1F_1$ is the confluent hypergeometric function of the first kind.

Proof. Substituting $x = ws$ and $y = s(1 - w)$ with the Jacobian $J(x, y \rightarrow w, s) = s$, in the joint density of X and Y , we obtain the joint density of W and S as

$$K(\alpha, \beta, \nu, \sigma, \phi) s^{\alpha+\beta+\nu-1} w^{\alpha-1} (1-w)^{\beta-1} \times \exp\left[-\left(\frac{w}{\sigma} + \frac{1-w}{\phi}\right)s\right], \quad (12)$$

where $0 < s < 1$ and $0 < w < 1$. Now, integrating appropriately by using the integral representation of confluent hypergeometric function (A.6), we obtain marginal densities of S and W . \square

Corollary 2.1.1. The density of $R = X/Y$ is given by

$$K(\alpha, \beta, \nu, \sigma, \phi) \Gamma(\alpha + \beta + \nu) \times \frac{r^{\alpha-1}(1+r)^\nu}{(r/\sigma + 1/\phi)^{\alpha+\beta+\nu}}, \quad r > 0.$$

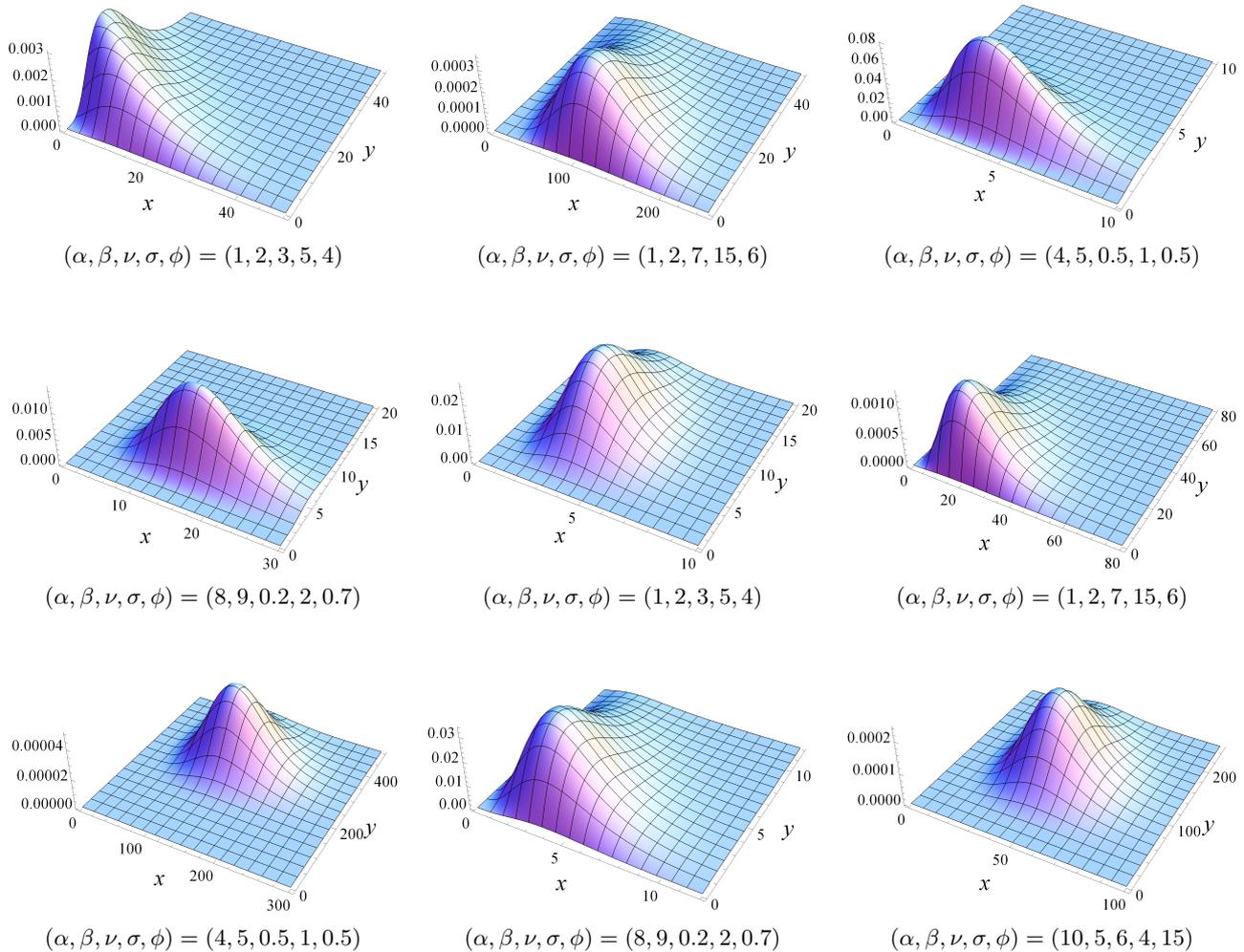


Figure 1: Plots of the pdf (1) for some selected values of parameters.

Proof. Use the transformation $W = R/(1+R)$ in the density of W given in the Theorem 2.1. \square

Corollary 2.1.2. If $\sigma = \phi$, then, the density of S is given by

$$\frac{s^{\alpha+\beta+\nu-1} \exp(-s/\sigma)}{\sigma^{\alpha+\beta+\nu} \Gamma(\alpha+\beta)}, s > 0.$$

The density of W is given by

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} w^{\alpha-1} (1-w)^{\beta-1}, 0 < w < 1$$

and the density of R is

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{r^{\alpha-1}}{(1+r)^{\alpha+\beta}}, r > 0.$$

Proof. Substitute $\sigma = \phi$ in the Theorem 2.1 and Corollary 2.1.1. \square

Corollary 2.1.3. Let X and Y be independent $X \sim Ga(\alpha, \sigma)$ and $Y \sim Ga(\beta, \phi)$. Then, the density of S is given by

$$\frac{s^{\alpha+\beta-1}}{\sigma^\alpha \phi^\beta \Gamma(\alpha+\beta)} \exp\left(-\frac{s}{\phi}\right) \times {}_1F_1\left(\alpha; \alpha+\beta; \left(\frac{1}{\phi} - \frac{1}{\sigma}\right) s\right), s > 0.$$

The density of W is given by

$$\frac{\Gamma(\alpha+\beta)}{\sigma^\alpha \phi^\beta \Gamma(\alpha)\Gamma(\beta)} \frac{w^{\alpha-1} (1-w)^{\beta-1}}{[w/\sigma + (1-w)/\phi]^{\alpha+\beta}}, 0 < w < 1$$

and the density of R is

$$\frac{\Gamma(\alpha+\beta)}{\sigma^\alpha \phi^\beta \Gamma(\alpha)\Gamma(\beta)} \frac{r^{\alpha-1}}{(r/\sigma + 1/\phi)^{\alpha+\beta}}, r > 0.$$

Proof. Substitute $\nu = 0$ in the Theorem 2.1 and the Corollary 2.1.1. \square

By using the above theorem and (A.5), it is straightforward to show that

$$E(S^r) = \frac{\phi^r \Gamma(\alpha + \beta + \nu + r)}{\Gamma(\alpha + \beta + \nu)} \times \frac{{}_2F_1(\alpha, \alpha + \beta + \nu + r; \alpha + \beta; 1 - \phi/\sigma)}{{}_2F_1(\alpha, \alpha + \beta + \nu; \alpha + \beta; 1 - \phi/\sigma)},$$

if $\phi \leq \sigma$ and

$$E(S^r) = \frac{\sigma^r \Gamma(\alpha + \beta + \nu + r)}{\Gamma(\alpha + \beta + \nu)} \times \frac{{}_2F_1(\alpha, \alpha + \beta + \nu + r; \alpha + \beta; 1 - \sigma/\phi)}{{}_2F_1(\alpha, \alpha + \beta + \nu; \alpha + \beta; 1 - \sigma/\phi)},$$

if $\phi > \sigma$. Further, by using the density of W given in Theorem 2.1, we derive $E(W^r)$ as

$$E(W^r) = K(\alpha, \beta, \nu, \sigma, \phi) \Gamma(\alpha + \beta + \nu) \times \int_0^1 \frac{w^{\alpha+r-1} (1-w)^{\beta-1}}{[w/\sigma + (1-w)/\phi]^{\alpha+\beta+\nu}} dw.$$

Now, writing

$$\frac{w}{\sigma} + \frac{1-w}{\phi} = \begin{cases} [1 - (1 - \frac{\phi}{\sigma})w]^{\frac{1}{\phi}}, & \frac{\phi}{\sigma} \leq 1 \\ [1 - (1 - \frac{\sigma}{\phi})(1-w)]^{\frac{1}{\sigma}}, & \frac{\sigma}{\phi} < 1, \end{cases}$$

and integrating w by using (A.7), we get

$$E(W^r) = \frac{\Gamma(\alpha + r) \Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\alpha + \beta + r)} \times \frac{{}_2F_1(\alpha + r, \alpha + \beta + \nu + r; \alpha + \beta; 1 - \phi/\sigma)}{{}_2F_1(\alpha, \alpha + \beta + \nu; \alpha + \beta; 1 - \phi/\sigma)}, \quad \phi/\sigma \leq 1,$$

$$= \frac{\Gamma(\beta + r) \Gamma(\alpha + \beta)}{\Gamma(\beta) \Gamma(\alpha + \beta + r)} \times \frac{{}_2F_1(\beta + r, \alpha + \beta + \nu + r; \alpha + \beta; 1 - \sigma/\phi)}{{}_2F_1(\beta, \alpha + \beta + \nu; \alpha + \beta; 1 - \sigma/\phi)}, \quad \phi/\sigma > 1.$$

In next two theorems, we derive marginal distributions of X and Y . It is interesting to note that these marginal distributions do not belong to the gamma family and differ by an additional factor containing the confluent hypergeometric function ψ . Figure 2 and Figure 3 illustrate the shape of the marginal density of X for selected of $\alpha, \beta, \nu, \sigma$ and ϕ . It can be observed that, for certain values of parameters, the marginal density of X tends to symmetry. A graphical comparison of the marginal density of X with the gamma density is also given in Figure 2 and Figure 3. It can also be seen that, compared to the gamma density, the curves of the marginal density of X are taller and have thinner tails. Further, the median of the marginal density of X is smaller than that of the gamma density.

Theorem 2.2. If $(X, Y) \sim \text{CSBKG}(\alpha, \beta, \nu, \sigma, \phi)$, then the marginal density of X is given by

$$K(\alpha, \beta, \nu, \sigma, \phi) \Gamma(\beta) x^{\alpha+\beta+\nu-1} \exp\left(-\frac{x}{\sigma}\right) \times \psi\left(\beta, \beta + \nu + 1, \frac{x}{\phi}\right), \quad x > 0,$$

where ψ is the confluent hypergeometric function of the second kind.

Proof. To find the marginal density of X , we integrate (1) with respect to y to get

$$K(\alpha, \beta, \nu, \sigma, \phi) x^{\alpha-1} \exp\left(-\frac{x}{\sigma}\right) \times \int_0^\infty y^{\beta-1} (x+y)^\nu \exp\left(-\frac{y}{\phi}\right) dy.$$

Substituting $z = y/x$ with $dy = x dz$ above, one obtains

$$K(\alpha, \beta, \nu, \sigma, \phi) x^{\alpha+\beta+\nu-1} \exp\left(-\frac{x}{\sigma}\right) \times \int_0^\infty z^{\beta-1} (1+z)^\nu \exp\left(-\frac{xz}{\phi}\right) dz.$$

Now, the desired result is obtained by using (A.8). \square

Theorem 2.3. If $(X, Y) \sim \text{CSBKG}(\alpha, \beta, \nu, \sigma, \phi)$, then the marginal density of Y is given by

$$K(\alpha, \beta, \nu, \sigma, \phi) \Gamma(\alpha) y^{\alpha+\beta+\nu-1} \exp\left(-\frac{y}{\phi}\right) \times \psi\left(\alpha, \alpha + \nu + 1, \frac{y}{\sigma}\right), \quad y > 0.$$

Proof. Similar to the proof of the Theorem 2.2. \square

Using the above theorem, the conditional density function of X given $Y = y, y > 0$, is obtained as

$$\frac{\exp(-x/\sigma) x^{\alpha-1} (x+y)^\nu}{\Gamma(\alpha) y^{\alpha+\nu} \psi(\alpha, \alpha + \nu + 1, y/\sigma)}, \quad x > 0.$$

Similarly, using Theorem 2.2, the conditional density function of Y given $X = x, x > 0$, is derived as

$$\frac{\exp(-y/\phi) y^{\beta-1} (x+y)^\nu}{\Gamma(\beta) x^{\beta+\nu} \psi(\beta, \beta + \nu + 1, x/\phi)}, \quad y > 0.$$

Further, using conditional densities given above, we derive

$$E(X^r | y) = \frac{y^r \Gamma(\alpha + r) \psi(\alpha + r, \alpha + \nu + r + 1, y/\sigma)}{\Gamma(\alpha) \psi(\alpha, \alpha + \nu + 1, y/\sigma)}$$

and

$$E(Y^r | x) = \frac{x^r \Gamma(\beta + r) \psi(\beta + r, \alpha + \nu + r + 1, x/\phi)}{\Gamma(\beta) \psi(\beta, \beta + \nu + 1, x/\phi)}.$$

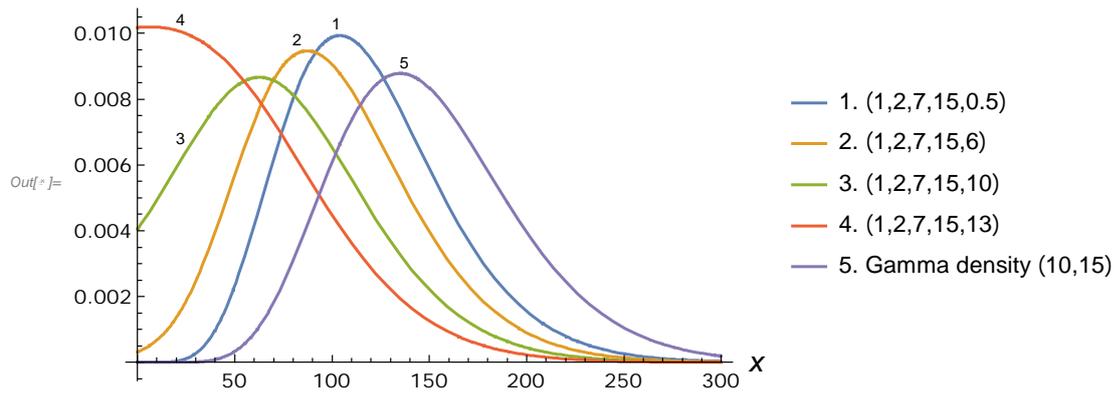


Figure 2: Comparison of the marginal density of X with the gamma density. Plots 1, 2, 3 and 4 are drawn by using the density of X given in Theorem 2.2 for $(\alpha, \beta, \nu, \sigma, \phi) = (1, 2, 7, 15, \phi)$ with $\phi/\sigma < 1$, and the plot of the gamma density with parameters $(1 + 2 + 7, 15)$ is given by 5.

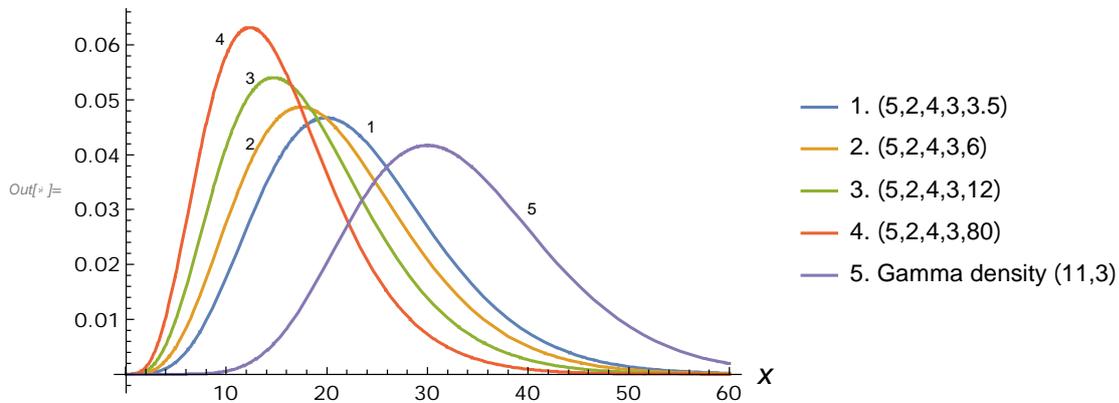


Figure 3: comparison of the marginal density of X with the gamma density. Plots 1, 2, 3 and 4 are drawn by using the density of X given in theorem 2.2 for $(\alpha, \beta, \nu, \sigma, \phi) = (5, 2, 4, 3, \phi)$ with $\sigma/\phi < 1$, and the plot of the gamma density with parameters $(5 + 2 + 4, 3)$ is given by 5.

Also, using (1), the joint (r, s) -th moment is obtained as

$$E(X^r Y^s) = K(\alpha, \beta, \nu, \sigma, \phi) \int_0^\infty \int_0^\infty x^{\alpha+r-1} y^{\beta+s-1} (x+y)^\nu \exp\left[-\left(\frac{x}{\sigma} + \frac{y}{\phi}\right)\right] dy dx$$

$$= \frac{K(\alpha, \beta, \nu, \sigma, \phi)}{K(\alpha + r, \beta + s, \nu, \sigma, \phi)}$$

$$= \sigma^{r+s} \frac{\Gamma(\alpha + r)\Gamma(\beta + s)\Gamma(\alpha + \beta + \nu + r + s)\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha + \beta + \nu)\Gamma(\alpha + \beta + r + s)} \times \frac{{}_2F_1(\beta + s, \alpha + \beta + \nu + r + s; \alpha + \beta + r + s; 1 - \sigma/\phi)}{{}_2F_1(\beta, \alpha + \beta + \nu; \alpha + \beta; 1 - \sigma/\phi)},$$

$\sigma/\phi < 1,$

where $\alpha + r > 0, \beta + s > 0$ and $\alpha + \beta + \nu + r + s > 0$. Now, substituting appropriately, we obtain, for $\phi/\sigma \leq 1,$

Further, substituting for $K(\alpha, \beta, \nu, \sigma, \phi)$, one gets

$$E(X^r Y^s) = \phi^{r+s} \frac{\Gamma(\alpha + r)\Gamma(\beta + s)\Gamma(\alpha + \beta + \nu + r + s)\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha + \beta + \nu)\Gamma(\alpha + \beta + r + s)} \times \frac{{}_2F_1(\alpha + r, \alpha + \beta + \nu + r + s; \alpha + \beta + r + s; 1 - \phi/\sigma)}{{}_2F_1(\alpha, \alpha + \beta + \nu; \alpha + \beta; 1 - \phi/\sigma)},$$

$\phi/\sigma \leq 1,$

$$E[(XY)^h] = \phi^{2h} \frac{\Gamma(\alpha + h)\Gamma(\beta + h)\Gamma(\alpha + \beta + \nu + 2h)\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha + \beta + \nu)\Gamma(\alpha + \beta + 2h)} \times \frac{{}_2F_1(\alpha + h, \alpha + \beta + \nu + 2h; \alpha + \beta + 2h; 1 - \phi/\sigma)}{{}_2F_1(\alpha, \alpha + \beta + \nu; \alpha + \beta; 1 - \phi/\sigma)},$$

$$E(X) = \phi \frac{\alpha(\alpha + \beta + \nu)}{\alpha + \beta} \times \frac{{}_2F_1(\alpha + 1, \alpha + \beta + \nu + 1; \alpha + \beta + 1; 1 - \phi/\sigma)}{{}_2F_1(\alpha, \alpha + \beta + \nu; \alpha + \beta; 1 - \phi/\sigma)},$$

$$E(Y) = \phi \frac{\beta(\alpha + \beta + \nu)}{\alpha + \beta} \times \frac{{}_2F_1(\alpha, \alpha + \beta + \nu + 1; \alpha + \beta + 1; 1 - \phi/\sigma)}{{}_2F_1(\alpha, \alpha + \beta + \nu; \alpha + \beta; 1 - \phi/\sigma)},$$

$$E(X^2) = \phi^2 \frac{\alpha(\alpha + 1)(\alpha + \beta + \nu)(\alpha + \beta + \nu + 1)}{(\alpha + \beta)(\alpha + \beta + 1)} \times \frac{{}_2F_1(\alpha + 2, \alpha + \beta + \nu + 2; \alpha + \beta + 2; 1 - \phi/\sigma)}{{}_2F_1(\alpha, \alpha + \beta + \nu; \alpha + \beta; 1 - \phi/\sigma)},$$

$$E(Y^2) = \phi^2 \frac{\beta(\beta + 1)(\alpha + \beta + \nu)(\alpha + \beta + \nu + 1)}{(\alpha + \beta)(\alpha + \beta + 1)} \times \frac{{}_2F_1(\alpha, \alpha + \beta + \nu + 2; \alpha + \beta + 2; 1 - \phi/\sigma)}{{}_2F_1(\alpha, \alpha + \beta + \nu; \alpha + \beta; 1 - \phi/\sigma)},$$

$$E(XY) = \phi^2 \frac{\alpha\beta(\alpha + \beta + \nu)(\alpha + \beta + \nu + 1)}{(\alpha + \beta)(\alpha + \beta + 1)} \times \frac{{}_2F_1(\alpha + 1, \alpha + \beta + \nu + 2; \alpha + \beta + 2; 1 - \phi/\sigma)}{{}_2F_1(\alpha, \alpha + \beta + \nu; \alpha + \beta; 1 - \phi/\sigma)},$$

$$\text{Var}(X) = \phi^2 \frac{\alpha(\alpha + \beta + \nu)}{\alpha + \beta} \left[\frac{(\alpha + 1)(\alpha + \beta + \nu + 1)}{\alpha + \beta + 1} \times \frac{{}_2F_1(\alpha + 2, \alpha + \beta + \nu + 2; \alpha + \beta + 2; 1 - \phi/\sigma)}{{}_2F_1(\alpha, \alpha + \beta + \nu; \alpha + \beta; 1 - \phi/\sigma)} - \frac{\alpha(\alpha + \beta + \nu)}{\alpha + \beta} \times \left\{ \frac{{}_2F_1(\alpha + 1, \alpha + \beta + \nu + 1; \alpha + \beta + 1; 1 - \phi/\sigma)}{{}_2F_1(\alpha, \alpha + \beta + \nu; \alpha + \beta; 1 - \phi/\sigma)} \right\}^2 \right],$$

$$\text{Var}(Y) = \phi^2 \frac{\beta(\alpha + \beta + \nu)}{\alpha + \beta} \left[\frac{(\beta + 1)(\alpha + \beta + \nu + 1)}{\alpha + \beta + 1} \times \frac{{}_2F_1(\alpha, \alpha + \beta + \nu + 2; \alpha + \beta + 2; 1 - \phi/\sigma)}{{}_2F_1(\alpha, \alpha + \beta + \nu; \alpha + \beta; 1 - \phi/\sigma)} - \frac{\beta(\alpha + \beta + \nu)}{\alpha + \beta} \times \left\{ \frac{{}_2F_1(\alpha, \alpha + \beta + \nu + 1; \alpha + \beta + 1; 1 - \phi/\sigma)}{{}_2F_1(\alpha, \alpha + \beta + \nu; \alpha + \beta; 1 - \phi/\sigma)} \right\}^2 \right],$$

and

$$\text{Cov}(X, Y) = \phi^2 \frac{\alpha\beta(\alpha + \beta + \nu)}{\alpha + \beta} \left[\frac{\alpha + \beta + \nu + 1}{\alpha + \beta + 1} \times \frac{{}_2F_1(\alpha + 1, \alpha + \beta + \nu + 2; \alpha + \beta + 2; 1 - \phi/\sigma)}{{}_2F_1(\alpha, \alpha + \beta + \nu; \alpha + \beta; 1 - \phi/\sigma)} - \frac{\alpha + \beta + \nu}{\alpha + \beta} \times \frac{{}_2F_1(\alpha + 1, \alpha + \beta + \nu + 1; \alpha + \beta + 1; 1 - \phi/\sigma)}{{}_2F_1(\alpha, \alpha + \beta + \nu; \alpha + \beta; 1 - \phi/\sigma)} \times \frac{{}_2F_1(\alpha, \alpha + \beta + \nu + 1; \alpha + \beta + 1; 1 - \phi/\sigma)}{{}_2F_1(\alpha, \alpha + \beta + \nu; \alpha + \beta; 1 - \phi/\sigma)} \right].$$

Notice that $E(XY)$, $E(X^2)$, $E(Y^2)$, $E(X)$ and $E(Y)$ involve ${}_2F_1(a, b; c; z)$ which can be computed by using a suitable software. Tables for correlations between X and Y can also be prepared by using the definition

$$\text{corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X) \text{Var}(Y)}}.$$

3 Entropies

In this section, Renyi and Shannon entropies are derived for the conditionally specified bivariate Kummer-gamma distribution defined in this article.

Let $(\mathcal{X}, \mathcal{B}, \mathcal{P})$ be a probability space. Consider a pdf f associated with \mathcal{P} , dominated by σ -finite measure μ on \mathcal{X} . The well-known Shannon entropy $H_{SH}(f)$ introduced by Shannon [39] is define by

$$H_{SH}(f) = - \int_{\mathcal{X}} f(x) \log f(x) d\mu. \quad (13)$$

One of the main extensions of the Shannon entropy was defined by Rényi [40]. This generalized entropy measure is given by

$$H_R(\eta, f) = \frac{\log G(\eta)}{1 - \eta} \quad (\text{for } \eta > 0 \text{ and } \eta \neq 1), \quad (14)$$

where

$$G(\eta) = \int_{\mathcal{X}} f^\eta d\mu.$$

The additional parameter η is used to describe complex behavior in probability models and the associated process under study. Rényi entropy is monotonically decreasing in η , while the Shannon entropy (13) is obtained from (14) for $\eta \uparrow 1$. These entropies have been used in information theory, science and engineering. For details see Nadarajah and Zografos [41], Zografos [42], and Zografos and Nadarajah [43].

First, we give the following lemma useful in deriving these entropies.

Lemma 3.1. Let $g(\alpha, \beta, \nu, \sigma, \phi) = \lim_{\eta \rightarrow 1} h(\eta)$, where

$$h(\eta) = \frac{d}{d\eta} {}_2F_1\left(\eta(\alpha - 1) + 1, \eta(\alpha + \beta + \nu - 2) + 2; \eta(\alpha + \beta - 2) + 2; 1 - \frac{\phi}{\sigma}\right).$$

with $\phi/\sigma < 1$. Then,

$$g(\alpha, \beta, \nu, \sigma, \phi) = \sum_{j=1}^{\infty} \frac{\Gamma(\alpha + j)\Gamma(\alpha + \beta + \nu + j)\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\alpha + \beta + \nu)\Gamma(\alpha + \beta + j)} \frac{(1 - \phi/\sigma)^j}{j!} \left[(\alpha - 1)\psi(\alpha + j) + (\alpha + \beta + \nu - 2)\psi(\alpha + \beta + \nu + j) + (\alpha + \beta - 2)\psi(\alpha + \beta) - (\alpha - 1)\psi(\alpha) - (\alpha + \beta + \nu - 2)\psi(\alpha + \beta + \nu) - (\alpha + \beta - 2)\psi(\alpha + \beta + j) \right], \quad (15)$$

where $\psi(\alpha) = \Gamma'(\alpha)/\Gamma(\alpha)$ is the digamma function.

Proof. Expanding ${}_2F_1$ in series form, we write

$$h(\eta) = \frac{d}{d\eta} \sum_{j=0}^{\infty} \Delta_j(\eta) \frac{(1 - \phi/\sigma)^j}{j!} = \sum_{j=0}^{\infty} \left[\frac{d}{d\eta} \Delta_j(\eta) \right] \frac{(1 - \phi/\sigma)^j}{j!}, \quad (16)$$

where

$$\Delta_j(\eta) = \frac{\Gamma[\eta(\alpha - 1) + 1 + j]}{\Gamma[\eta(\alpha - 1) + 1]} \times \frac{\Gamma[\eta(\alpha + \beta + \nu - 2) + 2 + j]}{\Gamma[\eta(\alpha + \beta + \nu - 2) + 2]} \times \frac{\Gamma[\eta(\alpha + \beta - 2) + 2]}{\Gamma[\eta(\alpha + \beta - 2) + 2 + j]}.$$

Now, differentiating the logarithm of $\Delta_j(\eta)$ w.r.t. to η , one obtains

$$\begin{aligned} & \frac{d}{d\eta} \Delta_j(\eta) \\ &= \Delta_j(\eta) \left[(\alpha - 1)\psi(\eta(\alpha - 1) + 1 + j) + (\alpha + \beta + \nu - 2)\psi(\eta(\alpha + \beta + \nu - 2) + 2 + j) + (\alpha + \beta - 2)\psi(\eta(\alpha + \beta - 2) + 2) - (\alpha - 1)\psi(\eta(\alpha - 1) + 1) - (\alpha + \beta + \nu - 2)\psi(\eta(\alpha + \beta + \nu - 2) + 2) - (\alpha + \beta - 2)\psi(\eta(\alpha + \beta - 2) + 2 + j) \right]. \quad (17) \end{aligned}$$

Finally, substituting (17) in (16) and taking $\eta \rightarrow 1$, one obtains the desired result. \square

Theorem 3.1. For the bivariate distribution defined by the pdf (1), the Rényi and the Shannon entropies are given by

$$H_R(\eta, f) = \frac{1}{1 - \eta} \left[\eta \ln K(\alpha, \beta, \nu, \sigma, \phi) + [\eta(\alpha + \beta + \nu - 2) + 2] \ln\left(\frac{\phi}{\eta}\right) + \ln \Gamma[\eta(\alpha - 1) + 1] + \ln \Gamma[\eta(\beta - 1) + 1] + \ln \Gamma[\eta(\alpha + \beta + \nu - 2) + 2] - \ln \Gamma[\eta(\alpha + \beta - 2) + 2] + \ln {}_2F_1\left(\eta(\alpha - 1) + 1, \eta(\alpha + \beta + \nu - 2) + 2; \eta(\alpha + \beta - 2) + 2; 1 - \frac{\phi}{\sigma}\right) \right]$$

and

$$H_{SH}(f) = -\ln K(\alpha, \beta, \nu, \sigma, \phi) - [(\alpha - 1)\psi(\alpha) + (\beta - 1)\psi(\beta) + (\alpha + \beta + \nu - 2)\psi(\alpha + \beta + \nu) - (\alpha + \beta - 2)\psi(\alpha + \beta)] - \frac{g(\alpha, \beta, \nu, \sigma, \phi)}{{}_2F_1(\alpha, \alpha + \beta + \nu; \alpha + \beta; 1 - \phi/\sigma)},$$

respectively, where $\psi(\alpha) = \Gamma'(\alpha)/\Gamma(\alpha)$ is the digamma function and $g(\alpha, \beta, \nu, \sigma, \phi)$ is given by (15).

Proof. For $\eta > 0$ and $\eta \neq 1$, using the joint density of X and Y given by (1), we have

$$\begin{aligned} G(\eta) &= \int_0^{\infty} \int_0^{\infty} f^{\eta}(x, y; \alpha, \beta, \nu, \sigma, \phi) dy dx \\ &= [K(\alpha, \beta, \nu, \sigma, \phi)]^{\eta} \times \int_0^{\infty} \int_0^{\infty} x^{\eta(\alpha-1)} y^{\eta(\beta-1)} (x + y)^{\eta\nu} \exp\left[-\eta\left(\frac{x}{\sigma} + \frac{y}{\phi}\right)\right] dy dx \\ &= \frac{[K(\alpha, \beta, \nu, \sigma, \phi)]^{\eta}}{K(\eta(\alpha - 1) + 1, \eta(\beta - 1) + 1, \eta\nu, \sigma/\eta, \phi/\eta)} \\ &= [K(\alpha, \beta, \nu, \sigma, \phi)]^{\eta} \frac{\Gamma[\eta(\alpha + \beta + \nu - 2) + 2]}{\Gamma[\eta(\alpha + \beta - 2) + 2]} \times \frac{\Gamma[\eta(\alpha - 1) + 1]\Gamma[\eta(\beta - 1) + 1]}{(\phi/\eta)^{-\eta(\alpha + \beta + \nu - 2) - 2}} \\ &\quad \times {}_2F_1\left(\eta(\alpha - 1) + 1, \eta(\alpha + \beta + \nu - 2) + 2; \eta(\alpha + \beta - 2) + 2; 1 - \frac{\phi}{\sigma}\right), \end{aligned}$$

where the last line has been obtained by using (A.9). Now, taking logarithm of $G(\eta)$ and using (14) we get $H_R(\eta, f)$. The Shannon entropy is obtained from $H_R(\eta, f)$ by taking $\eta \uparrow 1$ and using L'Hopital's rule. \square

4 Conclusion

We have considered a bivariate distribution of positive random variables X and Y defined by (1). It has been shown that conditional distributions of $X|y$ and $Y|x$ are Kummer-gamma thereby naming the bivariate distribution defined by (1) the *conditionally specified bivariate Kummer-gamma (CSBKG) distribution*. The CSBKG distribution may serve as an alternative to many existing bivariate distributions defined on $(0, \infty) \times (0, \infty)$. By using standard definitions and results, several properties of this distribution have also been derived. The distributions of $X + Y$ and $X/(X + Y)$ have also been obtained by using transformation of variables. It is interesting to note that most of the results (including the normalizing constant) derived in this article involve the well know Gauss' hypergeometric function/confluent hypergeometric function studied extensively in the literature making the CSBKG model mathematically and statistically interesting. The matrix variate generalization of the CSBKG model is also a captivating topic to explore.

Appendix

The Pochhammer symbol $(a)_n$ is defined by $(a)_n = a(a + 1) \cdots (a + n - 1) = (a)_{n-1}(a + n - 1)$ for $n = 1, 2, \dots$, and $(a)_0 = 1$. The generalized hypergeometric function of scalar argument is defined by

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!}, \quad (A.1)$$

where $a_i, i = 1, \dots, p; b_j, j = 1, \dots, q$ are complex numbers with suitable restrictions and z is a complex variable. Conditions for the convergence of the series in (A.1) are available in the literature, see Luke [32]. From (A.1) it is easy to see that

$${}_1F_1(a; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} \frac{z^k}{k!}, \quad (A.2)$$

and

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \quad |z| < 1. \quad (A.3)$$

Also, under suitable conditions, we have (Luke [32, Eq. 3.6(10)]),

$$\int_0^1 z^{\alpha-1} (1-z)^{\beta-1} {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; zy) dz = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \times {}_{p+1}F_{q+1}(a_1, \dots, a_p, \alpha; b_1, \dots, b_q, \alpha+\beta; y). \quad (A.4)$$

and (Luke [32, Eq. 3.6(13)]),

$$\int_0^{\infty} \exp(-\delta z) z^{\alpha-1} {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; zy) dz = \delta^{-\alpha} \Gamma(\alpha) \times {}_{p+1}F_q(a_1, \dots, a_p, \alpha; b_1, \dots, b_q; \delta^{-1}y). \quad (A.5)$$

The integral representations of the confluent hypergeometric function (first kind) and the Gauss hypergeometric function are given as

$${}_1F_1(a; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \times \int_0^1 t^{a-1} (1-t)^{c-a-1} \exp(zt) dt, \quad (A.6)$$

and

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \times \int_0^1 \frac{t^{a-1} (1-t)^{c-a-1}}{(1-zt)^b} dt, \quad (A.7)$$

respectively, where $\text{Re}(a) > 0$ and $\text{Re}(c-a) > 0$. Note that, the series expansions for ${}_1F_1$ and ${}_2F_1$ given in (A.2) and (A.3) can be obtained by expanding $\exp(zt)$ and $(1-zt)^{-b}$, $|zt| < 1$, in (A.6) and (A.7) and integrating t .

The integral representations of the confluent hypergeometric function of the second kind is defined by the integral

$$\psi(a, b, z) = \frac{1}{\Gamma(a)} \int_0^{\infty} \exp(-zt) t^{a-1} (1+t)^{b-a-1} dt, \quad \text{Re}(a) > 0. \quad (A.8)$$

For properties and further results on these functions the reader is referred to Luke [32].

Lemma A.1. *Let*

$$C(\alpha, \beta, \nu, \sigma, \phi) = \int_0^{\infty} \int_0^{\infty} x^{\alpha-1} y^{\beta-1} (x+y)^{\nu} \exp\left[-\left(\frac{x}{\sigma} + \frac{y}{\phi}\right)\right] dx dy, \quad (A.9)$$

where $\alpha > 0, \beta > 0, \sigma > 0, \phi > 0$ and $\alpha + \beta + \nu > 0$. Then

$$C(\alpha, \beta, \nu, \sigma, \phi) = \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta+\nu)}{\phi^{-(\alpha+\beta+\nu)}\Gamma(\alpha+\beta)} \times {}_2F_1\left(\alpha, \alpha+\beta+\nu; \alpha+\beta; 1-\frac{\phi}{\sigma}\right)$$

for $\phi/\sigma < 1$ and

$$C(\alpha, \beta, \nu, \sigma, \phi) = \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta+\nu)}{\sigma^{-(\alpha+\beta+\nu)}\Gamma(\alpha+\beta)} \times {}_2F_1\left(\beta, \alpha+\beta+\nu; \alpha+\beta; 1-\frac{\sigma}{\phi}\right)$$

for $\sigma/\phi < 1$.

Proof. Substituting $s = x + y$ and $r = x/(x + y)$ with $dx dy = s ds dr$ in (A.9) and integrating s , one gets

$$C(\alpha, \beta, \nu, \sigma, \phi) = \Gamma(\alpha + \beta + \nu) \int_0^1 r^{\alpha-1} (1-r)^{\beta-1} \left(\frac{r}{\sigma} + \frac{1-r}{\phi} \right)^{-(\alpha+\beta+\nu)} dr. \quad (A.10)$$

Now, writing

$$\left(\frac{r}{\sigma} + \frac{1-r}{\phi} \right)^{-(\alpha+\beta+\nu)} = \phi^{\alpha+\beta+\nu} \left[1 - r \left(1 - \frac{\phi}{\sigma} \right) \right]^{-(\alpha+\beta+\nu)}, \quad \frac{\phi}{\sigma} < 1$$

and integrating r by using (A.7) we get the result. Further if $\phi/\sigma > 1$, then $\sigma/\phi < 1$ and we write

$$\left(\frac{r}{\sigma} + \frac{1-r}{\phi} \right)^{-(\alpha+\beta+\nu)} = \sigma^{\alpha+\beta+\nu} \left[1 - (1-r) \left(1 - \frac{\sigma}{\phi} \right) \right]^{-(\alpha+\beta+\nu)}, \quad \frac{\sigma}{\phi} < 1$$

and (A.10) becomes

$$C(\alpha, \beta, \nu, \sigma, \phi) = \sigma^{\alpha+\beta+\nu} \Gamma(\alpha + \beta + \nu) \int_0^1 r^{\alpha-1} (1-r)^{\beta-1} \left[1 - (1-r) \left(1 - \frac{\sigma}{\phi} \right) \right]^{-(\alpha+\beta+\nu)} dr.$$

Now, application of (A.7) yield the desired result. □

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