Pairwise Strongly Lindelöf, Pairwise Nearly, Almost and Weakly Lindelöf Bitopological Spaces

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Abstract: - The main purposes of this article is to introduce new generalizations of the notion of pairwise Lindelöf spaces in bitopological spaces where new notions: pairwise strongly Lindelöf, pairwise nearly, pairwise almost and pairwise weakly strongly Lindelöf bitopological spaces depend on the new notion pairwise preopen countable covers. These covers where we focused on their importance in topology consist of countable subfamilies whose closures cover the bitopological spaces and we clarified how pairwise preopen countable covers effect on pairwise strongly Lindelöf spaces. The new concepts of pairwise strongly Lindelöf, pairwise nearly, pairwise almost and pairwise weakly strongly Lindelöf bitopological spaces are introduced and many definitions, propositions, characterizations and remarks concerning those notions are initiated, discussed and explored. Furthermore, the relationships between those bitopological spaces are examined and investigated. We illustrated the implications hold by these new bitopological spaces. We put some queries and claims, then we struggle to provide their proofs.

Key-Words: - Pairwise Strongly Lindelöf, Pairwise Almost Strongly Lindelöf.

Received: January 16, 2021. Revised: April 1, 2021. Accepted: April 5, 2021. Published: April 9, 2021.

1 Introduction

The study of bitopological spaces was initiated by J.C.Kelly [1] in 1963 in an article in London mathematical society titled “Bitopological spaces” and thereafter a lot of articles were proposed to generalize topological concepts to bitopological ones. Typically, If X is a non-empty set endowed with two different topologies τ_1 and τ_2 on it, then the triple (X,τ_1,τ_2) or simply X is said to be a bitopological space. Mashhour [2] initiated a topological space named strongly compact (resp. Lindelöf) in which every preopen cover of a space has a finite (resp. countable) subcover. After there, many articles discussing such spaces introduced by Ganster [3], Omari, Noiri and Noorani [4]. In addition, Hdeib and Sarsak [5] proposed new results concerning those notions. Examining the relationship between strongly compact spaces and strongly lindelöf spaces reveals a lot of class of such spaces. In a topological space (X,τ), a subset U is called preopen [6] if it is contained in the interior of its closure, hence every open subset of a topological space is preopen. Moreover, every dense subset of a topological space is preopen.

Ganster [7] showed that a subset U of X is preopen if it is the intersection of an open and dense subset. Few years later, Ganster [3] introduced the notion of d-Lindelöf spaces. In this space, each cover of a space X by dense subsets has a countable subcover. He pointed out that a topological space X is d- Lindelöf if and only if $X-I_X$ is countable where I_X is the set of all isolated points of a space X, or equivalently, $\overline{U}-U$ is countable $\forall U\subseteq X$. Typically, every strongly Lindelöf space is Lindelöf. Hdeib and Sarsak [8] called a subset U of a topological space (X,τ) strongly Lindelöf relative to X if each preopen cover of U admits a countable subcover of U. Some relevant studies can be found in [20] and [21].

2. Preliminaries

Salleh in his paper [9] called a subset U of X in the bitopological space (X, τ_1, τ_2) an (i,j) −preopen (resp. (i,j) −preclosed) if
$U \subseteq i - int\big(j - cl(U)\big)$ (resp. $i - cl\big(j - int(U)\big)$ $\forall i,j = 1,2 \quad i \neq j$.
If U is $(1,2)$ −preopen and $(2,1)$ −preopen, then U is pairwise preopen. The complement of a pairwise preopen subset is pairwise preclosed. Furthermore, Salleh [16] called a subset O of a bitopological space (X, τ_1, τ_2) an (i,j) −regular open (resp. (i,j) −regular closed) if
$O = i - int(j - cl(O))$ (resp. $O = i - cl(j - int(O))$. If O is $(1,2)$ −regular open and $(2,1)$ −regular open, then O is pairwise regular open. The complement of a pairwise regular open is pairwise regular closed. A subset A of a bitopological space (X, τ_1, τ_2) is said to be (i,j) −semi-open (resp. (i,j) −semi-closed) if
$A \subseteq i - cl(j - int(A))$ (resp. $A \subseteq i - int(j - cl(A))$ see [11]. If A is $(1,2)$ −semi-open and $(2,1)$ −semi-open, then A is a pairwise semi-open. The complement of a pairwise semi-open subset is pairwise semi-closed.

3.Pairwise Strongly Lindelöf

A bitopological space (X, τ_1, τ_2) is said to be (i,j) −strongly compact if for every (i,j) −preopen cover $\{u_\alpha : \alpha \in \Lambda\}$ of X, there exists a finite subset $\{\alpha_1, \alpha_2, \dots, \alpha_k\}_{k \in \mathbb{N}}$ of Λ such that $X = \bigcup_{n=1}^{k} u_{\alpha_n}$ $\forall i,j = 1,2 \quad i \neq j$. A bitopological space is called pairwise strongly compact if it is $(1,2)$ −strongly compact and $(-2,1)$ −strongly compact.
A bitopological space (X, τ_1, τ_2) is said to be (i,j) −strongly Lindelöf if for every (i,j) −preopen cover $\{u_\alpha : \alpha \in \Lambda\}$ of X, there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Λ such that $X = \bigcup_{n \in \mathbb{N}} (u_{\alpha_n})$ $\forall i,j = 1,2 \quad i \neq j$. A topological space is called pairwise strongly Lindelöf if it is $(1,2)$ −strongly Lindelöf and $(2,1)$ −strongly Lindelöf.
If U is a subset of a bitopological space (X, τ_1, τ_2), then U is said to be (i,j) −strongly Lindelöf relative to X if every (i,j) −preopen cover of U admits a countable subcover of it $\forall\, i,j = 1,2 \quad i \neq j$. A subset U is pairwise strongly Lindelöf relative to X if it is both $(1,2)$ −strongly Lindelöf relative to X and $(2,1)$ −strongly Lindelöf relative to X.

Remark3.1: If a subset U of a bitopological space (X, τ_1, τ_2) is pairwise strongly Lindelöf relative to X, then U is pairwise strongly Lindelöf. Bourbaki [13] called a topological space (X, τ) submaximal if each of its dense subsets is open. Reilly and Vamanamurthy [10] proved that X is submaximal if and only if each preopen subset of X is open. Ganster [4] showed that each submaximal Lindelöf space is strongly Lindelöf. A bitopological space (X, τ_1, τ_2) is said to be (i,j) −submaximal if each i −dense subset of X is j −open $\forall i,j = 1,2 \quad i \neq j$. If X is $(1,2)$ −submaximal and $(2,1)$ −submaximal, then X is pairwise submaximal.
A bitopological space (X, τ_1, τ_2) is said to be a pairwise maximal strongly Lindelöf if it is pairwise strongly Lindelöf and there exists no strictly finer pairwise strongly Lindelöf bitopological space.

Proposition3.2: A bitopological space is pairwise maximal strongly Lindelöf if and only if it is pairwise L-closed.

Proposition3.3: A countable discrete bitopological space is a pairwise maximal strongly Lindelöf.
Recall that a bitopological space (X, τ_1, τ_2) is p-Housdorff [16] if for each distinct points x and y in X, there exist two disjoint subsets $u \in (X, \tau_i)$ and $v \in (X, \tau_j)$ such that $x \in u$ and $y \in v$ $\forall i,j = 1,2 \quad i \neq j$. Moreover, a bitopological space (X, τ_1, τ_2) is a pairwise P-space if a countable intersection of τ_i −open subsets of X is τ_j −open subset $\forall i,j = 1,2 \quad i \neq j$.

Proposition3.4: If a bitopological space (X, τ_1, τ_2) is p-Housdorff pairwise strongly Lindelöf, then X is a pairwise maximal strongly Lindelöf if and only if it is a pairwise P-space.
Recall that a bitopological space (X, τ_1, τ_2) is p-normal if for a given τ_1 −closed subset F_1 and a τ_2 −closed subset F_2 of X such that $F_1 \cap F_2 = \emptyset$, there exists a τ_1 −open subset u and a τ_2 −open subset v such that $F_1 \subseteq u$ and $F_2 \subseteq v$ and $u \cap v = \emptyset$.

Proposition3.5: If a bitopological space (X, τ_1, τ_2) is p-Hausdorff pairwise maximal strongly Lindelöf , then X is p-normal.

Proposition3.6: Every pairwise maximal strongly Lindelöf is a pairwise submaximal.

Proposition3.7: If a bitopological space (X, τ_1, τ_2) is p-Housdorff pairwise strongly Lindelöf , then it is a pairwise strongly Lindelöf if and only if it is a P-space.

Proposition3.8: A bitopological space (X, τ_1, τ_2) is pairwise submaximal if and only if X is pairwise open and pairwise preopen.

Proposition3.9: The set of all rational numbers is pairwise Lindelöf but not pairwise submaximal.

Proposition3.10: Every pairwise strongly Lindelöf is a pairwise Lindelöf.
Proof: Suppose that a bitopological space (X, τ_1, τ_2) is pairwise strongly Lindelöf, and let $\{u_\alpha : \alpha \in \Lambda\}$ be a cover consisting of pairwise preopen subsets, then $\{u_\alpha : \alpha \in \Lambda\}$ is a cover consisting of pairwise open subsets since each preopen subset is open. Hence, there exists a countable subset $\{\alpha_1, \alpha_2, ...\}$ such that $X = \bigcup_{k=1}^{\infty} u_{\alpha_k}$.That is, X is pairwise Lindelöf.

4. Pairwise Nearly Strongly Lindelöf Spaces

Proposition4.1: In a bitopological space (X, τ_1, τ_2), X is pairwise nearly strongly Lindelöf if and only if for each family $\{F_\alpha : \alpha \in \Lambda\}$ of pairwise preclosed subsets of X with $\bigcap_{\alpha\in\Lambda} F_\alpha \neq \emptyset$, there exists a countable subset $\{\alpha_1, \alpha_2, ...\}$ such that $\bigcap_{k\in\mathbb{N}} precl_i(preint_j F_{\alpha_k}) \neq \emptyset$.
Proof: Let X be a pairwise nearly strongly Lindelöf and that a family $\{F_\alpha : \alpha \in \Lambda\}$ of pairwise preclosed subsets of X has a non-empty intersection, then $X = X - \bigcap_{\alpha\in\Lambda} F_\alpha = \bigcup_{\alpha\in\Lambda}(X - F_\alpha)$. So, the family $\{F_\alpha : \alpha \in \Lambda\}$ is a pairwise preopen cover of X. The bitopological space being pairwise nearly strongly Lindelöf, thus there exists a countable subset $\{\alpha_1, \alpha_2, ...\}$ such that

$$X = \bigcup_{k\in\mathbb{N}} preint_i(precl_j(X - F_{\alpha_k})) = \bigcup_{k\in\mathbb{N}} preint_i(X - precl_j(F_{\alpha_k})) = \bigcup_{k\in\mathbb{N}} \left(X - precl_i\left(preint_j(F_{\alpha_k})\right)\right) = X - \bigcap_{k\in\mathbb{N}} precl_i(preint_j(F_{\alpha_k})).$$

Thus , $\bigcap_{k\in\mathbb{N}} precl_i(preint_j(F_{\alpha_k})) \neq \emptyset$
For the converse, suppose that for each family $\{F_\alpha : \alpha \in \Lambda\}$ of pairwise preclosed subsets of X with $\bigcap_{\alpha\in\Lambda} F_\alpha \neq \emptyset$, there exists a countable subset $\{\alpha_1, \alpha_2, ...\}$ such that $\bigcap_{k\in\mathbb{N}} precl_i(preint_j F_{\alpha_k}) \neq \emptyset$, let $\{u_\alpha : \alpha \in \Lambda\}$ be a pairwise preopen cover of X , then $\emptyset \neq X - \bigcup_{\alpha\in\Lambda} u_\alpha$, so $\{X - u_\alpha : \alpha \in \Lambda\}$ is a family of pairwise preclosed subsets that has a non-empty intersection, by the assumption, there exists a countable subset such that $\{X - u_{\alpha_k} : k \in \mathbb{N}\}$ such that $\emptyset \neq \bigcap_{k\in\mathbb{N}} precl_i(preint_j(X - u_{\alpha_k})) = \bigcap_{k\in\mathbb{N}} precl_i\left(X - preint_j(u_{\alpha_k})\right) = X\, pre - \bigcup_{k\in\mathbb{N}} preint_i(precl_j(u_{\alpha_k}))$, so
$X = \bigcup_{k\in\mathbb{N}} preint_i(precl_j(u_{\alpha_k}))$, that is, X is a pairwise nearly strongly Lindelöf.

Proposition4.2: If a bitopological space (X, τ_1, τ_2) is a p-regular and every point has a pairwise nearly strongly Lindelöf neighbourhood, then X is pairwise nearly strongly Lindelöf.
Proof: Let F be a pairwise nearly strongly Lindelöf subset of X and suppose that $x \in X\ \ni$

$$x \in preint_i\left(precl_j(F)\right) - F\ \forall i, j = 1,2\ \ i \neq j,$$

then $\exists$ a pairwise preopen subset u containing x and a preopen set v containing F.
Now, u is a pairwise nearly strongly Lindelöf neighbourhood of x. But X is p-regular, so $\exists$ a preopen subset $w \ni$
$x \in w \subseteq preint_i(precl_j(w)) \subseteq u$ and $preint_i(precl_j(F)) \cap preint_i\left(precl_j(w)\right)$ is a pairwise nearly strongly Lindelöf subset of u. Therefore, $x \in F$ and F is pairwise closed.
For every pairwise preopen cover $\{u_\beta : \beta \in \Lambda\}$ of F, there exists a pairwise countable subset $\{\beta_1, \beta_2, ...\}$ such that $F \subseteq \bigcup_{k\in\mathbb{N}} preint_i(precl_j(u_{\beta_k}))$. Thus, X is a pairwise nearly strongly Lindelöf.

Proposition4.3: In a bitopological space (X, τ_1, τ_2), the following statements are equivalent:

1. X is pairwise nearly strongly Lindelöf.
2. For every family of p-regular closed subsets $\{F_\alpha : \alpha \in \Delta\}$ of $X \ni \bigcap F_\alpha = \emptyset$,
there is a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of $\Delta \ni \bigcap_{n\in\mathbb{N}} F_{\alpha_n} = \emptyset$.

Proposition4.4: If a bitopological space (X, τ_1, τ_2) is pairwise nearly strongly Lindelöf, then for each pairwise regularly preclosed subsets A and B of X, $\exists$ two disjoint preopen subsets U and $V \ni A \subseteq U$ and $B \subseteq V$.
Proof: Let X be a pairwise nearly strongly Lindelöf p-regular space, then for every $x \in A$, $\exists$ a pairwise preopen x −neighborhood $U_x \ni B \subseteq X - cl_i(U_x)$ $\forall i = 1,2$.

If $\widetilde{U} = \{U_x : x \in A\}$ is a pairwise preopen-cover of X, then $\exists$ a countable set of points of X say $\{x_1, x_2, \dots\} \ni X = \left(\bigcup_{k=1}^{\infty} int_i(cl_j(U_{x_k}))\right) \cup (X - A)$ because X is a pairwise nearly strongly Lindelöf space $\forall i, j = 1,2 \;\; i \neq j$.
Similarly, for every $y \in B$, $\exists$ a pairwise preopen y −neighborhood $G_y \ni B \subseteq X - cl_i(G_y)$ $\forall i = 1,2$. If $\tilde{G} = \{G_y : y \in B\}$ is a pairwise preopen-cover of X, then $\exists$ a countable set of points of X say $\{y_1, y_2, \dots\} \ni$
$X = \left(\bigcup_{k=1}^{\infty} int_i(cl_j(G_{y_k}))\right) \cup (X - B)$ because X is a pairwise nearly strongly Lindelöf. Now, let $L_k = O_{x_k}$-$\bigcup_{k=1}^{\infty} cl_i(G_{y_k})$ and $M_k = G_{y_k}$-$\bigcup_{k=1}^{\infty} cl_i(O_{x_k})$, then $O = \bigcup_{k=1}^{\infty} L_k$ and $G = \bigcup_{k=1}^{\infty} M_k$ are two disjoint pairwise preopen subsets of X.

Proposition4.5: In a bitopological space (X, τ_1, τ_2), the sum $\bigoplus_{\alpha\in\Lambda} X_\alpha$ where$X_\alpha \neq \emptyset$ for some $\alpha \in \Lambda$ has pairwise nearly strongly Lindelöf property if and only if all bitopological spaces X_α have the pairwise nearly strongly Lindelöf property and the set Λ is countable.
Proof: Suppose that the sum $\bigoplus_{\alpha\in\Lambda} X_\alpha$ where$X_\alpha \neq \emptyset$ for some $\alpha \in \Lambda$ has pairwise nearly strongly Lindelöf property, then all bitopological spaces X_α are pairwise nearly strongly Lindelöf the set Λ is countable.
On the other hand, suppose that all bitopological spaces X_α have the pairwise nearly strongly Lindelöf property and the set Λ is countable and $\{(X, {\tau_1}^k, {\tau_2}^k) : k \in \mathbb{N}\}$ is a family of pairwise nearly strongly Lindelöf, then $X = X_1 \oplus X_2 \oplus \dots$ is pairwise nearly strongly Lindelöf.

Example4.6: Let $X = \mathbb{N} \times [0,1]$ and the topology on X is a subspace of the plane. Now, every $\{a\} \times [0,1]$ is open and closed for every $a \in \mathbb{N}$ in X. A space X can be expressed as a sum of many copies of [0,1].

Proposition4.7: If a bitopological space (X, τ_1, τ_2) is pairwise nearly strongly Lindelöf and a subset A of X is pairwise locally countable subset of X, then A is discrete.

5.Pairwise Almost Strongly Lindelöf Spaces

A bitopological space $(X, \tau_1, \tau_2)$is called an τ_i −almost strongly Lindelöf if every pairwise preopen cover $\{u_\alpha : \alpha \in \Lambda\}$ of X admits a countable subset $\{\alpha_1, \alpha_2, \dots\}$ such that $X = \bigcup_{k\in\mathbb{N}} precl_i(u_{\alpha_k})$ $\forall i = 1,2$. Now if X is τ_1 −almost strongly Lindelöf and τ_2 −almost strongly Lindelöf, then X is a pairwise almost Lindelöf.

Proposition5.1: A bitopological space (X, τ_1, τ_2) is a pairwise nearly strongly, pairwise almost strongly Lindelöf if and only if every family of a pairwise preclosed subsets $\tilde{F} = \{F_\alpha : \alpha \in \Delta\}$ of X such that $\bigcap_{\alpha\in\Delta} F_\alpha = \emptyset$ admits a countable subfamily such that $\bigcap_{k=1}^{\infty} F_{\alpha_k} = \emptyset$.
Proof: Suppose that $\tilde{F} = \{F_\alpha : \alpha \in \Delta\}$ is a family of pairwise preclosed subsets of X such that $\bigcap_{\alpha\in\Delta} F_\alpha = \emptyset$, then $\{X - F_\alpha : \alpha \in \Delta\}$ is a cover of Xconsisting of pairwise preopen subsets.
Now, X is a pairwise nearly strongly and a pairwise almost strongly Lindelöf space, so there exists a countable subfamily $\{\alpha_1,\alpha_2, \dots\}$ such that $X = \bigcup_{k=1}^{\infty} precl_i\left((X - F_{\alpha_k})\right)$, that is $\bigcap_{k=1}^{\infty} preint_i(F_{\alpha_k}) = \emptyset$.
On the other hand, if $\widetilde{U} = \{u_\alpha : \alpha \in \Delta\}$ is a pairwise preopen cover of X, then $\{X - u_\alpha : \alpha \in \Delta\}$ is a family of pairwise preclosed subsets of X such that $\bigcap_{\alpha\in\Delta}(X - u_\alpha) = \emptyset$.
Using the assumption, there exists a countable subfamily $\{\alpha_1,\alpha_2, \dots\}$ such that $\bigcap_{k=1}^{\infty}(X - u_{\alpha_k}) = \emptyset$, i.e $X = \bigcup_{k=1}^{\infty} cl(u_{\alpha_k})$.

Propositio5.2: A bitopological space (X, τ_1, τ_2) is a pairwise almost strongly Lindelöf if every cover of X by pairwise preopen p-regular subsets has a countable refinement.

If a bitopological space (X, τ_1, τ_2) is pairwise almost strongly Lindelöf p-regular, then a subspace (Y, τ'_1, τ'_2) is said to be pairwise almost strongly Lindelöf p-regular relative to X if for each family $\widetilde{U} = \{u_\alpha : \alpha \in \Lambda\}$ consisting of pairwise preopen subsets of $X \ni Y \subseteq \bigcup_{\alpha\in\Lambda} precl_i u_\alpha$, $\exists$ a regularly-closed subset F_α of $X \ni F_\alpha \subseteq u_\alpha$ and $Y \subseteq \bigcup_{\alpha\in\Lambda} preint_i(F_\alpha)$, $\exists$ a countable subset $\{\alpha_k : k \in \mathbb{N}\}$ $\ni Y \subseteq \bigcup_{k\in\mathbb{N}} precl_i(u_{\alpha_k})$.

Proposition5.3: If a bitopological space (X, τ_1, τ_2) is pairwise almost strongly Lindelöf p-regular, and (Y, τ'_1, τ'_2) is a pairwise almost strongly Lindelöf p-regular subspace of X, then Y is pairwise almost strongly Lindelöf p-regular relative to X.
Proof: Let $\widetilde{U} = \{u_\alpha : \alpha \in \Lambda\}$ be a cover of pairwise preopen subsets of Y, $\exists$ a pairwise regularly-closed subset F_α of $X \ni F_\alpha \subseteq u_\alpha$ and $Y \subseteq \bigcup_{\alpha\in\Lambda} preint_i(F_\alpha)$ $\forall i = 1,2$, $\exists$ a countable subset $\{\alpha_k : k \in \mathbb{N}\} \ni Y \subseteq \bigcup_{k\in\mathbb{N}} precl_i(u_{\alpha_k})$.
Now, $V_\alpha = preint_i(F_\alpha) \cap Y$ and $W_\alpha = u_\alpha \cap Y$ are pairwise preopen subsets of Y and $precl_i(V_\alpha)$ is a p-regularly closed subset of Y, so $precl_i(V_\alpha) \subseteq (F_\alpha \cap Y) \subseteq u_\alpha \cap Y$ $\forall \alpha \in \Lambda$. $Y = \bigcap_{\alpha\in\Lambda} precl_i V_\alpha$ and $V_\alpha \subseteq int(cl(V_\alpha))$ in Y, so $Y = \bigcup_{\alpha\in\Lambda} preint_i(precl_j(V_\alpha))$, but Y is a pairwise almost strongly Lindelöf p-regular subspace of X, so $\exists$ a countable subset $\{\alpha_1, \alpha_2, \ldots\}$ $\ni$ $Y = \bigcup_{k\in\mathbb{N}} precl_i(W_{\alpha_k})$. Since $precl_i(W_{\alpha_k}) \subseteq precl_i u_{\alpha_k}$, hence $Y = \bigcup_{k\in\mathbb{N}} precl_i u_{\alpha_k}$. Thus, Y is a pairwise almost strongly Lindelöf p-regular subspace relative to X.

Popsition5.4: In a bitopological space (X, τ_1, τ_2), if every pairwise proper preclosed subset is a pairwise almost strongly Lindelöf relative to X, then X is pairwise almost strongly Lindelöf.
Proof: Suppose that $\{u_\alpha : \alpha \in \Lambda\}$ be a cover of X consisting of pairwise preopen subsets. Let F_α be a pairwise proper preclosed subset that is pairwise almost strongly Lindelöf relative to X such that it is covered by $\{u_\alpha : \alpha \in \Lambda\} - F_\alpha$. There exists a countable subset say $\{\alpha_1, \alpha_2, \ldots\}$ such that $F_\alpha \subset \bigcup_{k\in\mathbb{N}} precl_i(u_{\alpha_k})$. Now,
$X = \bigcup_{k\in\mathbb{N}} precl_i(u_{\alpha_k}) \cup precl_i(X - F_\alpha)$ $\forall i = 1,2$ which implies that X is a pairwise almost strongly Lindelöf.

Proposition5.5: In a bitopological space (X, τ_1, τ_2), if every pairwise preopen subset is a pairwise almost strongly Lindelöf relative to X, then the subset is pairwise almost strongly Lindelöf.

Corollary5.6: If every proper p-regularly-closed subset of a bitopological space (X, τ_1, τ_2) is pairwise almost strongly Lindelöf p-regular, then X is pairwise almost strongly Lindelöf p-regular space.

Proposition5.7: A pairwise strongly p-regular pairwise almost strongly Lindelöf is pairwise strongly Lindelöf.

A bitopological space (X, τ_1, τ_2) is said to be a hereditarily pairwise almost strongly Lindelöf space if every subspace of X is pairwise almost strongly Lindelöf.

Proposition5.8: For hereditarily pairwise almost strongly Lindelöf spaces, the following statements are equivalent:
1. X is countable discrete.
2. X is hereditarily pairwise Lindelöf.

Proposition5.9:In a bitopological space (X, τ_1, τ_2), if U is a proper pairwise preopen subset of X and F is a proper pairwise preclosed subset of X each of which is pairwise almost strongly Lindelöf relative to X, then X is pairwise almost strongly Lindelöf.
Proof: Suppose that U be a pairwise preopen subset of X and $\{u_\alpha : \alpha \in \Lambda\}$ be a cover of it consisting of pairwise preopen subsets, then $\{u_\alpha : \alpha \in \Lambda\} \cup (X - U)$ is a cover of X consisting of pairwise preopen subsets, there exists a countable subset $\{\alpha_1, \alpha_2, \ldots\}$ such that $X = \bigcup_{k\in\mathbb{N}} precl_i(u_{\alpha_k}) \cup precl_i(X - U)$. Now, $U \subset \bigcup_{k\in\mathbb{N}} precl_i(u_{\alpha_k})$, hence U is a pairwise almost strongly Lindelöf relative to X. Thus, it is a pairwise almost strongly Lindelöf subspace of X.

Proposition5.10: A bitopological space (X, τ_1, τ_2) is a pairwise almost strongly Lindelöf pairwise P-space if and only if every locally countable family of subsets of X has a closure preserving property.
Proof: Let that X be a pairwise P-space, if $\tilde{A} = \{A_\alpha : \alpha \in \Lambda\}$ is a locally countable family of subsets of X, and $\tilde{A}_1$ is a subfamily of $\tilde{A}$, then $\bigcup cl_i(A_\alpha)$ is pairwise preclosed $\forall \alpha \in \Lambda$ because X is pairwise almost strongly Lindelöf pairwise P-space and $\tilde{A}$ is locally countable. Hence, $\tilde{A}$ has a closure preserving property $\forall i = 1,2$.
Conversely, suppose that $\tilde{A}$ is a locally countable family, and F_k is a pairwise preclosed subset of X. Let $F = \bigcup_{k\in\mathbb{N}} F_k$ and $x \in F_k$ for some $k \in \mathbb{N}$. Hence, $F_k \cap A_\alpha$ is countable since $\tilde{A}$ is locally countable. Now, X has a closure preserving property, thus

$$cl_i(\bigcup\{(F_k \cap A_\alpha): A_\alpha \in \tilde{A}_1\}) = \bigcup\{cl_i(F_k \cap A_\alpha): A \in \tilde{A}_1\} \quad \forall i = 1,2$$

, hence X is a pairwise P-space.

6. Pairwise Weakly Strongly Lindelöf Spaces

In 1996, Commoroto and Santoro [1] initiated a topological space (X, τ) that is called a weakly Lindelöf in which for each open-cover $\tilde{O} = \{O_\alpha : \alpha \in \Delta\}$ of subsets of X $\exists$ a countable subset $\{\alpha_1, \alpha_2, \ldots\} \ni X = cl(\bigcup_{k=1}^{\infty} O_{\alpha_k})$.

Definition6.1: A bitopological space (X, τ_1, τ_2) is said to be τ_i −weakly strongly Lindelöf space if for each pairwise preopen-cover $\tilde{O} = \{O_\alpha : \alpha \in \Delta\}$ of subsets of X ,$\exists$ a countable subset $\{\alpha_1, \alpha_2, \ldots\} \ni X = precl_i(\bigcup_{k=1}^{\infty} O_{\alpha_k})$ $\forall i = 1,2$. If X is τ_1 −weakly strongly Lindelöf and τ_2 −weakly strongly Lindelöf, then X is called a pairwise weakly strongly Lindelöf bitopological space.

Proposition6.2: A bitopological space (X, τ_1, τ_2) is pairwise weakly strongly Lindelöf if and only if for any family of pairwise preclosed subsets of X say $\widetilde{F} = \{F_\alpha : \alpha \in \Delta\} \ni \bigcap_{\alpha\in\Delta} F_\alpha = \emptyset$, $\exists$ a countable subset $\{\alpha_1, \alpha_2, \ldots\} \ni preint_i(\bigcap_{k=1}^{\infty} F_{\alpha_k}) = \emptyset$.

Proof: Let $\widetilde{F} = \{F_\alpha : \alpha \in \Delta\}$ be a family of pairwise preclosed subsets of $X \ni \bigcap_{\alpha\in\Delta} F_\alpha = \emptyset$. Then $X = \bigcup_{\alpha\in\Delta}(X - F_\alpha)$, $\exists$ a countable subset $\{\alpha_1, \alpha_2, \ldots\} \ni X = precl_i(X - F_{\alpha_k})$. So, $X - precl_i(\bigcup_{k=1}^{\infty}(X - F_{\alpha_k})) = \emptyset$.

That is, $preint_i(X - \bigcup_{k=1}^{\infty}(X - F_{\alpha_k})) = preint_i(\bigcap_{k=1}^{\infty} F_{\alpha_k}) = \emptyset$ $\forall i = 1,2$.

The converse is clear.

Definition 6.3: A bitopological space (X, τ_1, τ_2) is said to be pairwise regularly weakly strongly Lindelöf space if every cover of X by p-regular pairwise preopen subsets $\{u_\alpha : \alpha \in \Lambda\}$, $\exists$ a countable subset $\{\alpha_1, \alpha_2, \ldots\} \ni X = precl_i(\bigcup_{k=1}^{\infty} u_{\alpha_k})$ $\forall i = 1,2$.

In a bitopological space (X, τ_1, τ_2), a subset A is called pairwise weakly strongly Lindelöf relative to X if for every cover $\{v_\alpha : \alpha \in \Lambda\}$ of X consisting of pairwise preopen subsets, there exists a countable subset $\{\alpha_1, \alpha_2, \ldots\}$ such that $A \subseteq precl_i(\bigcup_{k\in\mathbb{N}} v_{\alpha_k})$.

Proposition6.4: In a bitopological space (X, τ_1, τ_2), if A is a subset of X, then A is pairwise weakly strongly Lindelöf if and only if for every family $\{F_\alpha : \alpha \in \mathbb{N}\}$ of pairwise preclosed subsets of X such that $(\bigcap_{k\in\mathbb{N}} preint_i(F_{\alpha_k})) \cap A = \emptyset$, there exists a countable subset $\{\alpha_1, \alpha_2, \ldots\}$ such that $(\bigcap_{k\in\mathbb{N}} preint_i(F_{\alpha_k}) \cap A) = \emptyset$.

Proposition6.5: In a bitopological space (X, τ_1, τ_2), if A is a subset of X such that it is a pairwise weakly strongly Lindelöf subset relative to X, then A is pairwise weakly strongly Lindelöf subspace of X.

Proof: Suppose that $\{u_\alpha : \alpha \in \Lambda\}$ is a cover of A consisting of pairwise preopen subsets of A, then $\{u_\alpha : \alpha \in \Lambda\} \cup (X - A)$ is a cover of X consisting of pairwise preopen subsets, so there exists a countable subset $\{\alpha_1, \alpha_2, \ldots\}$ such that $A \subseteq preint_i(\bigcup_{k\in\mathbb{N}} u_{\alpha_k})$ $\forall i = 1,2$. Thus, A is a pairwise weakly strongly Lindelöf subspace of X.

7.Conclusions

In this article, we clarified the importance of preopen countable covers and their subfamilies and we display the role of such covers in formatting bitopological spaces we introduce and how implications are generated. Bitopological spaces we introduce are: pairwise strongly Lindelöf, pairwise nearly, pairwise almost and pairwise weakly strongly Lindelöf bitopological spaces.

Acknowledgements

The authors gratefully thank Deanship of Scientific Research and Graduate Studies in their universities and the three anonymous referees for their deductive comments and useful recommendations that helped to enhance the quality of this paper.

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Contribution of individual authors to the creation of a scientific article (ghostwriting policy)

Eman Almuhur carried out proofs of prepositions of sections 3 and 4, and she got counter examples of implications.
Manal Al-labadi carried out proofs of prepositions of sections 5 and 6.

Sources of funding for research presented in a scientific article or scientific article itself

Funding; there is no financial support.

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